Third homework assignment tentatively due on Wed 11 Mar.

1. The state $|j, m = j\rangle$ is an angular-momentum eigenstate of $\boldsymbol{J}^2$ and of J_3 with eigenvalues $\hbar^2 j(j+1)$ and $\hbar j$. Suppose we right-handedly rotate it by ϵ about the y-axis

$$|j,j\rangle' = \exp(-i\epsilon J_2/\hbar)\,|j,j\rangle. \tag{1}$$

Find the probability

$$P = |\langle j,j|j,j\rangle'|^2 \tag{2}$$

that the rotated system is still in the state $|j,j\rangle$ to order ϵ^2. *Hint*: The relation

$$J_2 = \frac{1}{2i}\,(J_+ - J_-) \tag{3}$$

may help.

2. Suppose $\boldsymbol{V}$ is a vector operator in the sense that

$$[V_i, J_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} V_k. \tag{4}$$

Define

$$J_\pm = J_1 \pm i J_2 \quad \text{and} \quad V_\pm = V_1 \pm i V_2. \tag{5}$$

(a) Find the commutators

$$[V_\pm, J_3]. \tag{6}$$

(b) If s and s' are eigenvalues of a scalar operator S that commutes with $\boldsymbol{J}$, then most of the matrix elements

$$\begin{gathered}\langle s,j,m|V_3|s',j',m'\rangle \\ \langle s,j,m|V_+|s',j',m'\rangle \\ \langle s,j,m|V_-|s',j',m'\rangle\end{gathered} \tag{7}$$

are necessarily zero. Use the answer to part a to find selection rules that identify the values of the difference $m-m'$ that allow each of these matrix elements to be non-zero. (c) Find the commutator

$$[V_+, J_+]. \tag{8}$$

(d) Use it to show that

$$\langle s, j, m+1|V_+|s, j, m\rangle = f_+(s, j)\langle s, j, m+1|J_+|s, j, m\rangle \tag{9}$$

in which $f_+(s, j)$ does **not** depend upon m. (e) Show that d implies that

$$\langle s, j, m|V_+|s, j, m'\rangle = f_+(s, j)\langle s, j, m|J_+|s, j, m'\rangle. \tag{10}$$

(f) Now show that

$$\langle s, j, m|V_-|s, j, m'\rangle = f_-(s, j)\langle s, j, m|J_-|s, j, m'\rangle. \tag{11}$$

(g) Evaluate the commutator

$$[V_+, J_-]. \tag{12}$$

(h) Use it and the result of e to show that

$$\langle s, j, m|V_3|s, j, m\rangle = m\hbar f_+(s, j). \tag{13}$$

(i) Now show that

$$\langle s, j, m|V_3|s, j, m\rangle = m\hbar f_-(s, j). \tag{14}$$

(j) Show that $f_+(s, j) = f_-(s, j) \equiv f(s, j)$ and that

$$\langle s, j, m|V_3|s, j, m\rangle = f(s, j)\langle s, j, m|J_3|s, j, m\rangle. \tag{15}$$

(k) Now show that

$$\langle s, j, m|\boldsymbol{V}|s, j, m'\rangle = f(s, j)\langle s, j, m|\boldsymbol{J}|s, j, m'\rangle. \tag{16}$$

3. Compute the quadratic Zeeman effect in the ground state of atomic hydrogen due to the term

$$V = \frac{q^2}{2m}\boldsymbol{A}^2 \tag{17}$$

in SI units. Use

$$\boldsymbol{A} = -\frac{1}{2}\boldsymbol{r} \times \boldsymbol{B} \tag{18}$$

in which $\boldsymbol{B}$ is a uniform static field. The wave-function of the ground state $|100\rangle$ of atomic hydrogen is

$$\langle \boldsymbol{r}|100\rangle = \frac{1}{\sqrt{\pi a_0^3}} \, e^{-r/a_0} \tag{19}$$

in which $r = \sqrt{\boldsymbol{r} \cdot \boldsymbol{r}}$, $a_0 = \hbar^2/(me^2)$, and $e = q/\sqrt{4\pi\epsilon_0}$. If we write the energy shift as

$$\Delta_{100}^{(2)} = -\frac{1}{2} \, \chi \boldsymbol{B}^2 \tag{20}$$

what is the **diamagnetic susceptibility** χ? The formula

$$\int_0^\infty e^{-cr} \, r^n \, dr = \frac{n!}{c^{n+1}} \tag{21}$$

may help.