1. Compute the matrix element of the third component z of the position operator $\boldsymbol{r}$ between the $3p$ state with $m = 0$ and the ground state of atomic hydrogen:

$$\langle 100|z|310\rangle. \tag{1}$$

We use

$$\langle r,\theta,\phi|100\rangle = \frac{1}{\sqrt{\pi}}\frac{1}{a_0^{3/2}}e^{-r/a_0}$$

and

$$\langle r\theta\phi|310\rangle = \frac{1}{81}\sqrt{\frac{2}{\pi}}\frac{1}{a_0^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0}e^{-r/3a_0}\cos\theta.$$

where

$$a_0 = \frac{\hbar^2}{m_e e^2} = \frac{1}{\alpha}\frac{\hbar c}{mc^2} \cong 0.53\,\mathring{A}.$$

Now $z = r\cos\theta$, so

$$\langle 100|z|310\rangle = \int_0^\infty r^2 dr\int_{-1}^{1}dx\, 2\pi\,(r\,x^2)\,\frac{\sqrt{2}}{81\pi a_0^3}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0}$$

$$\times\, e^{-r/3a_0}\, e^{-r/a_0}$$

$$= \frac{2\sqrt{2}}{81}a_0^3\int_0^\infty dr\, r^3\left(6-\frac{r}{a_0}\right)\frac{r}{a_0}e^{-\frac{4r}{3a_0}}\int_{-1}^{1}x^2dx$$

$$= \frac{4\sqrt{2}}{3\cdot 81}\frac{1}{a_0^3}\int_0^\infty \frac{r^4}{a_0}\left(6-\frac{r}{a_0}\right)e^{-4r/3a_0}\,dr.$$

Let $u = r/a_0$. Then

$$\langle 100|z|310\rangle = \frac{2^2\sqrt{2}a_0}{3^5}\int_0^\infty u^4(6-u)e^{-4u/3}\,du$$

$$= \frac{2^2\sqrt{2}\,a_0}{3^5}\left[6\int_0^\infty u^4 e^{-4u/3}\,du - \int_0^\infty u^5 e^{-4u/3}\,du\right]$$

$$= \frac{2^2\sqrt{2}}{3^5}\left(6\,\frac{4!\,3^5}{4^5} - \frac{5!\,3^6}{4^6}\right)a_0$$

$$= 2^2\sqrt{2}\left(\frac{6\cdot3\cdot2}{2^8} - \frac{5\cdot3\cdot2\cdot3}{2^{10}}\right)a_0$$

$$= 2^2\sqrt{2}\left(\frac{3^2}{2^6} - \frac{5\cdot3^2}{2^9}\right)a_0$$

$$= \frac{\sqrt{2}\,3^2}{2^4}\left(1 - \frac{5}{2^3}\right)a_0$$

$$= \frac{\sqrt{2}\,3^2}{2^4}\left(\frac{3}{8}\right)a_0 = \frac{\sqrt{2}\,3^3}{2^7}\,a_0$$

$$= \frac{27}{64\sqrt{2}}\,a_0 = \frac{27}{64\sqrt{2}}\,\frac{1}{\alpha}\,\frac{\hbar c}{m_e c^2}.$$

2. Compute the rate of transitions from the $3p$ state with $m = 0$ (directly) to the ground state of atomic hydrogen. What is the partial lifetime of the $3p$ state, excluding decays to the $2s$ state?

The amplitude is

$$\langle 100k|S(t,0)|310\rangle = \frac{q}{m}\left(\frac{\hbar}{2\epsilon_0 V\omega_k}\right)^{\frac{1}{2}} \langle 100|\epsilon^* \cdot \vec{p}\,|310\rangle \frac{e^{i\Delta E t/\hbar}-1}{\Delta E}$$

in the dipole approximation. So the probability is

$$P(t) = |\langle 100k|S(t,0)|310\rangle|^2 = \left(\frac{q}{m}\right)^2 \frac{\hbar}{2\epsilon_0 V\omega_k} |\langle 100|\epsilon^* \cdot p|310\rangle|^2 \frac{2\pi t}{\hbar}\delta(\Delta E)$$

and the rate is

$$\hat{w} = \frac{2\pi}{\hbar}\left(\frac{q}{m}\right)^2 \frac{\hbar}{2\epsilon_0 V\omega_k} |\langle 100|\epsilon^* \cdot p|310\rangle|^2 \delta(\Delta E)$$

where $\Delta E = E_{100} + \hbar\omega_k - E_{310}$. Now

$$\vec{p} = \frac{m}{i\hbar}[x, H_{OM}]$$

so the matrix element is

$$|\langle 100|\epsilon^* \cdot p|310\rangle|^2 = \frac{m^2}{\hbar^2}(E_{310}-E_{100})^2 |\langle 100|\vec{\epsilon}_r^{\,*} \cdot \vec{x}\,|310\rangle|^2.$$

$$= \frac{m^2}{\hbar^2}(E_{310}-E_{100})^2 |\langle 100|z|310\rangle|^2 |\epsilon_{r3}|^2.$$

