1. The state $|j, m = j\rangle$ is an angular-momentum eigenstate of $\boldsymbol{J}^2$ and of J_3 with eigenvalues $\hbar^2 j(j+1)$ and $\hbar j$. Suppose we right-handedly rotate it by ϵ about the y-axis

$$|j,j\rangle' = \exp(-i\epsilon J_2/\hbar)\,|j,j\rangle. \tag{1}$$

Find the probability

$$P = |\langle j,j|j,j\rangle'|^2 \tag{2}$$

that the rotated system is still in the state $|j,j\rangle$ to order ϵ^2.

1. $\langle jj | e^{-i\epsilon J_y/\hbar} | jj \rangle \approx \langle jj | 1 - \frac{i\epsilon J_y}{\hbar^2} - \frac{\epsilon^2 J_y^2}{2\hbar^2} | jj \rangle.$

Now J_y is a linear combination of J_+ and J_-, so $\langle jj | J_y | jj \rangle = 0$.

$J_y = \frac{J_+ - J_-}{2i}$ so

$\langle jj | J_y^2 | jj \rangle = -\frac{1}{4} \langle jj | -J_+J_- - J_-J_+ | jj \rangle$

$= \frac{1}{4} \langle jj | J_+ J_- | jj \rangle$. Now

$J_+J_- |jj\rangle = J_+ \sqrt{2j}\, \hbar\, |j, j-1\rangle = \hbar\sqrt{2j}\, J_+ |j, j-1\rangle$

$= \hbar\sqrt{2j}\sqrt{2j}\,\hbar\, |j,j\rangle = \hbar^2 2j\, |j,j\rangle.$

Thus $\langle jj | J_y^2 | jj \rangle = \frac{\hbar^2 j}{2}$ and so

$\langle jj | e^{-i\epsilon J_y/\hbar} | jj \rangle \approx 1 - \frac{\epsilon^2}{\hbar^2}\frac{\hbar^2 j}{4} = 1 - \frac{\epsilon^2 j}{4}.$

Thus the probability is

$P(jj) = |\langle jj | e^{-i\epsilon J_y/\hbar} | jj \rangle|^2 \approx 1 - \frac{\epsilon^2}{2} j.$

2. Suppose $\boldsymbol{V}$ is a vector operator in the sense that

$$[V_i, J_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} V_k. \tag{4}$$

Define

$$J_\pm = J_1 \pm iJ_2 \quad \text{and} \quad V_\pm = V_1 \pm iV_2. \tag{5}$$

(a) Find the commutators

$$[V_\pm, J_3]. \tag{6}$$

(b) If s and s' are eigenvalues of a scalar operator S that commutes with $\boldsymbol{J}$, then most of the matrix elements

$$\begin{aligned} &\langle s, j, m|V_3|s', j', m'\rangle \\ &\langle s, j, m|V_+|s', j', m'\rangle \\ &\langle s, j, m|V_-|s', j', m'\rangle \end{aligned} \tag{7}$$

are necessarily zero. Use the answer to part a to find selection rules that identify the values of the difference $m - m'$ that allow each of these matrix elements to be non-zero. (c) Find the commutator

$$[V_+, J_+]. \tag{8}$$

(d) Use it to show that

$$\langle s, j, m+1|V_+|s, j, m\rangle = f_+(s, j)\langle s, j, m+1|J_+|s, j, m\rangle \tag{9}$$

in which $f_+(s, j)$ does **not** depend upon m. (e) Show that d implies that

$$\langle s, j, m|V_+|s, j, m'\rangle = f_+(s, j)\langle s, j, m|J_+|s, j, m'\rangle. \tag{10}$$

(f) Now show that

$$\langle s, j, m|V_-|s, j, m'\rangle = f_-(s, j)\langle s, j, m|J_-|s, j, m'\rangle. \tag{11}$$

(g) Evaluate the commutator

$$[V_+, J_-]. \tag{12}$$

(h) Use it and the result of e to show that

$$\langle s, j, m|V_3|s, j, m\rangle = m\hbar f_+(s, j). \tag{13}$$

(i) Now show that

$$\langle s, j, m|V_3|s, j, m\rangle = m\hbar f_-(s, j). \tag{14}$$

(j) Show that $f_+(s, j) = f_-(s, j) \equiv f(s, j)$ and that

$$\langle s, j, m|V_3|s, j, m\rangle = f(s, j)\langle s, j, m|J_3|s, j, m\rangle. \tag{15}$$

(k) Now show that

$$\langle s, j, m|\boldsymbol{V}|s, j, m'\rangle = f(s, j)\langle s, j, m|\boldsymbol{J}|s, j, m'\rangle. \tag{16}$$

Answers to parts (a–k) of problem 2 are given in these pages on the Wigner-Eckart theorem.

