In chapter III, we stated the postulates of non-relativistic quantum mechanics, and in chapter IX, we concentrated on those which concern spin degrees of freedom. Here, we shall see (§A) that, in reality, these postulates are not sufficient when we are dealing with systems containing many identical particles since, in this case, their application leads to ambiguities in the physical predictions. To eliminate these ambiguities, it is necessary to introduce a new postulate, concerning the quantum mechanical description of systems of identical particles. We shall state this postulate in §C and discuss its physical implications in §D. Before we do so, however, we shall (in §B) define and study permutation operators, which considerably facilitate the reasoning and the calculations.

A. STATEMENT OF THE PROBLEM

1. Identical particles: definition

Two particles are said to be identical if all their intrinsic properties (mass, spin, charge, etc.) are exactly the same : no experiment can distinguish one from the other. Thus, all the electrons in the universe are identical, as are all the protons and all the hydrogen atoms. On the other hand, an electron and a positron are not identical, since, although they have the same mass and the same spin, they have different electrical charges.

An important consequence can be deduced from this definition : when a physical system contains two identical particles, there is no change in its properties or its evolution if the roles of these two particles are exchanged.

COMMENT :

Note that this definition is independent of the experimental conditions. Even if, in a given experiment, the charges of the particles are not measured, an electron and a positron can never be treated like identical particles.

2. Identical particles in classical mechanics

In classical mechanics, the presence of identical particles in a system poses no particular problems. This special case is treated just like the general case. Each particle moves along a well-defined trajectory, which enables us to distinguish it from the others and "follow" it throughout the evolution of the system.

To treat this point in greater detail, we shall consider a system of two identical particles. At the initial time t_0, the physical state of the system is defined by specifying the position and velocity of each of the two particles; we denote this initial data by $\{ \mathbf{r}_0, \mathbf{v}_0 \}$ and $\{ \mathbf{r}'_0, \mathbf{v}'_0 \}$. To describe this physical state and calculate its evolution, we number the two particles: $\mathbf{r}_1(t)$ and $\mathbf{v}_1(t)$ denote the position and velocity of particle (1) at time t, and $\mathbf{r}_2(t)$ and $\mathbf{v}_2(t)$, those of particle (2). This numbering has no physical foundation, as it would if we were dealing with two particles having different natures. It follows that the initial physical state which we have just defined may, in theory, be described by two different "mathematical states" as we can set, either:

$$\begin{aligned} \mathbf{r}_1(t_0) &= \mathbf{r}_0 \qquad \mathbf{r}_2(t_0) = \mathbf{r}'_0 \\ \mathbf{v}_1(t_0) &= \mathbf{v}_0 \qquad \mathbf{v}_2(t_0) = \mathbf{v}'_0 \end{aligned} \tag{A-1}$$

or:

$$\begin{aligned} \mathbf{r}_1(t_0) &= \mathbf{r}'_0 \qquad \mathbf{r}_2(t_0) = \mathbf{r}_0 \\ \mathbf{v}_1(t_0) &= \mathbf{v}'_0 \qquad \mathbf{v}_2(t_0) = \mathbf{v}_0 \end{aligned} \tag{A-2}$$

Now, let us consider the evolution of the system. Suppose that the solution of the equations of motion defined by initial conditions (A-1) can be written:

$$\mathbf{r}_1(t) = \mathbf{r}(t) \qquad \mathbf{r}_2(t) = \mathbf{r}'(t) \tag{A-3}$$

where $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are two vector functions. The fact that the two particles are identical implies that the system is not changed if they exchange roles. Consequently, the Lagrangian $\mathscr{L}(\mathbf{r}_1, \mathbf{v}_1; \mathbf{r}_2, \mathbf{v}_2)$ and the classical Hamiltonian $\mathscr{H}(\mathbf{r}_1, \mathbf{p}_1; \mathbf{r}_2, \mathbf{p}_2)$ are invariant under exchange of indices 1 and 2. It follows that the solution of the equations of motion corresponding to the initial state (A-2) is:

$$\mathbf{r}_1(t) = \mathbf{r}'(t) \qquad \mathbf{r}_2(t) = \mathbf{r}(t) \tag{A-4}$$

where the functions $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are the same as in (A-3).

The two possible mathematical descriptions of the physical state under consideration are therefore perfectly equivalent, since they lead to the same physical predictions. The particle which started from $\{ \mathbf{r}_0, \mathbf{v}_0 \}$ at t_0 is at $\mathbf{r}(t)$ with the velocity $\mathbf{v}(t) = d\mathbf{r}/dt$ at time t, and the one which started from $\{ \mathbf{r}'_0, \mathbf{v}'_0 \}$ is at $\mathbf{r}'(t)$ with the velocity $\mathbf{v}'(t) = d\mathbf{r}'/dt$ (fig. 1). Under these conditions, all we need to do is choose, at the initial time, either one of the two possible "mathematical states" and ignore the existence of the other one. Thus, we treat the system as if the two particles were actually of different natures. The numbers (1) and (2), with which we label them

arbitrarily at t_0, then act like intrinsic properties to distinguish the two particles. Since we can follow each particle step-by-step along its trajectory (arrows in figure 1), we can determine the locations of the particle numbered (1) and the one numbered (2) at any time.

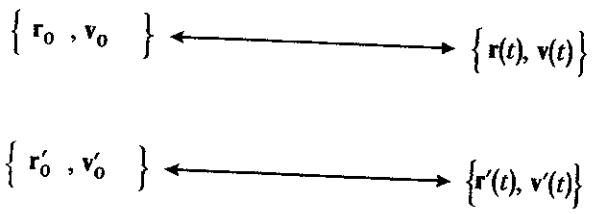

FIGURE 1

Position and velocity of each of the two particles at the initial time t_0 and at time t.

3. Identical particles in quantum mechanics: the difficulties of applying the general postulates

a. QUALITATIVE DISCUSSION OF A FIRST SIMPLE EXAMPLE

It is immediately apparent that the situation is radically different in quantum mechanics, since the particles no longer have definite trajectories. Even if, at t_0, the wave packets associated with two identical particles are completely separated in space, their subsequent evolution may mix them. We then "lose track" of the particles; when we detect one particle in a region of space in which both of them have a non-zero position probability, we have no way of knowing if the particle detected is the one numbered (1) or the one numbered (2). Except in special cases – for example, when the two wave packets never overlap – the numbering of the two particles becomes ambiguous when their positions are measured, since, as we shall see, there exist several distinct "paths" taking the system from its initial state to the state found in the measurement.

To investigate this point in greater detail, consider a concrete example: a collision between two identical particles in their center of mass frame (fig. 2). Before the collision, we have two completely separate wave packets, directed towards each other (fig. 2-a). We can agree, for example, to denote by (1) the particle on the left and by (2), the one on the right. During the collision (fig. 2-b), the two wave packets overlap. After the collision, the region of space in which the probability density of the two particles is non-zero* looks like a spherical shell whose radius increases over time (fig. 2-c). Suppose that a detector placed in the direction which makes an angle θ with the initial velocity of wave packet (1) detects a particle. It is then certain (because momentum is conserved in the collision) that the other particle is moving away in the opposite direction. However, it is impossible to know if the particle detected at D is the one initially numbered (1) or the one numbered (2). Thus, there

* The two-particle wave function depends on six variables (the components of the two particles coordinates $\mathbf{r}$ and $\mathbf{r}'$) and is not easily represented in 3 dimensions. Figure 2 is therefore very schematic: the grey regions are those to which both $\mathbf{r}$ and $\mathbf{r}'$ must belong for the wave function to take on significant values.

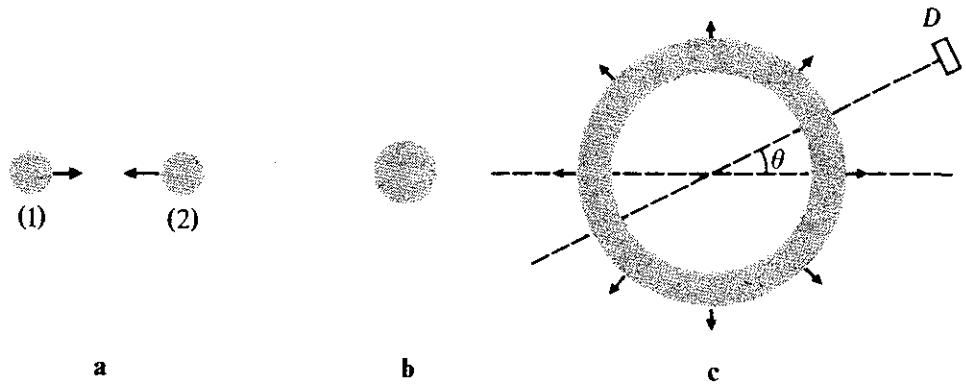

FIGURE 2

Collision between two identical particles in the center of mass frame: schematic representation of the probability density of the two particles.
Before the collision (fig. a), the two wave packets are clearly separated and can be labeled. During the collision (fig. b), the two wave packets overlap. After the collision (fig. c), the probability density is non-zero in a region shaped like a spherical shell whose radius increases over time. Because the two particles are identical, it is impossible, when a particle is detected at *D*, to know with which wave packet, (1) or (2), it was associated before the collision.

are two different "paths" that could have led the system from the initial state shown in figure 2-a to the final state found in the measurement. These two paths are represented schematically in figures 3-a and 3-b. Nothing enables us to determine which one was actually followed.

A fundamental difficulty then arises in quantum mechanics when using the postulates of chapter III. In order to calculate the probability of a given measurement result it is necessary to know the final state vectors associated with this result. Here, there are two, which correspond respectively to figures 3a and 3b. These two kets are distinct (and, furthermore, orthogonal). Nevertheless, they are

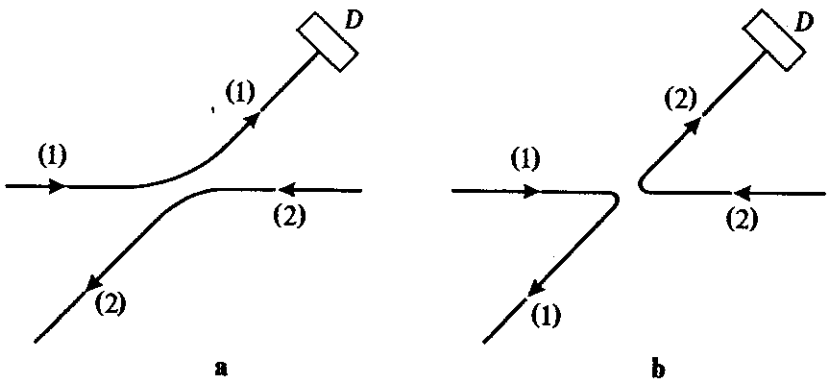

FIGURE 3

Schematic representation of two types of "paths" which the system could have followed in going from the initial state to the state found in the measurement. Because the two particles are identical, we cannot determine the path that was actually followed.

associated with a single physical state since it is impossible to imagine a more complete measurement that would permit distinguishing between them. Under these conditions, should one calculate the probability using path 3a, path 3b or both? In the latter case, should one take the sum of the probabilities associated with each path, or the sum of their probability amplitudes (and in this case, with what sign)? These different possibilities lead, as we shall verify later, to different predictions.

The answer to the preceding questions will be given in §D after we have stated the symmetrization postulate. Before going on, we shall study another example which will aid us in understanding the difficulties related to the indistinguishability of two particles.

b. ORIGIN OF THE DIFFICULTIES: EXCHANGE DEGENERACY

In the preceding example, we considered two wave packets which, initially, did not overlap, which enabled us to label each of them arbitrarily with a number, (1) or (2). Ambiguities appeared, however, when we tried to determine the mathematical state (or ket) associated with a given result of a position measurement. Actually, the same difficulty arises in the choice of the mathematical ket used to describe the initial physical state. This type of difficulty is related to the concept of "exchange degeneracy" which we shall introduce in this section. To simplify the reasoning, we shall first consider a different example, so as to confine ourselves to a finite-dimensional space. Then, we shall generalize the concept of exchange degeneracy, showing that it can be generalized to all quantum mechanical systems containing identical particles.

α. *Exchange degeneracy for a system of two spin 1/2 particles*

Let us consider a system composed of two identical spin 1/2 particles, confining ourselves to the study of its spin degrees of freedom. As in §A-2, we shall distinguish between the physical state of the system and its mathematical description (a ket in state space).

It would seem natural to suppose that, if we made a complete measurement of each of the two spins, we would then know the physical state of the total system perfectly. Here, we shall assume that the component along Oz of one of them is equal to $+\hbar/2$ and that of the other one, $-\hbar/2$ (this is the equivalent for the two spins of the specification of $\{ \mathbf{r}_0, \mathbf{v}_0 \}$ and $\{ \mathbf{r}'_0, \mathbf{v}'_0 \}$ in §A-2).

To describe the system mathematically, we number the particles: $\mathbf{S}_1$ and $\mathbf{S}_2$ denote the two spin observables, and $\{ | \varepsilon_1, \varepsilon_2 \rangle \}$ (where ε_1 and ε_2 can be equal to $+$ or $-$) is the orthonormal basis of the state space formed by the common eigenkets of S_{1z} (eigenvalue $\varepsilon_1\hbar/2$) and S_{2z} (eigenvalue $\varepsilon_2\hbar/2$).

Just as in classical mechanics, two different "mathematical states" could be associated with the same physical state. Either one of the two orthogonal kets:

$$| \varepsilon_1 = +, \varepsilon_2 = - \rangle \qquad \text{(A-5-a)}$$

$$| \varepsilon_1 = -, \varepsilon_2 = + \rangle \qquad \text{(A-5-b)}$$

can, *a priori*, describe the physical state considered here.