From problem (1), we have

$$\langle 100|z|310\rangle = \frac{27}{64\sqrt{2}} \frac{1}{\alpha} \frac{\hbar c}{mc^2}.$$

And

$$(E_{310} - E_{100})^2 = \left(\frac{1}{2}mc^2\alpha^2\right)^2 \left(1 - \frac{1}{3^2}\right)^2 = \frac{16}{81}(mc^2\alpha^2)^2$$

And $\delta(\Delta E)$ ensures that

$$\hbar\omega_k = E_{310} - E_{100} = \frac{4}{9} mc^2\alpha^2.$$

So since $q^2 = 4\pi\epsilon_0 \hbar c\alpha$, the rate $\hat{w}$ is

$$\hat{w} = \frac{2\pi}{\hbar} \frac{4\pi\epsilon_0\hbar c\alpha}{m^2} \frac{\hbar\, 9\hbar}{2\epsilon_0 V\, 4mc^2\alpha^2} \frac{m^2}{\hbar^2} \frac{16}{81}(mc^2\alpha^2)^2 \delta(\Delta E)$$

$$\times \left(\frac{27}{64\sqrt{2}} \frac{1}{\alpha} \frac{\hbar c}{mc^2}\right)^2 \epsilon^*_{r3}(k)\cdot\epsilon_{r3}(k)$$

$$= \frac{3^4}{2^9} \pi^2 \frac{\hbar^2 c\,\alpha}{mV} \epsilon^*_{r3}(k)\cdot\epsilon_{r3}(k)\, \delta(\Delta E).$$

Now

$$\sum_{r=1}^{2} \epsilon^*_{r3}(k)\cdot\epsilon_{r3}(k) = \hat{z}^T\left(1 - \hat{k}\hat{k}^T\right)\hat{z}$$

$$= 1 - (\hat{z}\cdot\hat{k})^2 = 1 - \cos^2\theta = \sin^2\theta.$$

We now integrate over final states

$$w = V \int \frac{d^3k}{(2\pi)^3} \hat{w}$$

$$= V \int_0^\infty \frac{d\hbar\omega}{(2\pi)^2} \frac{k^2}{\hbar c} \int_{-1}^{1} dx \, \frac{3^4 \pi^2}{2^9} \frac{\hbar c^2 \alpha}{m V} (1-x^2)\, \delta(\Delta E)$$

$$\int_{-1}^{1} dx\,(1-x^2) = 2 - \left[\frac{x^3}{3}\right]_{-1}^{1} = 2 - \left(\frac{1}{3}+\frac{1}{3}\right) = \frac{4}{3}$$

So

$$W = \frac{1}{(2\pi)^2} \frac{k^2}{\hbar c} \frac{3^4}{2^9} \frac{\pi^2 \hbar c^2 \alpha}{m} \frac{4}{3}$$

$$= \frac{3^3}{2^9} \frac{\hbar k^2 \alpha}{m} = \frac{3^3}{2^9} \frac{\alpha}{m} \frac{(\hbar\omega_k)^2}{\hbar c^2}$$

$$= \frac{3^3}{2^9} \frac{\alpha}{m} \frac{1}{\hbar c^2} \left(\frac{4}{9} m c^2 \alpha^2\right)^2$$

$$= \frac{1}{3 \cdot 2^5} \frac{mc^2}{\hbar} \alpha^5$$ and the partial

lifetime is

$$\tau = 3 \cdot 2^5 \frac{\hbar}{mc^2} \alpha^{-5} \simeq 3 \cdot 2^5 \, \frac{6.582 \times 10^{-22}}{0.511} (137.036)^5 \quad s$$

That is,

$$\tau \approx 5.98 \text{ ns}$$

is an upper bound on the lifetime of the 3p state of atomic hydrogen.

The actual lifetime of the 3p state is

$$\tau = 5.41 \pm 0.18 \quad \text{ns}$$

as reported by Etherton et al., Phys. Rev. A2, 2177-2179 (1970).