Complement D_X

VECTOR OPERATORS : THE WIGNER-ECKART THEOREM

In complement B_{VI} (*cf.* §5-b), we defined the concept of a scalar operator: it is an operator A which commutes with the angular momentum **J** of the system under study. An important property of these operators was then given (*cf.* §6-c-β of that complement): in a standard basis, $\{ | k, j, m \rangle \}$, the non-zero matrix elements $\langle k, j, m | A | k', j', m' \rangle$ of a scalar operator must satisfy the conditions $j = j'$ and $m = m'$; in addition, these elements do not depend on m*, which allows us to write:

$$\langle k, j, m | A | k', j', m' \rangle = a_j(k, k')\delta_{jj'}\delta_{mm'} \tag{1}$$

In particular, if the values of k and j are fixed, which amounts to considering the "restriction" of A (*cf.* complement B_{II}, § 3) to the subspace $\mathscr{E}(k, j)$ spanned by the $(2j + 1)$ kets $| k, j, m \rangle$ $(m = -j, -j + 1, ..., +j)$, we obtain a very simple $(2j + 1) \times (2j + 1)$ matrix: it is diagonal and all its elements are equal.

Now consider another scalar operator B. The matrix corresponding to it in the subspace $\mathscr{E}(k, j)$ possesses the same property: it is proportional to the unit matrix. Therefore, the matrix corresponding to B can easily be obtained from the one associated with A, by multiplying all the (diagonal) elements by the same constant. We therefore see that the restrictions of two scalar operators A and B to a subspace $\mathscr{E}(k, j)$ are always proportional. Denoting by $P(k, j)$ the projector onto the subspace $\mathscr{E}(k, j)$, we can write this result in the form**:

$$P(k, j)\, B\, P(k, j) = \lambda(k, j)\, P(k, j)\, A\, P(k, j) \tag{2}$$

The aim of this complement is to study another type of operator which possesses properties analogous to the ones just recalled : the vector operator. We shall see that if **V** and **V**′ are vectorial, their matrix elements also obey selection rules, which we shall establish. Moreover, we shall show that the restrictions of **V** and **V**′ to $\mathscr{E}(k, j)$ are always proportional:

$$P(k, j)\, \mathbf{V}'\, P(k, j) = \mu(k, j)\, P(k, j)\, \mathbf{V}\, P(k, j) \tag{3}$$

These results constitute the Wigner-Eckart theorem for vector operators.

* The proof of these properties was outlined in complement B_{VI}. We shall return to this point in this complement (§3-a) when we study the matrix elements of a scalar Hamiltonian.

** For two given operators A and B, the proportionality coefficient generally depends on the subspace $\mathscr{E}(k, j)$ chosen; this is why we write $\lambda(k, j)$.

COMMENT:

Actually, the Wigner-Eckart theorem is much more general. For example, it enables us to obtain selection rules for the matrix elements of **V** between two kets belonging to two different subspaces $\mathscr{E}(k, j)$ and $\mathscr{E}(k', j')$, or to relate these elements to the corresponding elements of $\mathbf{V}'$. The Wigner-Eckart theorem can also be applied to a whole class of operators, of which scalars and vectors merely represent special cases: the irreducible tensor operators (*cf.* exercise 8 of complement G_X), which we shall not treat here.

1. Definition of vector operators; examples

In §5-c of complement B_{VI}, we showed that an observable **V** is a vector if its three components V_x, V_y and V_z in an orthonormal frame $Oxyz$ satisfy the following commutation relations:

$$[J_x, V_x] = 0 \tag{4-a}$$
$$[J_x, V_y] = i\hbar V_z \tag{4-b}$$
$$[J_x, V_z] = -i\hbar V_y \tag{4-c}$$

as well as those obtained by cyclic permutation of the indices x, y and z.