These two kets span a two-dimensional subspace whose normalized vectors are of the form:

$$\alpha | +, - \rangle + \beta | -, + \rangle \qquad \text{(A-6)}$$

with:

$$|\alpha|^2 + |\beta|^2 = 1 \tag{A-7}$$

By the superposition principle, all mathematical kets (A-6) can represent the same physical state as (A-5-a) or (A-5-b) (one spin pointing up and the other one pointing down). This is called "*exchange degeneracy*".

Exchange degeneracy creates fundamental difficulties, since application of the postulates of chapter III to the various kets (A-6) can lead to physical predictions which depend on the ket chosen. Let us determine, for example, the probability of finding the components of the two spins along Ox both equal to $+\hbar/2$. With this measurement result is associated a single ket of the state space. According to formula (A-20) of chapter IV, this ket can be written:

$$\frac{1}{\sqrt{2}}[|\varepsilon_1 = +\rangle + |\varepsilon_1 = -\rangle] \otimes \frac{1}{\sqrt{2}}[|\varepsilon_2 = +\rangle + |\varepsilon_2 = -\rangle]$$

$$= \frac{1}{2}[|+,+\rangle + |-,+\rangle + |+,-\rangle + |-,-\rangle] \tag{A-8}$$

Consequently, the desired probability, for the vector (A-6), is equal to:

$$\left|\frac{1}{2}(\alpha + \beta)\right|^2 \tag{A-9}$$

This probability does depend on the coefficients α and β. It is not possible, therefore, to describe the physical state under consideration by the set of kets (A-6) or by any one of them chosen arbitrarily. The exchange degeneracy must be removed. That is, we must indicate unambiguously which of the kets (A-6) is to be used.

COMMENT:

In this example, exchange degeneracy appears only in the initial state, since we chose the same value for the components of the two spins in the final state. In the general case (for example, if the measurement result corresponds to two different eigenvalues of S_x), exchange degeneracy appears in both the initial and the final state.

β. *Generalization*

The difficulties related to exchange degeneracy arise in the study of all systems containing an arbitrary number N of identical particles ($N > 1$).

Consider, for example, a three-particle system. With each of the three particles, taken separately, are associated a state space and observables acting in this space. Thus, we are led to number the particles: $\mathscr{E}(1)$, $\mathscr{E}(2)$ and $\mathscr{E}(3)$ will denote the three

one-particle state spaces, and the corresponding observables will be labeled by the same indices. The state space of the three-particle system is the tensor product:

$$\mathcal{E} = \mathcal{E}(1) \otimes \mathcal{E}(2) \otimes \mathcal{E}(3) \tag{A-10}$$

Now, consider an observable $B(1)$, initially defined in $\mathcal{E}(1)$. We shall assume that $B(1)$ alone constitutes a C.S.C.O. in $\mathcal{E}(1)$ [or that $B(1)$ actually denotes several observables which form a C.S.C.O.]. The fact that the three particles are identical implies that the observables $B(2)$ and $B(3)$ exist and that they constitute C.S.C.O.'s in $\mathcal{E}(2)$ and $\mathcal{E}(3)$ respectively. $B(1)$, $B(2)$ and $B(3)$ have the same spectrum, $\{b_n; n = 1, 2, \ldots\}$. Using the bases which define these three observables in $\mathcal{E}(1)$, $\mathcal{E}(2)$ and $\mathcal{E}(3)$, we can construct, by taking the tensor product, an orthonormal basis of $\mathcal{E}$, which we shall denote by:

$$\{|\, 1 : b_i; 2 : b_j; 3 : b_k \rangle; i, j, k = 1, 2, \ldots\} \tag{A-11}$$

The kets $|\, 1 : b_i; 2 : b_j; 3 : b_k \rangle$ are common eigenvectors of the extensions of $B(1)$, $B(2)$ and $B(3)$ in $\mathcal{E}$, with respective eigenvalues b_i, b_j and b_k.

Since the three particles are identical, we cannot measure $B(1)$ or $B(2)$ or $B(3)$, since the numbering has no physical significance. However, we can measure the physical quantity B for each of the three particles. Suppose that such a measurement has resulted in three different eigenvalues, b_n, b_p and b_q. Exchange degeneracy then appears, since the state of the system after this measurement can, *a priori*, be represented by any one of the kets of the subspace of $\mathcal{E}$ spanned by the six basis vectors:

$$\begin{array}{lll} |\, 1 : b_n; 2 : b_p; 3 : b_q \rangle, & |\, 1 : b_q; 2 : b_n; 3 : b_p \rangle, & |\, 1 : b_p; 2 : b_q; 3 : b_n \rangle, \\ |\, 1 : b_n; 2 : b_q; 3 : b_p \rangle, & |\, 1 : b_p; 2 : b_n; 3 : b_q \rangle, & |\, 1 : b_q; 2 : b_p; 3 : b_n \rangle \end{array} \tag{A-12}$$

Therefore, *a complete measurement on each of the particles does not permit the determination of a unique ket* of the state space of the system.

COMMENT:

The indeterminacy due to exchange degeneracy is, of course, less important if two of the eigenvalues found in the measurement are equal. This indeterminacy disappears in the special case in which the three results are identical.

B. PERMUTATION OPERATORS

Before stating the additional postulate which enables us to remove the indeterminacy related to exchange degeneracy, we shall study certain operators, defined in the total state space of the system under consideration, which actually permute the various particles of the system. The use of these permutation operators will simplify the calculations and reasoning in §§ C and D.

1. Two-particle systems

a. DEFINITION OF THE PERMUTATION OPERATOR P_{21}

Consider a system composed of two particles with the same spin s. Here it is not necessary for these two particles to be identical; it is sufficient that their individual state spaces be isomorphic. Therefore, to avoid the problems which arise when the two particles are identical, we shall assume that they are not: the numbers (1) and (2) with which they are labeled indicate their natures. For example, (1) will denote a proton and (2), an electron.

We choose a basis, $\{ | u_i \rangle \}$, in the state space $\mathcal{E}(1)$ of particle (1). Since the two particles have the same spin, $\mathcal{E}(2)$ is isomorphic to $\mathcal{E}(1)$, and it can be spanned by the same basis. By taking the tensor product, we construct, in the state space $\mathcal{E}$ of the system, the basis:

$$\{ | 1 : u_i; 2 : u_j \rangle \} \tag{B-1}$$

Since the order of the vectors is of no importance in a tensor product, we have:

$$| 2 : u_j; 1 : u_i \rangle \equiv | 1 : u_i; 2 : u_j \rangle \tag{B-2}$$

However, note that:

$$| 1 : u_j; 2 : u_i \rangle \neq | 1 : u_i; 2 : u_j \rangle \qquad \text{if} \quad i \neq j \tag{B-3}$$

The permutation operator P_{21} is then defined as the linear operator whose action on the basis vectors is given by:

$$P_{21} | 1 : u_i; 2 : u_j \rangle = | 2 : u_i; 1 : u_j \rangle = | 1 : u_j; 2 : u_i \rangle \tag{B-4}$$

Its action on any ket of $\mathcal{E}$ can easily be obtained by expanding this ket on the basis (B-1)★.

COMMENT:

If we choose the basis formed by the common eigenstates of the position observable $\mathbf{R}$ and the spin component S_z, (B-4) can be written:

$$P_{21} | 1 : \mathbf{r}, \varepsilon; 2 : \mathbf{r}', \varepsilon' \rangle = | 1 : \mathbf{r}', \varepsilon'; 2 : \mathbf{r}, \varepsilon \rangle \tag{B-5}$$

Any ket $| \psi \rangle$ of the state space $\mathcal{E}$ can be represented by a set of $(2s + 1)^2$ functions of six variables:

$$| \psi \rangle = \sum_{\varepsilon, \varepsilon'} \int d^3r \, d^3r' \, \psi_{\varepsilon,\varepsilon'}(\mathbf{r}, \mathbf{r}') \, | 1 : \mathbf{r}, \varepsilon; 2 : \mathbf{r}', \varepsilon' \rangle \tag{B-6}$$

with:

$$\psi_{\varepsilon,\varepsilon'}(\mathbf{r}, \mathbf{r}') = \langle 1 : \mathbf{r}, \varepsilon; 2 : \mathbf{r}', \varepsilon' | \psi \rangle \tag{B-7}$$

★ It can easily be shown that the operator P_{21} so defined does not depend on the $\{ | u_i \rangle \}$ basis chosen.

We then have:

$$P_{21} \mid \psi \rangle = \sum_{\varepsilon,\varepsilon'} \int d^3r\, d^3r'\, \psi_{\varepsilon,\varepsilon'}(\mathbf{r}, \mathbf{r}') \mid 1 : \mathbf{r}', \varepsilon' ; 2 : \mathbf{r}, \varepsilon \rangle \tag{B-8}$$

By changing the names of the dummy variables:

$$\begin{aligned} &\varepsilon \longleftrightarrow \varepsilon' \\ &\mathbf{r} \longleftrightarrow \mathbf{r}' \end{aligned} \tag{B-9}$$

we transform formula (B-8) into:

$$P_{21} \mid \psi \rangle = \sum_{\varepsilon,\varepsilon'} \int d^3r\, d^3r'\, \psi_{\varepsilon',\varepsilon}(\mathbf{r}', \mathbf{r}) \mid 1 : \mathbf{r}, \varepsilon ; 2 : \mathbf{r}', \varepsilon' \rangle \tag{B-10}$$

Consequently, the functions:

$$\psi'_{\varepsilon,\varepsilon'}(\mathbf{r}, \mathbf{r}') = \langle 1 : \mathbf{r}, \varepsilon ; 2 : \mathbf{r}', \varepsilon' \mid P_{21} \mid \psi \rangle \tag{B-11}$$

which represent the ket $\mid \psi' \rangle = P_{21} \mid \psi \rangle$ can be obtained from the functions (B-7) which represent the ket $\mid \psi \rangle$ by inverting $(\mathbf{r}, \varepsilon)$ and $(\mathbf{r}', \varepsilon')$:

$$\psi'_{\varepsilon,\varepsilon'}(\mathbf{r}, \mathbf{r}') = \psi_{\varepsilon',\varepsilon}(\mathbf{r}', \mathbf{r}) \tag{B-12}$$

b. PROPERTIES OF P_{21}

We see directly from definition (B-4) that:

$$(P_{21})^2 = 1 \tag{B-13}$$

The operator P_{21} is its own inverse.

It can easily be shown that P_{21} *is Hermitian*:

$$P_{21}^{\dagger} = P_{21} \tag{B-14}$$

The matrix elements of P_{21} in the $\{ \mid 1 : u_i ; 2 : u_j \rangle \}$ basis are:

$$\begin{aligned} \langle 1 : u_{i'} ; 2 : u_{j'} \mid P_{21} \mid 1 : u_i ; 2 : u_j \rangle &= \langle 1 : u_{i'} ; 2 : u_{j'} \mid 1 : u_j ; 2 : u_i \rangle \\ &= \delta_{i'j}\, \delta_{j'i} \end{aligned} \tag{B-15}$$

Those of $P_{21}^{\dagger}$ are, by definition:

$$\begin{aligned} \langle 1 : u_{i'} ; 2 : u_{j'} \mid P_{21}^{\dagger} \mid 1 : u_i ; 2 : u_j \rangle &= (\langle 1 : u_i ; 2 : u_j \mid P_{21} \mid 1 : u_{i'} ; 2 : u_{j'} \rangle)^* \\ &= (\langle 1 : u_i ; 2 : u_j \mid 1 : u_{j'} ; 2 : u_{i'} \rangle)^* \\ &= \delta_{ij'}\, \delta_{ji'} \end{aligned} \tag{B-16}$$

Each of the matrix elements of $P_{21}^{\dagger}$ is therefore equal to the corresponding matrix element of P_{21}. This leads to relation (B-14).

It follows from (B-13) and (B-14) that P_{21} is also *unitary*:

$$P_{21}^{\dagger} P_{21} = P_{21} P_{21}^{\dagger} = 1 \tag{B-17}$$

c. SYMMETRIC AND ANTISYMMETRIC KETS. SYMMETRIZER AND ANTISYMMETRIZER

According to (B-14), the eigenvalues of P_{21} must be real. Since, according to (B-13), their squares are equal to 1, these eigenvalues are simply + 1 and − 1. The eigenvectors of P_{21} associated with the eigenvalue + 1 are called *symmetric*, those corresponding to the eigenvalue − 1, *antisymmetric*:

$$\begin{aligned} P_{21}\,|\,\psi_S\,\rangle &= \quad |\,\psi_S\,\rangle \qquad \Longrightarrow |\,\psi_S\,\rangle \text{ symmetric} \\ P_{21}\,|\,\psi_A\,\rangle &= -\,|\,\psi_A\,\rangle \qquad \Longrightarrow |\,\psi_A\,\rangle \text{ antisymmetric} \end{aligned} \tag{B-18}$$

Now consider the two operators:

$$S = \frac{1}{2}(1 + P_{21}) \tag{B-19-a}$$

$$A = \frac{1}{2}(1 - P_{21}) \tag{B-19-b}$$

These operators are *projectors*, since (B-13) implies that:

$$S^2 = S \tag{B-20-a}$$

$$A^2 = A \tag{B-20-b}$$

and, in addition, (B-14) enables us to show that:

$$S^\dagger = S \tag{B-21-a}$$

$$A^\dagger = A \tag{B-21-b}$$

S and A are projectors onto orthogonal subspaces, since, according to (B-13):

$$SA = AS = 0 \tag{B-22}$$

These subspaces are supplementary, since definitions (B-19) yield:

$$S + A = 1 \tag{B-23}$$

If $|\,\psi\,\rangle$ is an arbitrary ket of the state space $\mathcal{E}$, $S\,|\,\psi\,\rangle$ is a symmetric ket and $A\,|\,\psi\,\rangle$, an antisymmetric ket, as it is easy to see, using (B-13) again, that:

$$\begin{aligned} P_{21}\,S\,|\,\psi\,\rangle &= S\,|\,\psi\,\rangle \\ P_{21}\,A\,|\,\psi\,\rangle &= -\,A\,|\,\psi\,\rangle \end{aligned} \tag{B-24}$$

For this reason, S and A are called, respectively, a *symmetrizer* and an *antisymmetrizer*.

