1. Quantum field theory would be easier to deal with if the physical operators were made of the "positive-frequency part"

$$\phi^+(x) = \int a(\boldsymbol{p})\, e^{ip\cdot x} \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \tag{1}$$

and its adjoint $\phi^{+\dagger}(x)$. For instance, we might imagine a hamiltonian of the form

$$H = c \int \phi^{+\dagger}(x)\phi^+(x) d^3x. \tag{2}$$

Here $a(\boldsymbol{p})$ annihilates a spinless particle of momentum $\boldsymbol{p}$, mass m, and charge q, and $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$. The annihilation and creation operators obey the continuum-limit commutation relations

$$\begin{aligned} [a(\boldsymbol{p}), a(\boldsymbol{p}')] &= 0 \\ [a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] &= \delta(\boldsymbol{p} - \boldsymbol{p}'). \end{aligned} \tag{3}$$

Show that

$$[\phi^+(x), \phi^{+\dagger}(y)] = \Delta_+(x - y) \tag{4}$$

where

$$\Delta_+(x) = \frac{1}{(2\pi)^3} \int e^{ip\cdot x} \frac{d^3p}{2p^0}. \tag{5}$$

This function is Lorentz invariant, and for space-like x, $x^2 > 0$, it is even

$$\Delta(-x) = \Delta(x). \tag{6}$$

The commutator is

$$[\phi^+(x), \phi^{+\dagger}(y)] = \int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{2p^0 2q^0}} [a(p), a^\dagger(q)]\, e^{i(px-qy)}$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{2p^0 2q^0}} \delta(\vec{p}-\vec{q})\, e^{i(px-qy)}$$

$$= \int \frac{d^3p}{(2\pi)^3 2p^0} e^{ip(x-y)} = \Delta_+(x-y).$$

2. Show that the commutator (4) is non-zero at space-like separations, such as at equal times $x^0 = y^0$ but different space points. This non-zero commutator would imply a violation of causality and so would represent a conflict between special relativity and quantum mechanics.

At equal times, $x^0 = y^0$, $x - y = \vec{x} - \vec{y}$, and so with $\vec{r} = \vec{x} - \vec{y}$

$$\Delta_+(x-y) = \int \frac{d^3p}{(2\pi)^3 2p^0} e^{i\vec{p}\cdot\vec{r}}$$

$$= \int \frac{d^3p \quad e^{i\vec{p}\cdot\vec{r}}}{(2\pi)^3 2\sqrt{\vec{p}^2 + m^2}}$$

$$= \frac{4\pi}{(2\pi)^3} \int_0^\infty \frac{p^2 dp}{2\sqrt{p^2 + m^2}} \frac{\sin p\sqrt{\vec{r}^2}}{p\sqrt{\vec{r}^2}}$$

where $p = |\vec{p}|$. With $u = p/m$, this is

$$\Delta_+(x-y) = \frac{m}{4\pi^2\sqrt{\vec{r}^2}} \int_0^\infty \frac{u\,du}{\sqrt{u^2+1}} \sin\left(mu\sqrt{\vec{r}^2}\right)$$

$$= \frac{m}{4\pi^2\sqrt{\vec{r}^2}} K_1\left(m\sqrt{r^2}\right)$$

in which K_1 is a Hankel function, which isn't identically zero.

3. We can avoid this conflict if we introduce anti-particles. Let $b^\dagger(\boldsymbol{p})$ create a spinless particle of momentum $\boldsymbol{p}$, mass m, and charge $-q$. Define the "negative-frequency part" of the field $\phi(x)$ as

$$\phi^-(x) = \int b^\dagger(\boldsymbol{p})\, e^{-ip\cdot x} \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \tag{7}$$

and the field $\phi(x)$ as

$$\phi(x) = \phi^+(x) + \phi^-(x). \tag{8}$$

Show now that the commutators

$$[\phi(x), \phi(y)] \tag{9}$$

and

$$[\phi(x), \phi^\dagger(y)] \tag{10}$$

vanish at space-like separations, that is, when $(x-y)^2 > 0$.

Thus, if we make the hamiltonian and other physical operators out of fields $\phi(x)$ and its adjoint $\phi^\dagger(x)$, then we can avoid the above conflict between special relativity and quantum mechanics.

It is true that we also could avoid this conflict by using fields of the form $\phi(x) = \phi^+(x) + \phi^{+\dagger}(x)$, but such a field would annihilate and create particles of charge q, and it is impossible to make operators that conserve charge out of such fields. So we need antiparticles to avoid the above conflict and also conserve charge.

If the particles of the field are neutral, then the field $\phi(x) = \phi^+(x) + \phi^{+\dagger}(x)$ allows us to avoid the above conflict without violating charge conservation. In this case, we say that the neutral particle is its own antiparticle.

These problems explain some of the logic behind some of the statements Cliff Burgess made in his colloquium on Friday, 8 May 2009.

With antiparticles, the commutator $[\phi(x), \phi(y)]$ is

$$[\phi(x), \phi(y)] = \int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{2p^0 2q^0}} \left[a(p)e^{ipx} + b^\dagger(p)e^{-ipx},\ a^\dagger(q)e^{iqy} + b(q)e^{-iqy} \right]$$

$$= 0$$

whatever x and y are.

The other commutator $[\phi(x), \phi^\dagger(y)]$ is

$$[\phi(x), \phi^\dagger(y)] = \int \frac{d^3p\, d^3q}{(2\pi)^3 \sqrt{2p^0 2q^0}} \Big[a(p) e^{ipx} + b^\dagger(p) e^{-ipx} , \; a^\dagger(q) e^{-iqy} + b(q) e^{iqy} \Big]$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^3 \sqrt{2p^0 2q^0}} \Big[\delta(\vec{p} - \vec{q})\, e^{ipx - iqy} - \delta(p-q)\, e^{-ipx + iqy} \Big]$$

$$= \int \frac{d^3p}{(2\pi)^3\, 2p^0} \left(e^{ip(x-y)} - e^{-ip(x-y)} \right)$$

which is not zero in general. It is

$$[\phi(x), \phi^\dagger(y)] = \Delta_+(x-y) - \Delta_+(y-x).$$

But since Δ_+ is even when $(x-y)^2 > 0$ i.e., when x and y are spacelike, the commutator does vanish

$$[\phi(x), \phi^\dagger(y)] = \Delta_+(x-y) - \Delta_+(x-y) = 0$$

when x and y are spacelike.