To give an idea of what this means, we shall give some examples of vector operators.

(*i*) The angular momentum **J** is itself a vector; replacing **V** by **J** in formulas (4), we simply obtain the relations which define an angular momentum (*cf.* chap. VI).

(*ii*) For a spinless particle whose state space is $\mathscr{E}_{\mathbf{r}}$, we have $\mathbf{J} = \mathbf{L}$. It is then simple to show that **R** and **P** are vector operators. We have, for example:

$$\begin{aligned} [L_x, X] &= [YP_z - ZP_y, X] = 0 \\ [L_x, Y] &= [-ZP_y, Y] = i\hbar Z \\ [L_x, Z] &= [YP_z, Z] = -i\hbar Y \end{aligned} \tag{5}$$

(*iii*) For a particle of spin **S**, whose state space is $\mathscr{E}_{\mathbf{r}} \otimes \mathscr{E}_s$, **J** is given by $\mathbf{J} = \mathbf{L} + \mathbf{S}$. In this case, the operators **L**, **S**, **R**, **P** are vectors. If we take into account the fact that all the spin operators (which act only in $\mathscr{E}_s$) commute with the orbital operators (which act only in $\mathscr{E}_{\mathbf{r}}$), the proof of these properties follows immediately from (*i*) and (*ii*).

On the other hand, operators of the type $\mathbf{L}^2$, $\mathbf{L} \cdot \mathbf{S}$, etc., are not vectors, but scalars [*cf.* comment (*i*) of complement B_{VI}, §5-c]. Other vector operators could, however, be constructed from those we have mentioned: $\mathbf{R} \times \mathbf{S}$, $(\mathbf{L} \cdot \mathbf{S})\mathbf{P}$, etc.

(*iv*) Consider the system (1) + (2), formed by the union of two systems: (1), of state space $\mathscr{E}_1$, and (2), of state space $\mathscr{E}_2$. If **V**(1) is an operator which acts only in $\mathscr{E}_1$, and if this operator is a vector [that is, satisfies commutation relations (4) with the angular momentum $\mathbf{J}_1$ of the first system], then the extension of **V**(1) into $\mathscr{E}_1 \otimes \mathscr{E}_2$ is also a vector. For example, for a two-electron system, the operators $\mathbf{L}_1$, $\mathbf{R}_1$, $\mathbf{S}_2$, etc. are vectors.

2. The Wigner-Eckart theorem for vector operators

a. NON-ZERO MATRIX ELEMENTS OF V IN A STANDARD BASIS

We introduce the operators V_+, V_-, J_+ and J_- defined by:

$$\begin{aligned} V_\pm &= V_x \pm iV_y \\ J_\pm &= J_x \pm iJ_y \end{aligned} \tag{6}$$

Using relations (4), we can easily show that:

$$[J_x, V_\pm] = \mp \hbar V_z \tag{7-a}$$
$$[J_y, V_\pm] = - i\hbar V_z \tag{7-b}$$
$$[J_z, V_\pm] = \pm \hbar V_\pm \tag{7-c}$$

from which we can deduce the commutation relations of $J_\pm$ and $V_\pm$:

$$[J_+, V_+] = 0 \tag{8-a}$$
$$[J_+, V_-] = 2\hbar V_z \tag{8-b}$$
$$[J_-, V_+] = - 2\hbar V_z \tag{8-c}$$
$$[J_-, V_-] = 0 \tag{8-d}$$

Now consider the matrix elements of **V** in a standard basis. We shall see that the fact that **V** is a vector implies that a large number of them are zero. First of all, we shall show that the matrix elements $\langle k, j, m \mid V_z \mid k', j', m' \rangle$ are necessarily zero whenever m is different from m'. It suffices to note that V_z and J_z commute [which follows, after cyclic permutation of the indices x, y and z, from relation (4-a)]. Therefore, the matrix elements of V_z between two vectors $\mid k, j, m \rangle$ corresponding to different eigenvalues $m\hbar$ of J_z are zero (*cf.* chap. II, §D-3-a-β).