COMMENT:

The same symmetric ket is obtained by applying S to $P_{21}\,|\,\psi\,\rangle$ or to $|\,\psi\,\rangle$ itself:

$$SP_{21}\,|\,\psi\,\rangle = S\,|\,\psi\,\rangle \tag{B-25}$$

For the antisymmetrizer, we have, similarly:

$$AP_{21}\,|\,\psi\,\rangle = -\,A\,|\,\psi\,\rangle \tag{B-26}$$

d. TRANSFORMATION OF OBSERVABLES BY PERMUTATION

Consider an observable $B(1)$, initially defined in $\mathscr{E}(1)$ and then extended into $\mathscr{E}$. It is always possible to construct the $\{ \mid u_i \rangle \}$ basis in $\mathscr{E}(1)$ from eigenvectors of $B(1)$ (the corresponding eigenvalues will be written b_i). Let us now calculate the action of the operator $P_{21}B(1)P_{21}^{\dagger}$ on an arbitrary basis ket of $\mathscr{E}$:

$$\begin{aligned} P_{21}B(1)P_{21}^{\dagger} \mid 1 : u_i ; 2 : u_j \rangle &= P_{21}B(1) \mid 1 : u_j ; 2 : u_i \rangle \\ &= b_j P_{21} \mid 1 : u_j ; 2 : u_i \rangle \\ &= b_j \mid 1 : u_i ; 2 : u_j \rangle \end{aligned} \tag{B-27}$$

We would obtain the same result by applying the observable $B(2)$ directly to the basis ket chosen. Consequently:

$$P_{21}B(1)P_{21}^{\dagger} = B(2) \tag{B-28}$$

The same reasoning shows that:

$$P_{21}B(2)P_{21}^{\dagger} = B(1) \tag{B-29}$$

In $\mathscr{E}$, there are also observables, such as $B(1) + C(2)$ or $B(1)C(2)$, which involve both indices simultaneously. We obviously have:

$$P_{21}[B(1) + C(2)]P_{21}^{\dagger} = B(2) + C(1) \tag{B-30}$$

Similarly, using (B-17), we find:

$$\begin{aligned} P_{21}B(1)C(2)P_{21}^{\dagger} &= P_{21}B(1)P_{21}^{\dagger}P_{21}C(2)P_{21}^{\dagger} \\ &= B(2)C(1) \end{aligned} \tag{B-31}$$

These results can be generalized to all observables in $\mathscr{E}$ which can be expressed in terms of observables of the type of $B(1)$ and $C(2)$, to be denoted by $\mathcal{O}(1, 2)$:

$$P_{21}\mathcal{O}(1, 2)P_{21}^{\dagger} = \mathcal{O}(2, 1) \tag{B-32}$$

$\mathcal{O}(2, 1)$ is the observable obtained from $\mathcal{O}(1, 2)$ by exchanging indices 1 and 2 throughout.

An observable $\mathcal{O}_S(1, 2)$ is said to be *symmetric* if:

$$\mathcal{O}_S(2, 1) = \mathcal{O}_S(1, 2) \tag{B-33}$$

According to (B-32), all symmetric observables satisfy:

$$P_{21}\mathcal{O}_S(1, 2) = \mathcal{O}_S(1, 2)P_{21} \tag{B-34}$$

that is:

$$[\mathcal{O}_S(1, 2), P_{21}] = 0 \tag{B-35}$$

Symmetric observables commute with the permutation operator.

2. Systems containing an arbitrary number of particles

In the state space of a system composed of N particles with the same spin (temporarily assumed to be of different natures), $N!$ permutation operators can be defined (one of which is the identity operator). If N is greater than 2, the properties of these operators are more complex than those of P_{21}. To have an idea of the changes involved when N is greater than 2, we shall briefly study the case in which $N = 3$.

a. DEFINITION OF THE PERMUTATION OPERATORS

Consider, therefore, a system of three particles which are not necessarily identical but have the same spin. As in § B-1-a, we construct a basis of the state space of the system by taking a tensor product:

$$\{ | 1 : u_i; 2 : u_j; 3 : u_k \rangle \} \tag{B-36}$$

In this case, there exist six permutation operators, which we shall denote by:

$$P_{123}, P_{312}, P_{231}, P_{132}, P_{213}, P_{321} \tag{B-37}$$

By definition, the operator P_{npq} (where n, p, q is an arbitrary permutation of the numbers 1, 2, 3) is the linear operator whose action on the basis vectors obeys:

$$P_{npq} | 1 : u_i; 2 : u_j; 3 : u_k \rangle = | n : u_i; p : u_j; q : u_k \rangle \tag{B-38}$$

For example:

$$\begin{aligned} P_{231} | 1 : u_i; 2 : u_j; 3 : u_k \rangle &= | 2 : u_i; 3 : u_j; 1 : u_k \rangle \\ &= | 1 : u_k; 2 : u_i; 3 : u_j \rangle \end{aligned} \tag{B-39}$$

P_{123} therefore coincides with the identity operator. The action of P_{npq} on any ket of the state space can easily be obtained by expanding this ket on the basis (B-36).

The $N!$ permutation operators associated with a system of N particles with the same spin could be defined analogously.

b. PROPERTIES

α. *The set of permutation operators constitutes a group*

This can easily be shown for the operators (B-37):

(*i*) P_{123} is the identity operator.

(*ii*) The product of two permutation operators is also a permutation operator. We can show, for example, that:

$$P_{312}P_{132} = P_{321} \tag{B-40}$$

To do so, we apply the left-hand side to an arbitrary basis ket:

$$\begin{aligned} P_{312}P_{132} | 1 : u_i; 2 : u_j; 3 : u_k \rangle &= P_{312} | 1 : u_i; 3 : u_j; 2 : u_k \rangle \\ &= P_{312} | 1 : u_i; 2 : u_k; 3 : u_j \rangle \\ &= | 3 : u_i; 1 : u_k; 2 : u_j \rangle \\ &= | 1 : u_k; 2 : u_j; 3 : u_i \rangle \end{aligned} \tag{B-41}$$

The action of P_{321} effectively leads to the same result:

$$P_{321}\,|\,1:u_i;\,2:u_j;\,3:u_k\,\rangle = |\,3:u_i;\,2:u_j;\,1:u_k\,\rangle = |\,1:u_k;\,2:u_j;\,3:u_i\,\rangle \tag{B-42}$$

(*iii*) Each permutation operator has an inverse, which is also a permutation operator. Reasoning as in (*ii*), we can easily show that:

$$P_{123}^{-1} = P_{123};\quad P_{312}^{-1} = P_{231};\quad P_{231}^{-1} = P_{312}$$
$$P_{132}^{-1} = P_{132};\quad P_{213}^{-1} = P_{213};\quad P_{321}^{-1} = P_{321} \tag{B-43}$$

Note that *the permutation operators do not commute with each other.*

For example:

$$P_{132}P_{312} = P_{213} \tag{B-44}$$

which, compared to (B-40), shows that the commutator of P_{132} and P_{312} is not zero.

β. *Transpositions. Parity of a permutation operator*

A *transposition* is a permutation which simply exchanges the roles of two of the particles, without touching the others. Of the operators (B-37), the last three are transposition operators*. Transposition operators are Hermitian, and each of them is the same as its inverse, so that they are also unitary [the proofs of these properties are identical to those for (B-14), (B-13) and (B-17)].

Any permutation operator can be broken down into a product of transposition operators. For example, the second operator (B-37) can be written:

$$P_{312} = P_{132}P_{213} = P_{321}P_{132} = P_{213}P_{321} = P_{132}P_{213}(P_{132})^2 = \ldots \tag{B-45}$$

This decomposition is not unique. However, for a given permutation, it can be shown that the parity of the number of transpositions into which it can be broken down is always the same : it is called the *parity of the permutation.* Thus, the first three operators (B-37) are even, and the last three, odd. For any N, there are always as many even permutations as odd ones.

γ. *Permutation operators are unitary*

Permutation operators, which are products of transposition operators, all of which are unitary, are therefore also unitary. However, they are not necessarily Hermitian, since transposition operators do not generally commute with each other.

Finally, note that the adjoint of a given permutation operator has the same parity as that of the operator, since it is equal to the product of the same transposition operators, taken in the opposite order.

* Of course, for $N = 2$, the only permutation possible is transposition.

c. COMPLETELY SYMMETRIC OR ANTISYMMETRIC KETS. SYMMETRIZER AND ANTISYMMETRIZER

Since the permutation operators do not commute for $N > 2$, it is not possible to construct a basis formed by common eigenvectors of these operators. Nevertheless, we shall see that there exist certain kets which are simultaneously eigenvectors of all the permutation operators.

We shall denote by P_α an arbitrary permutation operator associated with a system of N particles with the same spin; α represents an arbitrary permutation of the first N integers. A ket $|\psi_S\rangle$ such that:

$$P_\alpha |\psi_S\rangle = |\psi_S\rangle \tag{B-46}$$

for any permutation P_α, is said to be *completely symmetric*. Similarly, a *completely antisymmetric* ket $|\psi_A\rangle$ satisfies, by definition*:

$$P_\alpha |\psi_A\rangle = \varepsilon_\alpha |\psi_A\rangle \tag{B-47}$$

where:

$$\begin{aligned} &\varepsilon_\alpha = +1 \quad \text{if } P_\alpha \text{ is an even permutation} \\ &\varepsilon_\alpha = -1 \quad \text{if } P_\alpha \text{ is an odd permutation} \end{aligned} \tag{B-48}$$

The set of completely symmetric kets constitutes a vector subspace $\mathscr{E}_S$ of the state space $\mathscr{E}$; the set of completely antisymmetric kets, a subspace $\mathscr{E}_A$.

Now consider the two operators:

$$S = \frac{1}{N!} \sum_\alpha P_\alpha \tag{B-49}$$

$$A = \frac{1}{N!} \sum_\alpha \varepsilon_\alpha P_\alpha \tag{B-50}$$

where the summations are performed over the $N!$ permutations of the first N integers, and ε_α is defined by (B-48). We shall show that S and A are the projectors onto $\mathscr{E}_S$ and $\mathscr{E}_A$ respectively. For this reason, they are called a *symmetrizer* and an *antisymmetrizer*.

S and A are Hermitian operators:

$$S^\dagger = S \tag{B-51}$$

$$A^\dagger = A \tag{B-52}$$

The adjoint $P_\alpha^\dagger$ of a given permutation operator is, as we saw above (*cf.* §B-2-b-γ), another permutation operator, of the same parity (which coincides, furthermore, with P_α^{-1}). Taking the adjoints of the right-hand sides of the definitions of S and A therefore amounts simply to changing the order of the terms in the summations (since the set of the P_α^{-1} is again the permutation group).

* According to the property stated in §B-2-b-β, this definition can also be based solely on the transposition operators : any transposition operator leaves a completely symmetric ket invariant and transforms a completely antisymmetric ket into its opposite.

Also, if P_{α_0} is an arbitrary permutation operator, we have:

$$P_{\alpha_0} S = S P_{\alpha_0} = S \tag{B-53-a}$$

$$P_{\alpha_0} A = A P_{\alpha_0} = \varepsilon_{\alpha_0} A \tag{B-53-b}$$

This is due to the fact that $P_{\alpha_0} P_\alpha$ is also a permutation operator:

$$P_{\alpha_0} P_\alpha = P_\beta \tag{B-54}$$

such that:

$$\varepsilon_\beta = \varepsilon_{\alpha_0} \varepsilon_\alpha \tag{B-55}$$

If, for P_{α_0} fixed, we choose successively for P_α all the permutations of the group, we see that the P_β are each identical to one and only one of these permutations (in, of course, a different order). Consequently:

$$P_{\alpha_0} S = \frac{1}{N!} \sum_\alpha P_{\alpha_0} P_\alpha = \frac{1}{N!} \sum_\beta P_\beta = S \tag{B-56-a}$$

$$P_{\alpha_0} A = \frac{1}{N!} \sum_\alpha \varepsilon_\alpha P_{\alpha_0} P_\alpha = \frac{1}{N!} \varepsilon_{\alpha_0} \sum_\beta \varepsilon_\beta P_\beta = \varepsilon_{\alpha_0} A \tag{B-56-b}$$

Similarly, we could prove analogous relations in which S and A are multiplied by P_{α_0} from the right.

From (B-53), we see that:

$$S^2 = S$$
$$A^2 = A \tag{B-57}$$

and, moreover:

$$AS = SA = 0 \tag{B-58}$$

This is because:

$$S^2 = \frac{1}{N!} \sum_\alpha P_\alpha S = \frac{1}{N!} \sum_\alpha S = S$$

$$A^2 = \frac{1}{N!} \sum_\alpha \varepsilon_\alpha P_\alpha A = \frac{1}{N!} \sum_\alpha \varepsilon_\alpha^2 A = A \tag{B-59}$$

as each summation includes $N!$ terms; furthermore:

$$AS = \frac{1}{N!} \sum_\alpha \varepsilon_\alpha P_\alpha S = \frac{1}{N!} S \sum_\alpha \varepsilon_\alpha = 0 \tag{B-60}$$

since half the ε_α are equal to $+1$ and half equal to -1 (*cf.* § B-2-b-β).