For the matrix elements $\langle k, j, m \mid V_\pm \mid k', j', m' \rangle$ of $V_\pm$, we shall show that they are different from zero only if $m - m' = \pm 1$. Equation (7-c) indicates that:

$$J_z V_\pm = V_\pm J_z \pm \hbar V_\pm \tag{9}$$

Applying both sides of this relation to the ket $\mid k', j', m' \rangle$, we obtain:

$$\begin{aligned} J_z(V_\pm \mid k', j', m' \rangle) &= V_\pm J_z \mid k', j', m' \rangle \pm \hbar V_\pm \mid k', j', m' \rangle \\ &= (m' \pm 1)\hbar \; V_\pm \mid k', j', m' \rangle \end{aligned} \tag{10}$$

This relation indicates that $V_\pm \mid k', j', m' \rangle$ is an eigenvector of J_z with the eigenvalue $(m' \pm 1)\hbar$*. Since two eigenvectors of the Hermitian operator J_z associated

* It should not be concluded that $V_\pm \mid k, j, m \rangle$ is necessarily proportional to $\mid k, j, m \pm 1 \rangle$. In fact, the argument we have given shows only that:

$$V_\pm \mid k, j, m \rangle = \sum_{k'} \sum_{j'} c_{k',j'} \mid k', j', m \pm 1 \rangle.$$

For us to be able to omit, for example, the summation over j', it would be necessary for $V_\pm$ to commute with $\mathbf{J}^2$, which is not generally the case.

with different eigenvalues are orthogonal, it follows that the scalar product $\langle k, j, m \mid V_\pm \mid k', j', m' \rangle$ is zero if $m \neq m' \pm 1$.

Summing up, the selection rules obtained for the matrix elements of **V** are as follows:

$$V_z \Longrightarrow \Delta m = m - m' = 0 \quad \text{(11-a)}$$
$$V_+ \Longrightarrow \Delta m = m - m' = +1 \quad \text{(11-b)}$$
$$V_- \Longrightarrow \Delta m = m - m' = -1 \quad \text{(11-c)}$$

From these results, we can easily deduce the forms of the matrices which represent the restrictions of the components of **V** inside a subspace $\mathscr{E}(k, j)$. The one associated with V_z is diagonal, and those associated with $V_\pm$ have matrix elements only just above and just below the principal diagonal.

b. PROPORTIONALITY BETWEEN THE MATRIX ELEMENTS OF J AND V INSIDE A SUBSPACE $\mathscr{E}(k, j)$

α. *Matrix elements of V_+ and V_-*

Expressing the fact that the matrix element of the commutator (8-a) between the bra $\langle k, j, m+2 \mid$ and the ket $\mid k, j, m \rangle$ is zero, we have:

$$\langle k, j, m+2 \mid J_+ V_+ \mid k, j, m \rangle = \langle k, j, m+2 \mid V_+ J_+ \mid k, j, m \rangle \quad (12)$$

On both sides of this relation and between the operators J_+ and V_+, we insert the closure relation:

$$\sum_{k', j', m'} \mid k', j', m' \rangle \langle k', j', m' \mid = 1 \quad (13)$$

We thus obtain the matrix elements $\langle k, j, m \mid J_+ \mid k', j', m' \rangle$ of J_+; by the very construction of the standard basis $\{ \mid k, j, m \rangle \}$, they are different from zero only if $k = k'$, $j = j'$ and $m = m' + 1$. The summations over k', j' and m' are therefore unnecessary in this case, and (12) can be written:

$$\langle k, j, m+2 \mid J_+ \mid k, j, m+1 \rangle \langle k, j, m+1 \mid V_+ \mid k, j, m \rangle$$
$$= \langle k, j, m+2 \mid V_+ \mid k, j, m+1 \rangle \langle k, j, m+1 \mid J_+ \mid k, j, m \rangle \quad (14)$$

that is:

$$\frac{\langle k, j, m+1 \mid V_+ \mid k, j, m \rangle}{\langle k, j, m+1 \mid J_+ \mid k, j, m \rangle} = \frac{\langle k, j, m+2 \mid V_+ \mid k, j, m+1 \rangle}{\langle k, j, m+2 \mid J_+ \mid k, j, m+1 \rangle} \quad (15)$$

(as long as the bras and kets appearing in this relation exist, that is, as long as $j - 2 \geqslant m \geqslant -j$, we can show immediately that neither of the denominators can go to zero). Writing the relation thus obtained for $m = -j, -j+1, ..., j-2$, we get:

$$\frac{\langle k,j,-j+1 \mid V_+ \mid k,j,-j \rangle}{\langle k,j,-j+1 \mid J_+ \mid k,j,-j \rangle} = \frac{\langle k,j,-j+2 \mid V_+ \mid k,j,-j+1 \rangle}{\langle k,j,-j+2 \mid J_+ \mid k,j,-j+1 \rangle} = \dots$$
$$= \frac{\langle k,j,m+1 \mid V_+ \mid k,j,m \rangle}{\langle k,j,m+1 \mid J_+ \mid k,j,m \rangle} = \dots$$
$$= \frac{\langle k,j,j \mid V_+ \mid k,j,j-1 \rangle}{\langle k,j,j \mid J_+ \mid k,j,j-1 \rangle} \quad (16)$$

that is, if we call $\alpha_+(k,j)$ the common value of these ratios:

$$\langle k,j,m+1 \mid V_+ \mid k,j,m \rangle = \alpha_+(k,j) \langle k,j,m+1 \mid J_+ \mid k,j,m \rangle \quad (17)$$

where $\alpha_+(k,j)$ depends on k and on j, but not on m.

In addition, selection rule (11-b) implies that all the matrix elements $\langle k,j,m \mid V_+ \mid k,j,m' \rangle$ and $\langle k,j,m \mid J_+ \mid k,j,m' \rangle$ are zero if $\Delta m = m - m' \neq +1$. Therefore, whatever m and m', we have:

$$\langle k,j,m \mid V_+ \mid k,j,m' \rangle = \alpha_+(k,j) \langle k,j,m \mid J_+ \mid k,j,m' \rangle \quad (18\text{-a})$$

This result expresses the fact that all the matrix elements of V_+ inside $\mathscr{E}(k,j)$ are proportional to those of J_+.

An analogous argument can be made by taking the matrix element of the commutator (8-d) between the bra $\langle k,j,m-2 \mid$ and the ket $\mid k,j,m \rangle$ to be zero. We are thus led to:

$$\langle k,j,m \mid V_- \mid k,j,m' \rangle = \alpha_-(k,j) \langle k,j,m \mid J_- \mid k,j,m' \rangle \quad (18\text{-b})$$

an equation which expresses the fact that the matrix elements of V_- and J_- inside $\mathscr{E}(k,j)$ are proportional.

β. *Matrix elements of V_z*

To relate the matrix elements of V_z to those of J_z, we now place relation (8-c) between the bra $\langle k,j,m \mid$ and the ket $\mid k,j,m \rangle$:

$$\begin{aligned} -2\hbar \langle k,j,m \mid V_z \mid k,j,m \rangle \\ &= \langle k,j,m \mid (J_-V_+ - V_+J_-) \mid k,j,m \rangle \\ &= \hbar\sqrt{j(j+1) - m(m+1)} \langle k,j,m+1 \mid V_+ \mid k,j,m \rangle \\ &\quad - \hbar\sqrt{j(j+1) - m(m-1)} \langle k,j,m \mid V_+ \mid k,j,m-1 \rangle \end{aligned} \quad (19)$$

Using (18-a), we get:

$$\begin{aligned}\langle k,j,m \mid V_z \mid k,j,m \rangle &= -\frac{1}{2}\alpha_+(k,j)\Big\{\sqrt{j(j+1)-m(m+1)}\,\langle k,j,m+1 \mid J_+ \mid k,j,m\rangle \\ &\qquad - \sqrt{j(j+1)-m(m-1)}\,\langle k,j,m \mid J_+ \mid k,j,m-1\rangle\Big\} \\ &= -\frac{\hbar}{2}\alpha_+(k,j)\,\{\,j(j+1)-m(m+1)-j(j+1)+m(m-1)\,\}\end{aligned} \qquad (20)$$

that is:

$$\langle k,j,m \mid V_z \mid k,j,m \rangle = m\hbar\,\alpha_+(k,j) \qquad (21)$$