S and A are therefore *projectors*. They project respectively onto $\mathscr{E}_S$ and $\mathscr{E}_A$ since, according to (B-53), their action on any ket $|\psi\rangle$ of the state space yields a completely symmetric or completely antisymmetric ket:

$$P_{\alpha_0} S |\psi\rangle = S |\psi\rangle \tag{B-61-a}$$

$$P_{\alpha_0} A |\psi\rangle = \varepsilon_{\alpha_0} A |\psi\rangle \tag{B-61-b}$$

COMMENTS:

(*i*) The completely symmetric ket constructed by the action of S on $P_\alpha \mid \psi \rangle$, where P_α is an arbitrary permutation, is the same as that obtained from $\mid \psi \rangle$, since expressions (B-53) indicate that:

$$SP_\alpha \mid \psi \rangle = S \mid \psi \rangle \tag{B-62}$$

As for the corresponding completely antisymmetric kets, they differ at most by their signs:

$$AP_\alpha \mid \psi \rangle = \varepsilon_\alpha A \mid \psi \rangle \tag{B-63}$$

(*ii*) For $N > 2$, the symmetrizer and antisymmetrizer are not projectors onto supplementary subspaces. For example, when $N = 3$, it is easy to obtain [by using the fact that the first three permutations (B-37) are even and the others odd] the relation:

$$S + A = \frac{1}{3}(P_{123} + P_{231} + P_{312}) \neq 1 \tag{B-64}$$

In other words, the state space is not the direct sum of the subspace $\mathcal{E}_S$ of completely symmetric kets and the subspace $\mathcal{E}_A$ of completely antisymmetric kets.

d. TRANSFORMATION OF OBSERVABLES BY PERMUTATION

We have indicated (§ B-2-b-β) that any permutation operator of an N-particle system can be broken down into a product of transposition operators analogous to the operator P_{21} studied in § B-1. For these transposition operators, we can use the arguments of §B-1-d to determine the behavior of the various observables of the system when they are multiplied from the left by an arbitrary permutation operator P_α and from the right by $P_\alpha^\dagger$.

In particular, the observables $\mathcal{O}_S(1, 2, ..., N)$, which are completely symmetric under exchange of the indices 1, 2, ..., N, commute with all the transposition operators, and, therefore, with all the permutation operators:

$$[\mathcal{O}_S(1, 2, ..., N), P_\alpha] = 0 \tag{B-65}$$

C. THE SYMMETRIZATION POSTULATE

1. Statement of the postulate

When a system includes several identical particles, only certain kets of its state space can describe its physical states. Physical kets are, depending on the nature of the identical particles, either completely symmetric or completely antisymmetric with respect to permutation of these particles. Those particles for which the physical kets are symmetric are called *bosons*, and those for which they are antisymmetric, *fermions*.

The symmetrization postulate thus limits the state space for a system of identical particles. This space is no longer, as it was in the case of particles of different natures, the tensor product $\mathcal{E}$ of the individual state spaces of the particles constituting the system. It is only a subspace of $\mathcal{E}$, $\mathcal{E}_S$ or $\mathcal{E}_A$, depending on whether the particles are bosons or fermions.

From the point of view of this postulate, particles existing in nature are divided into two categories. All currently known particles obey the following *empirical rule**: particles of half-integral spin (electrons, positrons, protons, neutrons, muons, etc.) are fermions, and particles of integral spin (photons, mesons, etc.) are bosons.

COMMENT:

Once this rule has been verified for the particles which are called "elementary", it holds for all other particles as well, inasmuch as they are composed of these elementary particles. Consider a system of many identical composite particles. Permuting two of them is equivalent to simultaneously permuting all the particles composing the first one with the corresponding particles (necessarily identical to the aforementioned ones) of the second one. This permutation must leave the ket describing the state of the system unchanged if the composite particles being studied are formed only of elementary bosons or if each of them contains an even number of fermions (no sign change, or an even number of sign changes); in this case, the particles are bosons. On the other hand, composite particles containing an odd number of fermions are themselves fermions (an odd number of sign changes in the permutation). Now, the spin of these composite particles is necessarily integral in the first case and half-integral in the second one (chap. X, § C-3-c). They therefore obey the rule just stated. For example, atomic nuclei are known to be composed of neutrons and protons, which are fermions (spin 1/2). Consequently, nuclei whose mass number A (the total number of nucleons) is even are bosons, and those whose mass number is odd are fermions. Thus, the nucleus of the ^{3}He isotope of helium is a fermion, and that of the ^{4}He isotope, a boson.

2. Removal of exchange degeneracy

We shall begin by examining how this new postulate removes the exchange degeneracy and the corresponding difficulties.

The discussion of § A can be summarized in the following way. Let $|u\rangle$ be a ket which can mathematically describe a well-defined physical state of a system containing N identical particles. For any permutation operator P_α, $P_\alpha|u\rangle$ can describe this physical state as well as $|u\rangle$. The same is true for any ket belonging to the subspace $\mathcal{E}_u$ spanned by $|u\rangle$ and all its permutations $P_\alpha|u\rangle$. Depending on the ket $|u\rangle$ chosen, the dimension of $\mathcal{E}_u$ can vary between 1 and $N!$. If this dimension is greater than 1, several mathematical kets correspond to the same physical state: there is then an exchange degeneracy.

* The "spin-statistics theorem", proven in quantum field theory, makes it possible to consider this rule to be a consequence of very general hypotheses. However, these hypotheses may not all be correct, and discovery of a boson of half-integral spin or a fermion of integral spin remains possible. It is not inconceivable that, for certain particles, the physical kets might have more complex symmetry properties than those envisaged here.

The new postulate which we have introduced considerably restricts the class of mathematical kets able to describe a physical state : these kets must belong to $\mathscr{E}_S$ for bosons and to $\mathscr{E}_A$ for fermions. We shall be able to say that the difficulties related to exchange degeneracy are eliminated if we can show that $\mathscr{E}_u$ contains *a single* ket of $\mathscr{E}_S$ or *a single* ket of $\mathscr{E}_A$.

To do so, we shall use the relations $S = SP_\alpha$ or $A = \varepsilon_\alpha AP_\alpha$, proven in (B-53). We obtain :

$$S \mid u \rangle = SP_\alpha \mid u \rangle \tag{C-1-a}$$

$$A \mid u \rangle = \varepsilon_\alpha AP_\alpha \mid u \rangle \tag{C-1-b}$$

These relations express the fact that the projections onto $\mathscr{E}_S$ and $\mathscr{E}_A$ of the various kets which span $\mathscr{E}_u$ and, consequently, of all the kets of $\mathscr{E}_u$, are collinear. The symmetrization postulate thus unambiguously indicates (to within a constant factor) *the* ket of $\mathscr{E}_u$ which must be associated with the physical state considered : $S \mid u \rangle$ for bosons and $A \mid u \rangle$ for fermions. This ket is called the *physical ket*.

COMMENT:

It is possible for all the kets of $\mathscr{E}_u$ to have a zero projection onto $\mathscr{E}_A$ (or $\mathscr{E}_S$). In this case, the symmetrization postulate excludes the corresponding physical state. Later (§§3-b and 3-c), we shall see examples of such a situation when dealing with fermions.

3. Construction of physical kets

a. THE CONSTRUCTION RULE

The discussion of the preceding section leads directly to the following rule for the construction of the *unique* ket (the physical ket) corresponding to a given physical state of a system of N identical particles:

(*i*) Number the particles arbitrarily, and construct the ket $\mid u \rangle$ corresponding to the physical state considered and to the numbers given to the particles.

(*ii*) Apply S or A to $\mid u \rangle$, depending on whether the identical particles are bosons or fermions.

(*iii*) Normalize the ket so obtained.

We shall describe some simple examples which illustrate this rule.

b. APPLICATION TO SYSTEMS OF TWO IDENTICAL PARTICLES

Consider a system composed of two identical particles. Suppose that one of them is known to be in the individual state characterized by the normalized ket $\mid \varphi \rangle$, and the other one, in the individual state characterized by the normalized ket $\mid \chi \rangle$.

First of all, we shall envisage the case in which the two kets, $\mid \varphi \rangle$ and $\mid \chi \rangle$, are distinct. The preceding rule is applied in the following way:

(*i*) We label with the number 1, for example, the particle in the state $|\varphi\rangle$, and with the number 2, the one in the state $|\chi\rangle$. This gives:

$$|u\rangle = |1:\varphi;2:\chi\rangle \tag{C-2}$$

(*ii*) We symmetrize $|u\rangle$ if the particles are bosons:

$$S|u\rangle = \frac{1}{2}[|1:\varphi;2:\chi\rangle + |1:\chi;2:\varphi\rangle] \tag{C-3-a}$$

We antisymmetrize $|u\rangle$ if the particles are fermions:

$$A|u\rangle = \frac{1}{2}[|1:\varphi;2:\chi\rangle - |1:\chi;2:\varphi\rangle] \tag{C-3-b}$$

(*iii*) The kets (C-3-a) and (C-3-b), in general, are not normalized. If we assume $|\varphi\rangle$ and $|\chi\rangle$ to be orthogonal, the normalization constant is very simple to calculate. All we have to do to normalize $S|u\rangle$ or $A|u\rangle$ is replace the factor 1/2 appearing in formulas (C-3) by $1/\sqrt{2}$. The normalized physical ket, in this case, can therefore be written:

$$|\varphi;\chi\rangle = \frac{1}{\sqrt{2}}[|1:\varphi;2:\chi\rangle + \varepsilon|1:\chi;2:\varphi\rangle] \tag{C-4}$$

with $\varepsilon = +1$ for bosons and -1 for fermions.

We shall now assume that the two individual states, $|\varphi\rangle$ and $|\chi\rangle$, are identical:

$$|\varphi\rangle = |\chi\rangle \tag{C-5}$$

(C-2) then becomes:

$$|u\rangle = |1:\varphi;2:\varphi\rangle \tag{C-6}$$

$|u\rangle$ is already symmetric. If the two particles are bosons, (C-6) is then the physical ket associated with the state in which the two bosons are in the same individual state $|\varphi\rangle$. If, on the other hand, the two particles are fermions, we see that:

$$A|u\rangle = \frac{1}{2}[|1:\varphi;2:\varphi\rangle - |1:\varphi;2:\varphi\rangle] = 0 \tag{C-7}$$

Consequently, there exists no ket of $\mathcal{E}_A$ able to describe the physical state in which two fermions are in the same individual state $|\varphi\rangle$. Such a physical state is therefore excluded by the symmetrization postulate. We have thus established, for a special case, a fundamental result known as "*Pauli's exclusion principle*": *two identical fermions cannot be in the same individual state.* This result has some very important physical consequences which we shall discuss in § D-1.

c. GENERALIZATION TO AN ARBITRARY NUMBER OF PARTICLES

These ideas can be generalized to an arbitrary number N of particles. To see how this can be done, we shall first treat the case $N = 3$.

Consider a physical state of the system defined by specifying the three individual normalized states $|\varphi\rangle$, $|\chi\rangle$ and $|\omega\rangle$. The state $|u\rangle$ which enters into the rule of § a can be chosen in the form:

$$|u\rangle = |1:\varphi;2:\chi;3:\omega\rangle \tag{C-8}$$

We shall discuss the cases of bosons and fermions separately.

α. *The case of bosons*

The application of S to $|u\rangle$ gives:

$$\begin{aligned} S|u\rangle &= \frac{1}{3!}\sum_{\alpha} P_\alpha |u\rangle \\ &= \frac{1}{6}[|1:\varphi;2:\chi;3:\omega\rangle + |1:\omega;2:\varphi;3:\chi\rangle + |1:\chi;2:\omega;3:\varphi\rangle \\ &\quad + |1:\varphi;2:\omega;3:\chi\rangle + |1:\chi;2:\varphi;3:\omega\rangle + |1:\omega;2:\chi;3:\varphi\rangle] \end{aligned} \tag{C-9}$$

It then suffices to normalize the ket (C-9).

First of all, let us assume that the three kets $|\varphi\rangle$, $|\chi\rangle$ and $|\omega\rangle$ are orthogonal. The six kets appearing on the right-hand side of (C-9) are then also orthogonal. To normalize (C-9), all we must do is replace the factor 1/6 by $1/\sqrt{6}$.

If the two states $|\varphi\rangle$ and $|\chi\rangle$ coincide, while remaining orthogonal to $|\omega\rangle$, only three distinct kets now appear on the right-hand side of (C-9). It can easily be shown that the normalized physical ket can then be written:

$$\begin{aligned} |\varphi;\varphi;\omega\rangle = \frac{1}{\sqrt{3}}[&|1:\varphi;2:\varphi;3:\omega\rangle \\ &+ |1:\varphi;2:\omega;3:\varphi\rangle + |1:\omega;2:\varphi;3:\varphi\rangle] \end{aligned} \tag{C-10}$$

Finally, if the three states $|\varphi\rangle$, $|\chi\rangle$, $|\varphi\rangle$ are the same, the ket:

$$|u\rangle = |1:\varphi;2:\varphi;3:\varphi\rangle \tag{C-11}$$

is already symmetric and normalized.