Similarly, an analogous argument based on (8-b) and (18-b) leads to:

$$\langle k,j,m \mid V_z \mid k,j,m \rangle = m\hbar\,\alpha_-(k,j) \qquad (22)$$

Relations (21) and (22) show that $\alpha_+(k,j)$ and $\alpha_-(k,j)$ are necessarily equal; from now on, we shall call their common value $\alpha(k,j)$:

$$\alpha(k,j) = \alpha_+(k,j) = \alpha_-(k,j) \qquad (23)$$

In addition, these relations imply that:

$$\langle k,j,m \mid V_z \mid k,j,m' \rangle = \alpha(k,j)\langle k,j,m \mid J_z \mid k,j,m' \rangle \qquad (24)$$

γ. *Generalization to an arbitrary component of* **V**

Any component of **V** is a linear combination of V_+, V_- and V_z. Consequently, using relation (23), we can summarize (18-a), (18-b) and (24) by writing:

$$\langle k,j,m \mid \mathbf{V} \mid k,j,m' \rangle = \alpha(k,j)\langle k,j,m \mid \mathbf{J} \mid k,j,m' \rangle \qquad (25)$$

Therefore, *inside* $\mathscr{E}(k,j)$, *all the matrix elements of* **V** *are proportional to those of* **J**. This result expresses the Wigner-Eckart theorem, for a special case. Introducing the "restrictions" of **V** and **J** to $\mathscr{E}(k,j)$ (*cf.* complement B_{II}, §3), we can also write it:

$$P(k,j)\,\mathbf{V}\,P(k,j) = \alpha(k,j)\,P(k,j)\,\mathbf{J}\,P(k,j) \qquad (26)$$

COMMENT:

J commutes with $P(k,j)$ [*cf.* (27)]; since, moreover

$$[P(k,j)]^2 = P(k,j)$$

we can omit either one of the two projectors $P(k,j)$ on the right-hand side of (26).

c. CALCULATION OF THE PROPORTIONALITY CONSTANT; THE PROJECTION THEOREM

Consider the operator $\mathbf{J} \cdot \mathbf{V}$; its restriction to $\mathscr{E}(k, j)$ is $P(k, j)\mathbf{J} \cdot \mathbf{V}P(k, j)$. To transform this expression, we can use the fact that:

$$[\mathbf{J}, P(k, j)] = \mathbf{0} \tag{27}$$

a relation that can easily be verified by showing that the action of the commutators $[J_z, P(k, j)]$ and $[J_\pm, P(k, j)]$ on any ket of the $\{ | k, j, m \rangle \}$ basis yields zero. Using (26), we then get:

$$\begin{aligned} P(k, j)\,\mathbf{J} \cdot \mathbf{V}\, P(k, j) &= \mathbf{J} \cdot [P(k, j)\,\mathbf{V}\, P(k, j)] \\ &= \alpha(k, j)\,\mathbf{J}^2 P(k, j) \\ &= \alpha(k, j)\, j(j+1)\hbar^2\, P(k, j) \end{aligned} \tag{28}$$

The restriction to the space $\mathscr{E}(k, j)$ of the operator $\mathbf{J} \cdot \mathbf{V}$ is therefore equal to the identity operator* multiplied by $\alpha(k, j) j(j+1)\hbar^2$. Therefore, if $| \psi_{k,j} \rangle$ denotes an arbitrary normalized state belonging to the subspace $\mathscr{E}(k, j)$, the mean value $\langle \mathbf{J} \cdot \mathbf{V} \rangle_{k,j}$ of $\mathbf{J} \cdot \mathbf{V}$ is independent of the ket $| \psi_{k,j} \rangle$ chosen, since:

$$\langle \mathbf{J} \cdot \mathbf{V} \rangle_{k,j} = \langle \psi_{k,j} | \mathbf{J} \cdot \mathbf{V} | \psi_{k,j} \rangle = \alpha(k, j)\, j(j+1)\hbar^2 \tag{29}$$

If we substitute this relation into (26), we see that, *inside the subspace* $\mathscr{E}(k, j)$**:

$$\mathbf{V} = \frac{\langle \mathbf{J} \cdot \mathbf{V} \rangle_{k,j}}{\langle \mathbf{J}^2 \rangle_{k,j}}\,\mathbf{J} = \frac{\langle \mathbf{J} \cdot \mathbf{V} \rangle_{k,j}}{j(j+1)\hbar^2}\,\mathbf{J} \tag{30}$$

This result is often called the "projection theorem". Whatever the physical system being studied, as long as we are concerned only with states belonging to the same subspace $\mathscr{E}(k, j)$, we can assume that all vector operators are proportional to $\mathbf{J}$.

We can give the following classical physical interpretation of this property: if $\mathbf{j}$ denotes the total angular momentum of any isolated physical system, all the physical quantities attached to the system rotate about $\mathbf{j}$, which is a constant vector (*cf.* fig. 1). In particular, for a vector quantity $\mathbf{v}$, all that remains after averaging over time is its projection $\mathbf{v}_{/\!/}$ onto $\mathbf{j}$, that is, a vector parallel to $\mathbf{j}$, given by:

$$\mathbf{v}_{/\!/} = \frac{\mathbf{j} \cdot \mathbf{v}}{\mathbf{j}^2}\,\mathbf{j} \tag{31}$$

a formula which is indeed analogous to (30).

* Since $\mathbf{J} \cdot \mathbf{V}$ is a scalar, the fact that its restriction is proportional to the identity operator was to be expected.

** We shall say that an operator relation is valid only inside a given subspace when it is actually valid only for the restrictions of the operators being considered to this subspace. To be completely rigorous, we should therefore have to place both sides of relation (30) between two projectors $P(k, j)$.

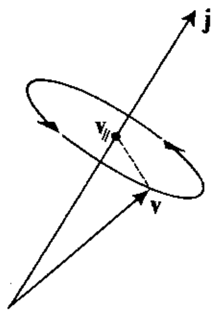

FIGURE 1

Classical interpretation of the projection theorem: since the vector v rotates very rapidly about the total angular momentum j, only its static component $v_{//}$ should be taken into account.

COMMENTS:

(*i*) It cannot be deduced from (30) that, in the total state space [the direct sum of all the subspaces $\mathscr{E}(k, j)$], **V** and **J** are proportional. It must be noted that the proportionality constant $\alpha(k, j)$ (or $\langle \mathbf{J} . \mathbf{V} \rangle_{k,j}$) depends on the subspace $\mathscr{E}(k, j)$ chosen. Moreover, any vector operator **V** may possess non-zero matrix elements between kets belonging to different subspaces $\mathscr{E}(k, j)$, while the corresponding elements of **J** are always zero.

(*ii*) Consider a second vector operator **W**. Its restriction inside $\mathscr{E}(k, j)$ is proportional to **J**, and therefore also to the restriction of **V**. Therefore, *inside a subspace $\mathscr{E}(k, j)$, all vector operators are proportional.*

However, to calculate the proportionality coefficient between **V** and **W**, we cannot simply replace **J** by **W** in (30) (which would give the value $\langle \mathbf{V} . \mathbf{W} \rangle_{k,j}/\langle \mathbf{W}^2 \rangle_{k,j}$). In the proof leading to relation (30), we used the fact that **J** commutes with $P(k, j)$ in (28), which is not generally the case for **W**. To calculate this proportionality coefficient correctly, we note that, inside the subspace $\mathscr{E}(k, j)$:

$$\mathbf{W} = \frac{\langle \mathbf{J} . \mathbf{W} \rangle_{k,j}}{\langle \mathbf{J}^2 \rangle_{k,j}} \mathbf{J} \tag{32}$$

This yields, with (30) taken into account:

$$\mathbf{V} = \frac{\langle \mathbf{J} . \mathbf{V} \rangle_{k,j}}{\langle \mathbf{J} . \mathbf{W} \rangle_{k,j}} \mathbf{W} \tag{33}$$

3. Application: calculation of the Landé g_J factor of an atomic level

In this section, we shall apply the Wigner-Eckart theorem to the calculation of the effect of a magnetic field **B** on the energy levels of an atom. We shall see that this theorem considerably simplifies the calculations and enables us to predict, in a very general way, that the magnetic field removes degeneracies, causing equidistant levels to appear (to first order in B). The energy difference of these states is proportional to B and to a constant g_J (the Landé factor) which we shall calculate.