β. *The case of fermions*

The application of A to $|u\rangle$, gives:

$$A|u\rangle = \frac{1}{3!}\sum_{\alpha} \varepsilon_\alpha P_\alpha |1:\varphi;2:\chi;3:\omega\rangle \tag{C-12}$$

The signs of the various terms of the sum (C-12) are determined by the same rule as those of a 3×3 determinant. This is why it is convenient to write $A|u\rangle$ in the form of a *Slater determinant*:

$$A|u\rangle = \frac{1}{3!}\begin{vmatrix} |1:\varphi\rangle & |1:\chi\rangle & |1:\omega\rangle \\ |2:\varphi\rangle & |2:\chi\rangle & |2:\omega\rangle \\ |3:\varphi\rangle & |3:\chi\rangle & |3:\omega\rangle \end{vmatrix} \tag{C-13}$$

$A \mid u \rangle$ is zero if two of the individual states $\mid \varphi \rangle$, $\mid \chi \rangle$ or $\mid \omega \rangle$ coincide, since the determinant (C-13) then has two identical columns. We obtain Pauli's exclusion principle, already mentioned in §C-3-b: the same quantum mechanical state cannot be simultaneously occupied by several identical fermions.

Finally, note that if the three states $\mid \varphi \rangle$, $\mid \chi \rangle$, $\mid \omega \rangle$ are orthogonal, the six kets appearing on the right-hand side of (C-12) are orthogonal. All we must then do to normalize $A \mid u \rangle$ is replace the factor $1/3!$ appearing in (C-12) or (C-13) by $1/\sqrt{3!}$.

If, now, the system being considered contains more than three identical particles, the situation actually remains similar to the one just described. It can be shown that, for N identical bosons, it is always possible to construct the physical state $S \mid u \rangle$ from arbitrary individual states $\mid \varphi \rangle$, $\mid \chi \rangle$, ... On the other hand, for fermions, the physical ket $A \mid u \rangle$ can be written in the form of an $N \times N$ Slater determinant; this excludes the case in which two individual states coincide (the ket $A \mid u \rangle$ is then zero). This shows, and we shall return to this in detail in §D, how different the consequences of the new postulate can be for fermion and boson systems.

d. CONSTRUCTION OF A BASIS IN THE PHYSICAL STATE SPACE

Consider a system of N identical particles. Starting with a basis, $\{ \mid u_i \rangle \}$, in the state space of a single particle, we can construct the basis:

$$\{ \mid 1 : u_i ; 2 : u_j ; \ldots ; N : u_p \rangle \}$$

in the tensor product space $\mathcal{E}$. However, since the physical state space of the system is not $\mathcal{E}$, but, rather, one of the subspaces, $\mathcal{E}_S$ or $\mathcal{E}_A$, the problem arises of how to determine a basis in this physical state space.

By application of S (or A) to the various kets of the basis:

$$\{ \mid 1 : u_i ; 2 : u_j ; \ldots ; N : u_p \rangle \}$$

we can obtain a set of vectors spanning $\mathcal{E}_S$ (or $\mathcal{E}_A$). Let $\mid \varphi \rangle$ be an arbitrary ket of $\mathcal{E}_S$, for example (the case in which $\mid \varphi \rangle$ belongs to $\mathcal{E}_A$ can be treated in the same way). $\mid \varphi \rangle$, which belongs to $\mathcal{E}$, can be expanded in the form:

$$\mid \varphi \rangle = \sum_{i,j,\ldots,p} a_{i,j,\ldots,p} \mid 1 : u_i ; 2 : u_j ; \ldots N : u_p \rangle \tag{C-14}$$

Since $\mid \varphi \rangle$, by hypothesis, belongs to $\mathcal{E}_S$, we have $S \mid \varphi \rangle = \mid \varphi \rangle$, and we simply apply the operator S to both sides of (C-14) to show that $\mid \varphi \rangle$ can be expressed in the form of a linear combination of the various kets $S \mid 1 : u_i ; 2 : u_j ; \ldots ; N : u_p \rangle$.

However, it must be noted that the various kets $S \mid 1 : u_i ; 2 : u_j ; \ldots ; N : u_p \rangle$ are not independent. Let us permute the roles of the various particles in one of the kets $\mid 1 : u_i ; 2 : u_j ; \ldots ; N : u_p \rangle$ of the initial basis (before symmetrization). On this new ket, application of S or A leads, according to (B-62) and (B-63), to the same ket of $\mathcal{E}_S$ or $\mathcal{E}_A$ (possibly with a change of sign).

Thus, we are led to introduce the concept of an *occupation number*: by definition, for the ket $\mid 1 : u_i ; 2 : u_j ; \ldots ; N : u_p \rangle$, the occupation number n_k of the individual state $\mid u_k \rangle$ is equal to the number of times that the state $\mid u_k \rangle$ appears in the

sequence $\{ | u_i \rangle, | u_j \rangle \ldots | u_p \rangle \}$, that is, the number of particles in the state $| u_k \rangle$ (we have, obviously, $\sum_k n_k = N$). Two different kets $| 1 : u_i; 2 : u_j; \ldots; N : u_p \rangle$ for which the occupation numbers are equal can be obtained from each other by the action of a permutation operator. Consequently, after the action of the symmetrizer S (or the antisymmetrizer A), they give the same physical state, which we shall denote by $| n_1, n_2, \ldots, n_k, \ldots \rangle$:

$$| n_1, n_2, \ldots, n_k, \ldots \rangle = c\, S | \underbrace{1 : u_1; 2 : u_1; \ldots n_1 : u_1}_{\substack{n_1 \text{ particles} \\ \text{in the state } | u_1 \rangle}}; \underbrace{n_1 + 1 : u_2; \ldots; n_1 + n_2 : u_2}_{\substack{n_2 \text{ particles} \\ \text{in the state } | u_2 \rangle}}; \ldots \rangle \tag{C-15}$$

For fermions, S would be replaced by A in (C-15) (c is a factor which permits the normalization of the state obtained in this way*). We shall not study the states $| n_1, n_2, \ldots, n_k, \ldots \rangle$ in detail here; we shall confine ourselves to giving some of their important properties:

(*i*) The scalar product of two kets $| n_1, n_2, \ldots, n_k, \ldots \rangle$ and $| n'_1, n'_2, \ldots, n'_k, \ldots \rangle$ is different from zero only if all the occupation numbers are equal ($n_k = n'_k$ for all k).

By using (C-15) and definitions (B-49) and (B-50) of S and A, we can obtain the expansion of the two kets under consideration on the orthonormal basis, $\{ | 1 : u_i; 2 : u_j; \ldots; N : u_p \rangle \}$. It is then easy to see that, if the occupation numbers are not all equal, these two kets cannot simultaneously have non-zero components on the same basis vector.

(*ii*) If the particles under study are bosons, the kets $| n_1, n_2, \ldots, n_k, \ldots \rangle$, in which the various occupation numbers n_k are arbitrary (with, of course $\sum_k n_k = N$), form an orthonormal basis of the physical state space.

Let us show that, for bosons, the kets $| n_1, n_2, \ldots, n_k, \ldots \rangle$ defined by (C-15) are never zero. To do so, we replace S by its definition (B-49). There then appear, on the right-hand side of (C-15), various orthogonal kets $| 1 : u_i; 2 : u_j; \ldots; N : u_p \rangle$, all with positive coefficients. $| n_1, n_2, \ldots, n_k, \ldots \rangle$ cannot, therefore, be zero.

The $| n_1, n_2, \ldots, n_k, \ldots \rangle$ form a basis in $\mathcal{E}_S$ since these kets span $\mathcal{E}_S$, are all non-zero, and are orthogonal to each other.

(*iii*) If the particles under study are fermions, a basis of the physical state space $\mathcal{E}_A$ is obtained by choosing the set of kets $| n_1, n_2, \ldots, n_k, \ldots \rangle$ in which all the occupation numbers are equal either to 1 or to 0 (again with $\sum_k n_k = N$).

* A simple calculation yields : $c = \sqrt{N!/n_1!\, n_2! \ldots}$ for bosons and $\sqrt{N!}$ for fermions.

The preceding proof is not applicable to fermions because of the minus signs which appear before the odd permutations in definition (B-50) of A. Furthermore, we saw in §c that two identical fermions cannot occupy the same individual quantum state : if any one of the occupation numbers is greater than 1, the vector defined by (C-15) is zero. On the other hand, it is never zero if all the occupation numbers are equal to one or zero, since two particles are then never in the same individual quantum state, so that the kets $|\, 1 : u_i; 2 : u_j; \ldots; N : u_p \,\rangle$ and $P_\alpha \,|\, 1 : u_i; 2 : u_j; \ldots; N : u_p \,\rangle$ are always distinct and orthogonal. Relation (C-15) therefore defines a non-zero physical ket in this case. The rest of the proof is the same as for bosons.

4. Application of the other postulates

It remains for us to show how the general postulates of chapter III can be applied in light of the symmetrization postulate introduced in § C-1, and to verify that no contradictions arise. More precisely, we shall see how measurement processes can be described with kets belonging only to either $\mathscr{E}_S$ or $\mathscr{E}_A$, and we shall show that the time evolution process does not take the ket $|\, \psi(t) \,\rangle$ associated with the state of the system out of this subspace. Thus, all the quantum mechanical formalism can be applied inside either $\mathscr{E}_S$ or $\mathscr{E}_A$.

a. MEASUREMENT POSTULATES

α. *Probability of finding the system in a given physical state*

Consider a measurement performed on a system of identical particles. The ket $|\, \psi(t) \,\rangle$ describing the quantum state of the system before the measurement must, according to the symmetrization postulate, belong to $\mathscr{E}_S$ or to $\mathscr{E}_A$, depending on whether the system is formed of bosons or fermions. To apply the postulates of chapter III concerning measurements, we must take the scalar product of $|\, \psi(t) \,\rangle$ with the ket $|\, u \,\rangle$ corresponding to the physical state of the system after the measurement. This ket $|\, u \,\rangle$ is to be constructed by applying the rule given in §C-3-a. The probability amplitude $\langle\, u \,|\, \psi(t) \,\rangle$ can therefore be expressed in terms of two vectors, both belonging either to $\mathscr{E}_S$ or to $\mathscr{E}_A$. In §D-2, we shall discuss a certain number of examples of such calculations.

If the measurement envisaged is a "complete" measurement (yielding, for example, the positions and spin components S_z for all the particles), the physical ket $|\, u \,\rangle$ is unique (to within a constant factor). On the other hand, if the measurement is "incomplete" (for example, a measurement of the spins only, or a measurement bearing on a single particle), several orthogonal physical kets are obtained, and the corresponding probabilities must then be summed.

β. *Physical observables; invariance of $\mathscr{E}_S$ and $\mathscr{E}_A$*

In certain cases, it is possible to specify the measurement performed on the system of identical particles by giving the explicit expression of the corresponding observable in terms of $\mathbf{R}_1$, $\mathbf{P}_1$, $\mathbf{S}_1$, $\mathbf{R}_2$, $\mathbf{P}_2$, $\mathbf{S}_2$, etc.

We shall give some concrete examples of observables which can be measured in a three-particle system:

– Position of the center of mass $\mathbf{R}_G$, total momentum $\mathbf{P}$ and total angular momentum $\mathbf{L}$:

$$\mathbf{R}_G = \frac{1}{3}(\mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_3) \tag{C-16}$$

$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 \tag{C-17}$$

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 \tag{C-18}$$

– Electrostatic repulsion energy:

$$W = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{|\mathbf{R}_1 - \mathbf{R}_2|} + \frac{1}{|\mathbf{R}_2 - \mathbf{R}_3|} + \frac{1}{|\mathbf{R}_3 - \mathbf{R}_1|}\right) \tag{C-19}$$

– Total spin:

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 \tag{C-20}$$

etc.

It is clear from these expressions that the observables associated with the physical quantities considered involve the various particles symmetrically. This important property follows directly from the fact that the particles are identical. In (C-16), for example, $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ have the same coefficient, since the three particles have the same mass. It is the equality of the charges which is at the basis of the symmetrical form of (C-19). In general, since no physical properties are modified when the roles of the N identical particles are permuted, these N particles must play a symmetrical role* in any actually measurable observable. Mathematically, the corresponding observable G, which we shall call a *physical* observable, must be invariant under all permutations of the N identical particles. It must therefore commute with all the permutation operators P_α of the N particles (*cf.* § B-2-d):

$$[G, P_\alpha] = 0 \qquad \text{for all} \quad P_\alpha \tag{C-21}$$

For a system of two identical particles, for example, the observable $\mathbf{R}_1 - \mathbf{R}_2$ (the vector difference of the positions of the two particles), which is not invariant under the effect of the permutation P_{21} ($\mathbf{R}_1 - \mathbf{R}_2$ changes signs) is not a physical observable; indeed, a measurement of $\mathbf{R}_1 - \mathbf{R}_2$ assumes that particle (1) can be distinguished from particle (2). On the other hand, we can measure the distance between the two particles, that is, $\sqrt{(\mathbf{R}_1 - \mathbf{R}_2)^2}$, which is symmetrical.

Relation (C-21) implies that $\mathscr{E}_S$ and $\mathscr{E}_A$ are both invariant under the action of a physical observable G. Let us show that, if $|\psi\rangle$ belongs to $\mathscr{E}_A$, $G|\psi\rangle$ also belongs to $\mathscr{E}_A$ (the same proof also applies, of course, to $\mathscr{E}_S$). The fact that $|\psi\rangle$ belongs to $\mathscr{E}_A$ means that:

$$P_\alpha |\psi\rangle = \varepsilon_\alpha |\psi\rangle \tag{C-22}$$

Now let us calculate $P_\alpha G|\psi\rangle$. According to (C-21) and (C-22), we have:

$$P_\alpha G|\psi\rangle = G P_\alpha |\psi\rangle = \varepsilon_\alpha G|\psi\rangle \tag{C-23}$$

* Note that this reasoning is valid for fermions as well as for bosons.

Since the permutation P_α is arbitrary, (C-23) expresses the fact that $G\,|\,\psi\,\rangle$ is completely antisymmetric and therefore belongs to $\mathscr{E}_A$.

All operations normally performed on an observable — in particular, the determination of eigenvalues and eigenvectors — can therefore be applied to G entirely within one of the subspaces, $\mathscr{E}_S$ or $\mathscr{E}_A$. Only the eigenkets of G belonging to the physical subspace, and the corresponding eigenvalues, are retained.

COMMENTS:

(*i*) All the eigenvalues of G which exist in the total space $\mathscr{E}$ are not necessarily found if we restrict ourselves to the subspace $\mathscr{E}_S$ (or $\mathscr{E}_A$). The effect of the symmetrization postulate on the spectrum of a symmetric observable G may therefore be to abolish certain eigenvalues. On the other hand, it adds no new eigenvalues to this spectrum, since, because of the global invariance of $\mathscr{E}_S$ (or $\mathscr{E}_A$) under the action of G, any eigenvector of G in $\mathscr{E}_S$ (or $\mathscr{E}_A$) is also an eigenvector of G in $\mathscr{E}$ with the same eigenvalue.

(*ii*) Consider the problem of writing mathematically, in terms of the observables $\mathbf{R}_i$, $\mathbf{P}_i$, $\mathbf{S}_i$, etc., the observables corresponding to the different types of measurement envisaged in § α. This problem is not always simple. For example, for a system of three identical particles, we shall try to write the observables corresponding to the simultaneous measurement of the three positions in terms of $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$. We can resolve this problem by considering several physical observables chosen such that we can, using the results obtained by measuring them, unambiguously deduce the position of each particle (without, of course, being able to associate a numbered particle with each position). For example, we can choose the set

$$X_1 + X_2 + X_3,\ X_1X_2 + X_2X_3 + X_3X_1,\ X_1X_2X_3$$

(and the corresponding observables for the Y and Z coordinates). However, this point of view is rather formal. Rather than trying to write the expressions for the observables in all cases, it is simpler to follow the method used in § α, in which we confined ourselves to using the physical eigenkets of the measurement.

b. TIME-EVOLUTION POSTULATES

The Hamiltonian of a system of identical particles must be a physical observable. We shall write, for example, the Hamiltonian describing the motion of the two electrons of the helium atom about the nucleus, assumed to be motionless★:

$$H(1,2) = \frac{\mathbf{P}_1^2}{2m_e} + \frac{\mathbf{P}_2^2}{2m_e} - \frac{2e^2}{R_1} - \frac{2e^2}{R_2} + \frac{e^2}{|\mathbf{R}_1 - \mathbf{R}_2|} \tag{C-24}$$

The first two terms represent the kinetic energy of the system; they are symmetrical because the two masses are equal. The next two terms are due to the attraction of the nucleus (whose charge is twice that of the proton). The electrons are obviously equally affected by this attraction. Finally, the last term describes

★ Here, we shall consider only the most important terms of this Hamiltonian. See complement B_{XIV} for a more detailed study of the helium atom.

the mutual interaction of the electrons. It is also symmetrical, since neither of the two electrons is in a privileged position. It is clear that this argument can be generalized to any system of identical particles. Consequently, all the permutation operators commute with the Hamiltonian of the system:

$$[H, P_\alpha] = 0 \tag{C-25}$$

Under these conditions, if the ket $|\psi(t_0)\rangle$ describing the state of the system at a given time t_0 is a physical ket, the same must be true of the ket $|\psi(t)\rangle$ obtained from $|\psi(t_0)\rangle$ by solving the Schrödinger equation. According to this equation:

$$|\psi(t + dt)\rangle = \left(1 + \frac{dt}{i\hbar} H\right) |\psi(t)\rangle \tag{C-26}$$

Now, applying P_α and using relation (C-25):

$$P_\alpha |\psi(t + dt)\rangle = \left(1 + \frac{dt}{i\hbar} H\right) P_\alpha |\psi(t)\rangle \tag{C-27}$$

If $|\psi(t)\rangle$ is an eigenvector of P_α, $|\psi(t + dt)\rangle$ is also an eigenvector of P_α, with the same eigenvalue. Since $|\psi(t_0)\rangle$, by hypothesis, is a completely symmetric or completely antisymmetric ket, this property is conserved over time.

The symmetrization postulate is therefore also compatible with the postulate which gives the time evolution of physical systems : the Schrödinger equation does not remove the ket $|\psi(t)\rangle$ from $\mathscr{E}_S$ or $\mathscr{E}_A$.

D. DISCUSSION

In this final section, we shall examine the consequences of the symmetrization postulate on the physical properties of systems of identical particles. First of all, we shall indicate the fundamental differences introduced by Pauli's exclusion principle between systems of identical fermions and systems of identical bosons. Then, we shall discuss the implications of the symmetrization postulate concerning the calculation of the probabilities associated with the various physical processes.

1. Differences between bosons and fermions. Pauli's exclusion principle

In the statement of the symmetrization postulate, the difference between bosons and fermions may appear insignificant. Actually, this simple sign difference in the symmetry of the physical ket has extremely important consequences. As we saw in §C-3 the symmetrization postulate does not restrict the individual states accessible to a system of identical bosons. On the other hand, it requires fermions to obey Pauli's exclusion principle : two identical fermions cannot occupy the same quantum mechanical state.

The exclusion principle was formulated initially in order to explain the properties of many-electron atoms (§D-1-a below and complement A_{XIV}). It can

now be seen to be more than a principle applicable only to electrons: it is a consequence of the symmetrization postulate, valid for all systems of identical fermions. Predictions based on this principle, which are often spectacular, have always been confirmed experimentally. We shall give some examples of them.

a. GROUND STATE OF A SYSTEM OF INDEPENDENT IDENTICAL PARTICLES

The Hamiltonian of a system of identical particles (bosons or fermions) is always symmetrical with respect to permutations of these particles (§ C-4). Consider such a system in which the various particles are independent, that is, do not interact with each other (at least in a first approximation). The corresponding Hamiltonian is then a sum of one-particle operators of the form:

$$H(1, 2, ..., N) = h(1) + h(2) + ... + h(N) \tag{D-1}$$

$h(1)$ is a function only of the observables associated with the particle numbered (1); the fact that the particles are identical [which implies a symmetrical Hamiltonian $H(1, 2, ..., N)$] requires this function h to be the same in the N terms of expression (D-1). In order to determine the eigenstates and eigenvalues of the total Hamiltonian $H(1, 2, ..., N)$, we simply calculate those of the individual Hamiltonian $h(j)$ in the state space $\mathscr{E}(j)$ of one of the particles:

$$h(j) \mid \varphi_n \rangle = \mathrm{e}_n \mid \varphi_n \rangle; \quad \mid \varphi_n \rangle \in \mathscr{E}(j) \tag{D-2}$$

For the sake of simplicity, we shall assume that the spectrum of $h(j)$ is discrete and non-degenerate.

If we are considering a system of identical bosons, the physical eigenvectors of the Hamiltonian $H(1, 2, ..., N)$ can be obtained by symmetrizing the tensor products of N arbitrary individual states $\mid \varphi_n \rangle$:

$$\mid \Phi^{(S)}_{n_1, n_2, ..., n_N} \rangle = c \sum_{\alpha} P_\alpha \mid 1 : \varphi_{n_1}; 2 : \varphi_{n_2}; ...; N : \varphi_{n_N} \rangle \tag{D-3}$$

where the corresponding energy is the sum of the N individual energies:

$$E_{n_1, n_2, ..., n_N} = \mathrm{e}_{n_1} + \mathrm{e}_{n_2} + ... + \mathrm{e}_{n_N} \tag{D-4}$$

[it can easily be shown that each of the kets appearing on the right-hand side of (D-3) is an eigenket of H with the eigenvalue (D-4); this is also true of their sum]. In particular, if e_1 is the smallest eigenvalue of $h(j)$, and $\mid \varphi_1 \rangle$ is the associated eigenstate, the ground state of the system is obtained when the N identical bosons are all in the state $\mid \varphi_1 \rangle$. The energy of this ground state is therefore:

$$E_{1,1,...,1} = N \mathrm{e}_1 \tag{D-5}$$

and its state vector is:

$$\mid \varphi^{(S)}_{1,1,...,1} \rangle = \mid 1 : \varphi_1; 2 : \varphi_1; ...; N : \varphi_1 \rangle \tag{D-6}$$

Now, suppose that the N identical particles considered are fermions. It is no longer possible for these N particles all to be in the individual state $\mid \varphi_1 \rangle$. To obtain

the ground state of the system, Pauli's exclusion principle must be taken into account. If the individual energies e_n are arranged in increasing order:

$$e_1 < e_2 < \ldots < e_{n-1} < e_n < e_{n+1} < \ldots, \tag{D-7}$$

the ground state of the system of N identical fermions has an energy of:

$$E_{1,2,\ldots,N} = e_1 + e_2 + \ldots + e_N \tag{D-8}$$

and it is described by the normalized physical ket:

$$|\Phi^{(A)}_{1,2,\ldots,N}\rangle = \frac{1}{\sqrt{N!}} \begin{vmatrix} |1:\varphi_1\rangle & |1:\varphi_2\rangle \ldots & |1:\varphi_N\rangle \\ |2:\varphi_1\rangle & |2:\varphi_2\rangle \ldots & |2:\varphi_N\rangle \\ \vdots & & \\ |N:\varphi_1\rangle & |N:\varphi_2\rangle \ldots & |N:\varphi_N\rangle \end{vmatrix} \tag{D-9}$$

The highest individual energy e_N found in the ground state is called the *Fermi energy* of the system.

Pauli's exclusion principle thus plays a role of primary importance in all domains of physics in which many-electron systems are involved, such as atomic and molecular physics (*cf.* complements A_{XIV} and B_{XIV}) and solid state physics (*cf.* complement C_{XIV}), and in all those in which many-proton and many-neutron systems are involved, such as nuclear physics★.

COMMENT:

In most cases, the individual energies e_n are actually degenerate. Each of them can then enter into a sum such as (D-8) a number of times equal to its degree of degeneracy.

b. QUANTUM STATISTICS

The object of statistical mechanics is to study systems composed of a very large number of particles (in numerous cases, the mutual interactions between these particles are weak enough to be neglected in a first approximation). Since we do not know the microscopic state of the system exactly, we content ourselves with describing it globally by its macroscopic properties (pressure, temperature, density, etc.). A particular macroscopic state corresponds to a whole set of microscopic states. We then use probabilities: the statistical weight of a macroscopic state is proportional to the number of distinct microscopic states which correspond to it, and the system, at thermodynamic equilibrium, is in its most probable macroscopic state (with any constraints that may be imposed taken into account). To study the macroscopic properties of the system, it is therefore essential to

★ The ket representing the state of a nucleus must be antisymmetric both with respect to the set of protons and with respect to the set of neutrons.

determine how many different microscopic states possess certain characteristics and, in particular, a given energy.

In classical statistical mechanics (Maxwell-Boltzmann statistics), the N particles of the system are treated as if they were of different natures, even if they are actually identical. Such a microscopic state is defined by specifying the individual state of each of the N particles. Two microscopic states are considered to be distinct when these N individual states are the same but the permutation of the particles is different.

In quantum statistical mechanics, the symmetrization postulate must be taken into account. A microscopic state of a system of identical particles is characterized by the enumeration of the N individual states which form it, the order of these states being of no importance since their tensor product must be symmetrized or antisymmetrized. The numbering of the microscopic states therefore does not lead to the same result as in classical statistical mechanics. In addition, Pauli's principle radically differentiates systems of identical bosons and systems of identical fermions: the number of particles occupying a given individual state cannot exceed one for fermions, while it can take on any value for bosons (*cf.* § C-3). Different statistical properties result: bosons obey *Bose-Einstein statistics* and fermions, *Fermi-Dirac statistics*. This is the origin of the terms "bosons" and "fermions".

The physical properties of systems of identical fermions and systems of identical bosons are very different. These differences can be observed, for example, at low temperatures. The particles then tend to accumulate in the lowest-energy individual states, as is possible for identical bosons (this phenomenon is called *Bose condensation*), while identical fermions are subject to the restrictions of Pauli's principle. Bose condensation is at the origin of the remarkable properties (superfluidity) of the ^{4}He isotope of helium, while the ^{3}He isotope, which is a fermion (*cf.* comment of § C-1), does not possess the same properties.