Let **L** be the total orbital angular momentum of the electrons of an atom (the sum of their individual orbital angular momenta $\mathbf{L}_i$), and let **S** be their total

3. Compute the quadratic Zeeman effect in the ground state of atomic hydrogen due to the term

$$V = \frac{q^2}{2m} \boldsymbol{A}^2 \tag{17}$$

in SI units. Use

$$\boldsymbol{A} = -\frac{1}{2} \boldsymbol{r} \times \boldsymbol{B} \tag{18}$$

in which $\boldsymbol{B}$ is a uniform static field. The wave-function of the ground state $|100\rangle$ of atomic hydrogen is

$$\langle \boldsymbol{r}|100\rangle = \frac{1}{\sqrt{\pi a_0^3}}\, e^{-r/a_0} \tag{19}$$

in which $r = \sqrt{\boldsymbol{r} \cdot \boldsymbol{r}}$, $a_0 = \hbar^2/(me^2)$, and $e = q/\sqrt{4\pi\epsilon_0}$. If we write the energy shift as

$$\Delta_{100}^{(2)} = -\frac{1}{2} \chi \boldsymbol{B}^2 \tag{20}$$

what is the **diamagnetic susceptibility** χ? The formula

$$\int_0^\infty e^{-cr}\, r^n \, dr = \frac{n!}{c^{n+1}} \tag{21}$$

may help.

First-order perturbation theory tells us that the change in the energy of the ground state is

$$\Delta E = \langle 100 | V | 100 \rangle = \frac{q^2}{2m} \langle 100 | \vec{A}^2 | 100 \rangle$$

where

$$\vec{A} = -\frac{1}{2} r \times B = -\frac{B}{2} r \times \hat{z}$$

whence

$$\vec{A}^2 = A_1^2 + A_2^2 = \frac{B^2}{4}(x^2+y^2) = \frac{B^2}{4}(r^2 - z^2).$$

So

$$\Delta = \frac{q^2 B^2}{8m} \int d^3r \, |\psi_{100}(r)|^2 (r^2 - z^2) \quad \text{where}$$

$$\psi_{100}(r) = \left(\frac{Z}{a_0}\right)^{3/2} 2\, e^{-Zr/a_0} \frac{1}{\sqrt{4\pi}} \quad \text{with } Z = 1.$$

So

$$\Delta = \frac{q^2 B^2}{8\,m}\left(\frac{1}{a_0}\right)^3 \frac{1}{\pi}\int d^3r\, e^{-\frac{2r}{a_0}}\,(r^2 - z^2)$$

$$= \frac{q^2 B^2}{8\pi m a_0^3}\,\frac{2}{3}\int d^3r\; e^{-2r/a_0}\; r^2$$

$$= \frac{q^2 B^2}{3\,m a_0^3}\int_0^\infty dr\;\; r^4\, e^{-2r/a_0}$$

$$= \frac{q^2 B^2}{3\,m\,a_0^3}\;\frac{4!}{\left(\frac{2}{a_0}\right)^5} \;=\; \frac{q^2 B^2 a_0^2\, 4\cdot 3\cdot 2}{3\cdot 4\cdot 2\cdot 4\;m}$$

$$= \frac{1}{4}\, q^2\frac{B^2 a_0^2}{m}.$$

If $\quad \Delta = \dfrac{q^2 B^2 a_0^2}{4\,m} = -\dfrac{1}{2}\chi B^2,$

then

$$\chi = -\frac{q^2 a_0^2}{2\,m}.$$

Incidentally, in SI units

$$\chi = -\frac{1}{2}\,\frac{(1.6\times 10^{-19}\,C)^2\,(0.53\times 10^{-10}\,m)^2}{9.11\times 10^{-31}\,kg} = -3.95\times 10^{-29}\,\frac{J}{T^2} = -2.47\times 10^{-1\ldots}\,eV/T^2.$$