2. The consequences of particle indistinguishability on the calculation of physical predictions

In quantum mechanics, all the predictions concerning the properties of a system are expressed in terms of probability amplitudes (scalar products of two state vectors) or matrix elements of an operator. It is then not surprising that the symmetrization or antisymmetrization of state vectors causes special *interference effects* to appear in systems of identical particles. First, we shall specify these effects, and then we shall see how they disappear under certain conditions (the particles of the system, although identical, then behave as if they were of different natures). To simplify the discussion, we shall confine ourselves to systems containing only two identical particles.

a. INTERFERENCES BETWEEN DIRECT AND EXCHANGE PROCESSES

α. *Predictions concerning a measurement on a system of identical particles: the direct term and the exchange term*

Consider a system of two identical particles, one of which is known to be in the individual state $|\varphi\rangle$ and the other, in the individual state $|\chi\rangle$. We shall

assume $|\varphi\rangle$ and $|\chi\rangle$ to be orthogonal, so that the state of the system is described by the normalized physical ket [*cf.* formula (C-4)]:

$$|\varphi;\chi\rangle = \frac{1}{\sqrt{2}}[1 + \varepsilon P_{21}]\,|1:\varphi;2:\chi\rangle \tag{D-10}$$

where:

$$\begin{aligned} \varepsilon &= +1 \quad \text{if the particles are bosons} \\ \varepsilon &= -1 \quad \text{if the particles are fermions} \end{aligned} \tag{D-11}$$

With the system in this state, suppose that we want to measure on each of the two particles the same physical quantity B with which the observables $B(1)$ and $B(2)$ are associated. For the sake of simplicity, we shall assume that the spectrum of B is entirely discrete and non-degenerate:

$$B\,|u_i\rangle = b_i\,|u_i\rangle \tag{D-12}$$

What is the probability of finding certain given values in this measurement (b_n for one of the particles and $b_{n'}$ for the other one)? We shall begin by assuming b_n and $b_{n'}$ to be different, so that the corresponding eigenvectors $|u_n\rangle$ and $|u_{n'}\rangle$ are orthogonal. Under these conditions, the normalized physical ket defined by the result of this measurement can be written:

$$|u_n;u_{n'}\rangle = \frac{1}{\sqrt{2}}[1 + \varepsilon P_{21}]\,|1:u_n;2:u_{n'}\rangle \tag{D-13}$$

which gives the probability amplitude associated with this result:

$$\langle u_n;u_{n'}\,|\,\varphi;\chi\rangle = \frac{1}{2}\langle 1:u_n;2:u_{n'}\,|\,(1 + \varepsilon P_{21}^{\dagger})(1 + \varepsilon P_{21})\,|\,1:\varphi;2:\chi\rangle \tag{D-14}$$

Using properties (B-13) and (B-14) of the operator P_{21}, we can write:

$$\frac{1}{2}(1 + \varepsilon P_{21}^{\dagger})(1 + \varepsilon P_{21}) = 1 + \varepsilon P_{21} \tag{D-15}$$

(D-14) then becomes:

$$\langle u_n;u_{n'}\,|\,\varphi;\chi\rangle = \langle 1:u_n;2:u_{n'}\,|(1 + \varepsilon P_{21})|\,1:\varphi;2:\chi\rangle \tag{D-16}$$

Letting $1 + \varepsilon P_{21}$ act on the bra, we obtain:

$$\begin{aligned} \langle u_n;u_{n'}\,|\,\varphi;\chi\rangle &= \langle 1:u_n;2:u_{n'}\,|\,1:\varphi;2:\chi\rangle \\ &\qquad + \varepsilon\langle 1:u_{n'};2:u_n\,|\,1:\varphi;2:\chi\rangle \\ &= \langle 1:u_n\,|\,1:\varphi\rangle\langle 2:u_{n'}\,|\,2:\chi\rangle \\ &\qquad + \varepsilon\langle 1:u_{n'}\,|\,1:\varphi\rangle\langle 2:u_n\,|\,2:\chi\rangle \\ &= \langle u_n\,|\,\varphi\rangle\langle u_{n'}\,|\,\chi\rangle + \varepsilon\langle u_{n'}\,|\,\varphi\rangle\langle u_n\,|\,\chi\rangle \end{aligned} \tag{D-17}$$

The numbering has disappeared from the probability amplitude, which is now expressed directly in terms of the scalar products $\langle u_n | \varphi \rangle \ldots \langle u_n | \chi \rangle$. Also, the probability amplitude appears either as a sum (for bosons) or a difference (for fermions) of two terms, with which we can associate the diagrams of figures 4-a and 4-b.

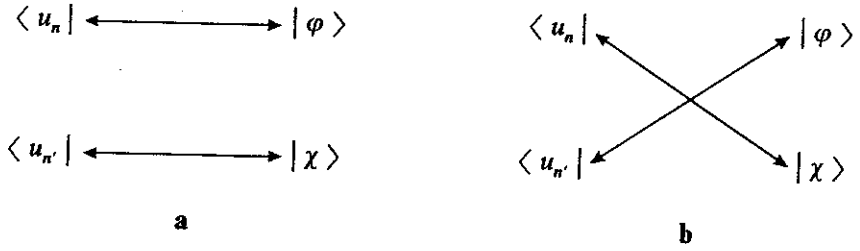

FIGURE 4

Schematic representation of the direct term and the exchange term associated with a measurement performed on a system of two identical particles. Before the measurement, one of the particles is known to be in the state $|\varphi\rangle$ and the other one, in the state $|\chi\rangle$. The measurement result obtained corresponds to a situation in which one particle is in the state $|u_n\rangle$ and the other one, in the state $|u_{n'}\rangle$. Two probability amplitudes are associated with such a measurement; they are represented schematically by figures a and b. These amplitudes interfere with a + sign for bosons and with a − sign for fermions.

We can interpret result (D-17) in the following way. The two kets $|\varphi\rangle$ and $|\chi\rangle$ associated with the initial state can be connected to the two bras $\langle u_n|$ and $\langle u_{n'}|$ associated with the final state by two different "paths", represented schematically by figures 4-a and 4-b. With each of these paths is associated a probability amplitude, $\langle u_n | \varphi \rangle \langle u_{n'} | \chi \rangle$ or $\langle u_{n'} | \varphi \rangle \langle u_n | \chi \rangle$, and *these two amplitudes interfere with a + sign for bosons and a − sign for fermions.* Thus, we obtain the answer to the question posed in § A-3-a above: the desired probability $\mathcal{P}(b_n; b_{n'})$ is equal to the square of the modulus of (D-17):

$$\mathcal{P}(b_n; b_{n'}) = |\langle u_n | \varphi \rangle \langle u_{n'} | \chi \rangle + \varepsilon \langle u_{n'} | \varphi \rangle \langle u_n | \chi \rangle|^2 \qquad \text{(D-18)}$$

One of the two terms on the right-hand side of (D-17), the one which corresponds, for example, to path 4-a, is often called the *direct term*. The other term is called the *exchange term*.

COMMENT:

Let us examine what happens if the two particles, instead of being identical, are of different natures. We shall then choose as the initial state of the system the tensor product ket:

$$|\psi\rangle = |1 : \varphi\rangle |2 : \chi\rangle \qquad \text{(D-19)}$$

Now, consider a measurement instrument which, although the two particles, (1) and (2), are not identical, is not able to distinguish between them. If it

yields the results b_n and $b_{n'}$, we do not know if b_n is associated with particle (1) or particle (2) (for example, for a system composed of a muon μ^- and an electron e^-, the measurement device may be sensitive only to the charge of the particles, giving no information about their masses). The two eigenstates $|\, 1 : u_n ; 2 : u_{n'} \rangle$ and $|\, 1 : u_{n'} ; 2 : u_n \rangle$ (which, in this case, represent different physical states) then correspond to the same measurement result. Since they are orthogonal, we must add the corresponding probabilities, which gives:

$$\begin{aligned} \mathscr{P}'(b_n ; b_{n'}) &= |\langle\, 1 : u_n ; 2 : u_{n'} \,|\, 1 : \varphi ; 2 : \chi \,\rangle|^2 \\ &\qquad + |\langle\, 1 : u_{n'} ; 2 : u_n \,|\, 1 : \varphi ; 2 : \chi \,\rangle|^2 \\ &= |\langle\, u_n \,|\, \varphi \,\rangle|^2 \, |\langle\, u_{n'} \,|\, \chi \,\rangle|^2 + |\langle\, u_{n'} \,|\, \varphi \,\rangle|^2 \, |\langle\, u_n \,|\, \chi \,\rangle|^2 \end{aligned} \tag{D-20}$$

Comparison of (D-18) with (D-20) clearly reveals the significant difference in the physical predictions of quantum mechanics depending on whether the particles under consideration are identical or not.

Now consider the case in which the two states $|\, u_n \rangle$ and $|\, u_{n'} \rangle$ are the same. When the two particles are fermions, the corresponding physical state is excluded by Pauli's principle, and the probability $\mathscr{P}(b_n ; b_n)$ is zero. On the other hand, if the two particles are bosons, we have:

$$|\, u_n ; u_n \,\rangle = |\, 1 : u_n ; 2 : u_n \,\rangle \tag{D-21}$$

and, consequently:

$$\begin{aligned} \langle\, u_n ; u_n \,|\, \varphi ; \chi \,\rangle &= \frac{1}{\sqrt{2}} \langle\, 1 : u_n ; 2 : u_n \,|(1 + P_{21})|\, 1 : \varphi ; 2 : \chi \,\rangle \\ &= \sqrt{2}\, \langle\, u_n \,|\, \varphi \,\rangle \langle\, u_n \,|\, \chi \,\rangle \end{aligned} \tag{D-22}$$

which gives:

$$\mathscr{P}(b_n ; b_n) = 2\, |\langle\, u_n \,|\, \varphi \,\rangle \langle\, u_n \,|\, \chi \,\rangle|^2 \tag{D-23}$$

COMMENTS:

(*i*) Let us compare this result with the one which would be obtained in the case, already considered above, in which the two particles are different. We must then replace $|\, \varphi ; \chi \,\rangle$ by $|\, 1 : \varphi ; 2 : \chi \,\rangle$ and $|\, u_n ; u_n \,\rangle$ by $|\, 1 : u_n ; 2 : u_n \,\rangle$, which gives the value for the probability amplitude:

$$\langle\, u_n \,|\, \varphi \,\rangle \langle\, u_n \,|\, \chi \,\rangle \tag{D-24}$$

and, consequently:

$$\mathscr{P}'(b_n ; b_n) = |\langle\, u_n \,|\, \varphi \,\rangle \langle\, u_n \,|\, \chi \,\rangle|^2 \tag{D-25}$$

(*ii*) For a system containing N identical particles, there are, in general, $N!$ distinct exchange terms which add (or subtract) in the probability amplitude. For example, consider a

system of three identical particles in the individual states $|\varphi\rangle$, $|\chi\rangle$ and $|\omega\rangle$, and the probability of finding, in a measurement, the results b_n, $b_{n'}$ and $b_{n''}$. The possible "paths" are then shown in figure 5. There are six such paths (all different if the three eigenvalues,

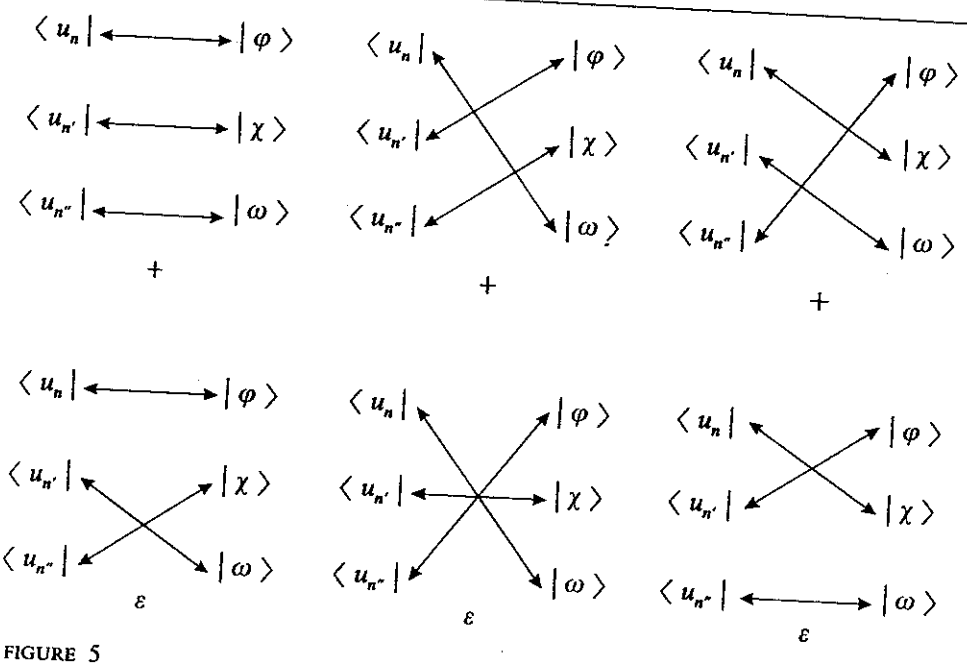

FIGURE 5

Schematic representation of the six probability amplitudes associated with a system of three identical particles. Before the measurement, one particle is known to be in the state $|\varphi\rangle$, another, in the state $|\chi\rangle$, and the last one, in the state $|\omega\rangle$. The result obtained corresponds to a situation in which one particle is in the state $|u_n\rangle$, another, in the state $|u_{n'}\rangle$, and the last one, in the state $|u_{n''}\rangle$. The six amplitudes interfere with a sign which is shown beneath each one ($\varepsilon = +1$ for bosons, -1 for fermions).

b_n, $b_{n'}$ and $b_{n''}$, are different). Some always contribute to the probability amplitude with a + sign, others with an ε sign (+ for bosons and − for fermions).

β. *Example : elastic collision of two identical particles*

To understand the physical meaning of the exchange term, let us examine a concrete example (already alluded to in §A-3-a): that of the elastic collision of two identical particles in their center of mass frame★. Unlike the situation in §α above, here we must take into account the evolution of the system between the initial time when it is in the state $|\psi_i\rangle$ and the time t when the measurement is performed. However, as we shall see, this evolution does not change the problem radically, and the exchange term enters the problem as before.

★ We shall give a simplified treatment of this problem, intended only to illustrate the relation between the direct term and the exchange term. In particular, we ignore the spin of the two particles. However, the calculations of this section remain valid in the case in which the interactions are not spin-dependent and the two particles are initially in the same spin state.

In the initial state of the system (fig. 6-a), the two particles are moving towards each other with opposite momenta. We choose the Oz axis along the direction of these momenta, and we denote their modulus by p. One of the particles thus possesses the momentum $p\mathbf{e}_z$, and the other one, the momentum $-p\mathbf{e}_z$ (where $\mathbf{e}_z$ is the unit vector of the Oz axis). We shall write the physical ket $|\psi_i\rangle$ representing this initial state in the form:

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}(1 + \varepsilon P_{21})\,|\,1 : p\mathbf{e}_z ; 2 : -p\mathbf{e}_z\,\rangle \tag{D-26}$$

$|\psi_i\rangle$ describes the state of the system at t_0 before the collision.

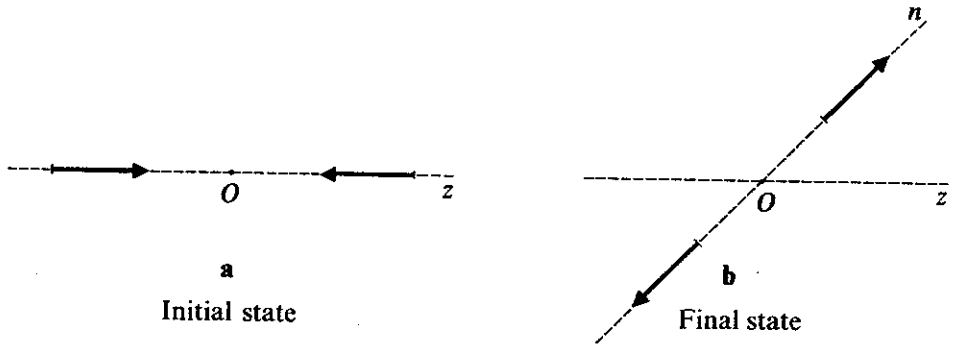

FIGURE 6

Collision between two identical particles in the center of mass frame : the momenta of the two particles in the initial state (fig. a) and in the final state found in the measurement (fig. b) are represented. For the sake of simplicity, we ignore the spin of the particles.

The Schrödinger equation which governs the time evolution of the system is linear. Consequently, there exists a linear operator $U(t, t')$, which is a function of the Hamiltonian H, such that the state vector at time t is given by:

$$|\psi(t)\rangle = U(t, t_0)\,|\psi_i\rangle \tag{D-27}$$

(complement F_{III}). In particular, after the collision, the state of the system at time t_1 is represented by the physical ket:

$$|\psi(t_1)\rangle = U(t_1, t_0)\,|\psi_i\rangle \tag{D-28}$$

Note that, since the Hamiltonian H is symmetric, the evolution operator U commutes with the permutation operator:

$$[U(t, t'), P_{21}] = 0 \tag{D-29}$$

Now, let us calculate the probability amplitude of the result envisaged in § A-3-a, in which the particles are detected in the two opposite directions of the On axis,

of unit vector $\mathbf{n}$ (fig. 6-b). We denote the physical ket associated with this final state by:

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}(1 + \varepsilon P_{21})\,|\,1 : p\mathbf{n}; 2 : -p\mathbf{n}\,\rangle \tag{D-30}$$

The desired probability amplitude can therefore be written:

$$\begin{aligned}\langle \psi_f | \psi(t_1)\rangle &= \langle \psi_f | U(t_1, t_0) | \psi_i\rangle \\ &= \frac{1}{2}\langle 1 : p\mathbf{n}; 2 : -p\mathbf{n} | (1 + \varepsilon P_{21}^{\dagger}) U(t_1, t_0)(1 + \varepsilon P_{21}) | 1 : pe_z; 2 : -pe_z\rangle\end{aligned} \tag{D-31}$$

According to relation (D-29) and the properties of the operator P_{21}, we finally obtain:

$$\begin{aligned}&\langle \psi_f | U(t_1, t_0) | \psi_i\rangle \\ &\quad = \langle 1 : p\mathbf{n}; 2 : -p\mathbf{n} | (1 + \varepsilon P_{21}^{\dagger}) U(t_1, t_0) | 1 : pe_z; 2 : -pe_z\rangle \\ &\quad = \langle 1 : p\mathbf{n}; 2 : -p\mathbf{n} | U(t_1, t_0) | 1 : pe_z; 2 : -pe_z\rangle \\ &\qquad + \varepsilon\langle 1 : -p\mathbf{n}; 2 : p\mathbf{n} | U(t_1, t_0) | 1 : pe_z; 2 : -pe_z\rangle\end{aligned} \tag{D-32}$$

The direct term corresponds, for example, to the process shown in figure 7-a, and the exchange term is then represented by figure 7-b. Again, the probability amplitudes associated with these two processes must be added or subtracted. This causes an

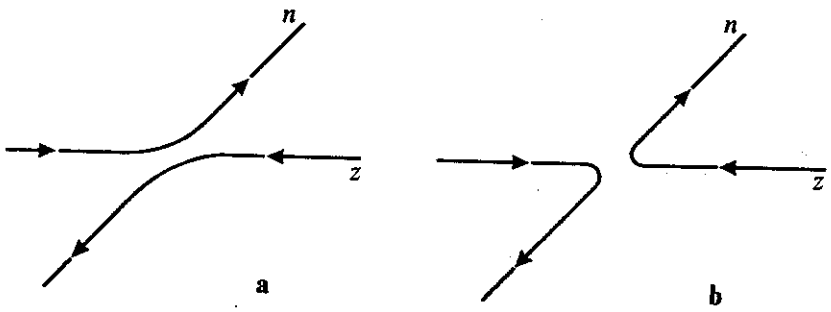

FIGURE 7

Collision between two identical particles in the center of mass frame : schematic representation of the physical processes corresponding to the direct term and the exchange term. The scattering amplitudes associated with these two processes interfere with a + sign for bosons and a − sign for fermions.

interference term to appear when the square of the modulus of expression (D-32) is taken. Note also that this expression is simply multiplied by ε if $\mathbf{n}$ is changed to $-\mathbf{n}$, so that the corresponding probability is invariant under this change.

b. SITUATIONS IN WHICH THE SYMMETRIZATION POSTULATE CAN BE IGNORED

If application of the symmetrization postulate were always indispensable, it would be impossible to study the properties of a system containing a restricted number of particles, because it would be necessary to take into account all the particles in the universe which are identical to those in the system. We shall see in this section that this is not the case. In fact, under certain special conditions, identical particles behave as if they were actually different, and it is not necessary to take the symmetrization postulate into account in order to obtain correct physical predictions. It seems natural to expect, considering the results of §D-2-a, that such a situation would arise whenever the exchange terms introduced by the symmetrization postulate are zero. We shall give two examples.

α. *Identical particles situated in two distinct regions of space*

Consider two identical particles, one of which is in the individual state $|\varphi\rangle$ and the other, in the state $|\chi\rangle$. To simplify the notation, we shall ignore their spin. Suppose that the domain of the wave functions representing the kets $|\varphi\rangle$ and $|\chi\rangle$ are well separated in space:

$$\begin{cases} \varphi(\mathbf{r}) = \langle \mathbf{r} | \varphi \rangle = 0 & \text{if} \quad \mathbf{r} \notin D \\ \chi(\mathbf{r}) = \langle \mathbf{r} | \chi \rangle = 0 & \text{if} \quad \mathbf{r} \notin \Delta \end{cases} \tag{D-33}$$

where the domains D and Δ do not overlap. The situation is analogous to the classical mechanical one (§A-2): as long as the domains D and Δ do not overlap, each of the particles can be "followed"; we therefore expect application of the symmetrization postulate to be unnecessary.

In this case, we can envisage measuring an observable related to one of the two particles. All we need is a measurement device placed so that it cannot record what happens in the domain D, or in the domain Δ. If it is D which is excluded in this way, the measurement will only concern the particle in Δ, an vice versa.

Now, imagine a measurement concerning the two particles simultaneously, but performed with two distinct measurement devices, one of which is not sensitive to phenomena occurring in Δ, and the other, to those in D. How can the probability of obtaining a given result be calculated? Let $|u\rangle$ and $|v\rangle$ be the individual states associated respectively with the results of the two measurement devices. Since the two particles are identical, the symmetrization postulate must, in theory, be taken into account. In the probability amplitude associated with the measurement result, the direct term is then $\langle u | \varphi \rangle \langle v | \chi \rangle$, and the exchange term is $\langle u | \chi \rangle \langle v | \varphi \rangle$. Now, the spatial disposition of the measurement devices implies that:

$$\begin{aligned} u(\mathbf{r}) &= \langle \mathbf{r} | u \rangle = 0 \quad \text{if} \quad \mathbf{r} \in \Delta \\ v(\mathbf{r}) &= \langle \mathbf{r} | v \rangle = 0 \quad \text{if} \quad \mathbf{r} \in D \end{aligned} \tag{D-34}$$

According to (D-33) and (D-34), the wave functions $u(\mathbf{r})$ and $\chi(\mathbf{r})$ do not overlap; neither do $v(\mathbf{r})$ and $\varphi(\mathbf{r})$, so that:

$$\langle u | \chi \rangle = \langle v | \varphi \rangle = 0 \tag{D-35}$$

The exchange term is therefore zero. Consequently, it is unnecessary, in this situation, to use the symmetrization postulate. We obtain the desired result directly by reasoning as if the particles were of different natures, labeling, for example, the one in the domain D with the number 1, and the one situated in Δ with the number 2. Before the measurement, the state of the system is then described by the ket $|\, 1 : \varphi; 2 : \chi \,\rangle$, and with the measurement result envisaged is associated the ket $|\, 1 : u; 2 : v \,\rangle$. Their scalar product gives the probability amplitude $\langle u \,|\, \varphi \rangle \langle v \,|\, \chi \rangle$.

This argument shows that the existence of identical particles does not prevent the separate study of restricted systems, composed of a small number of particles.

COMMENT:

In the initial state chosen, the two particles are situated in two distinct regions of space. In addition, we have defined the state of the system by specifying two individual states. We might wonder if, after the system has evolved, it is still possible to study one of the two particles and ignore the other one. For this to be the case, it is necessary, not only that the two particles remain in two distinct regions of space, but also that they do not interact. Whether the particles are identical or not, an interaction always introduces correlations between them, and it is no longer possible to describe each of them by a state vector.

β. *Particles which can be identified by the direction of their spins*

Consider an elastic collision between two identical spin 1/2 particles (electrons, for example), assuming that spin-dependent interactions can be neglected, so that the spin states of the two particles are conserved during the collision. If these spin states are initially orthogonal, they enable us to distinguish between the two particles at all times, as if they were not identical; consequently, the symmetrization postulate should again have no effect here.

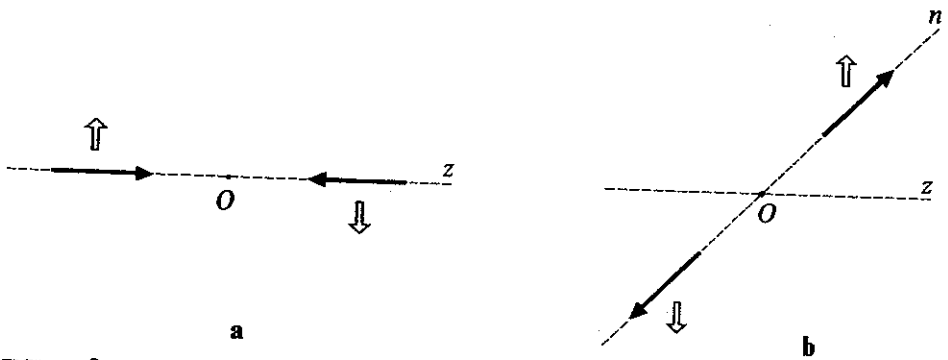

FIGURE 8

Collision between two identical spin 1/2 particles in the center of mass frame : a schematic representation of the momenta and spins of the two particles in the initial state (fig. a) and in the final state found in the measurement (fig. b). If the interactions between the two particles are spin-independent, the orientation of the spins does not change during the collision. When the two particles are not in the same spin state before the collision (the case of the figure), it is possible to determine the "path" followed by the system in arriving at a given final state. For example, the only scattering process which leads to the final state of figure b and which has a non-zero amplitude is of the type shown in figure 7-a.

We can show this, using the calculation of §D-2-a-β. The initial physical ket will be, for example (fig. 8-a):

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}(1 - P_{21})\,|\,1 : p\mathbf{e}_z, +; 2 : -p\mathbf{e}_z, -\rangle \tag{D-36}$$

(where the symbol + or − added after each momentum indicates the sign of the spin component along a particular axis). The final state which we are considering (fig. 8-b) will be described by:

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}(1 - P_{21})\,|\,1 : p\mathbf{n}, +; 2 : -p\mathbf{n}, -\rangle \tag{D-37}$$

Under these conditions, only the first term of (D-32) is different from zero, since the second one can be written:

$$\langle\, 1 : -p\mathbf{n}, -; 2 : p\mathbf{n}, +\,|\, U(t_1, t_0)\,|\,1 : p\mathbf{e}_z, +; 2 : -p\mathbf{e}_z, -\rangle \tag{D-38}$$

This is the matrix element of a spin-independent operator (by hypothesis) between two kets whose spin states are orthogonal; it is therefore zero. Consequently, we would obtain the same result if we treated the two particles directly as if they were different, that is, if we did not antisymmetrize the initial and final kets and if we associated index 1 with the spin state $|+\rangle$ and index 2 with the spin state $|-\rangle$. Of course, this is no longer possible if the evolution operator U, that is, the Hamiltonian H of the system, is spin-dependent.

References and suggestions for further reading:

The importance of interference between direct and exchange terms is stressed in Feynman III (1.2), § 3.4 and chap. 4.

Quantum statistics : Reif (8.4), Kittel (8.2).

Permutation groups : Messiah (1.17), app. D, § IV; Wigner (2.23), chap. 13; Bacry (10.31), §§ 41 and 42.

The effect of the symmetrization postulate on molecular spectra : Herzberg (12.4), Vol. I, chap. III, § 2f.

An article giving a popularized version: Gamow (1.27).