1. Consider an electron with orbital angular momentum $\ell = 1$ (and of course spin 1/2) in a state in which the total angular momentum $J = 3/2$ and $M = 1/2$. (a) What is the probability that the spin of the electron is up? (b) Same question, but if $J = 1/2$?

(a)

$$|\ell+\tfrac{1}{2}, \ell-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2\ell+1}}\left(\sqrt{2\ell}\,|\ell,\ell-1\rangle|\tfrac{1}{2},+\rangle + |\ell,\ell\rangle|\tfrac{1}{2},-\rangle\right)$$

Set $\ell = 1$

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\left(\sqrt{2}\,|1,0\rangle|\tfrac{1}{2}+\rangle + |1,1\rangle|\tfrac{1}{2}-\rangle\right).$$

So the probability that the electron's spin is up is

$$\left(\sqrt{\frac{2}{3}}\right)^2 = \frac{2}{3}.$$

(b) $$|\ell-\tfrac{1}{2}, \ell-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2\ell+1}}\left(\sqrt{2\ell}\,|\ell\ell\rangle|\tfrac{1}{2}-\rangle - |\ell\ell-1\rangle|\tfrac{1}{2}+\rangle\right)$$

Set $\ell = 1$

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\left(\sqrt{2}\,|1,1\rangle|\tfrac{1}{2}-\rangle - |1,0\rangle|\tfrac{1}{2}+\rangle\right).$$

Now the spin-up probability is

$$\left(-\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}.$$

2. The Δ resonance occurs in pion-nucleon scattering at about 1232 MeV. The pion has isopin 1, so that $|\pi^+\rangle = |1,1\rangle$, $|\pi^0\rangle = |1,0\rangle$, and $|\pi^-\rangle = |1,-1\rangle$. The proton-neutron system has isospin 1/2 with $|p\rangle = |1/2,1/2\rangle$ and $|n\rangle = |1/2,1/2\rangle$. This giant resonance has $J = 3/2$ and isospin $I = 3/2$. The $I_z = 1/2$ state of this resonance can decay into both $|\pi^0,p\rangle$ and $|\pi^+,n\rangle$. What is the ratio of these two decay modes?

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\left(\sqrt{2}\,|1,0\rangle|\tfrac{1}{2},+\rangle + |1,1\rangle|\tfrac{1}{2},-\rangle\right)$$

$$|\Delta^+\rangle = \frac{1}{\sqrt{3}}\left(\sqrt{2}\,|\pi^0\rangle|p\rangle + |\pi^+\rangle|n\rangle\right).$$

So

$$\langle\pi^0 p|\Delta^+\rangle = \sqrt{\frac{2}{3}}$$

and

$$\langle\pi^+ n|\Delta^+\rangle = \frac{1}{\sqrt{3}}.$$

So

$$\frac{|\langle\pi^0 p|\Delta^+\rangle|^2}{|\langle\pi^+ n|\Delta^+\rangle|^2} = 2.$$

3. (a) Compute the derivatives of the unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ with respect to the variables r, θ, and ϕ. Your formulas should express these derivatives in terms of the basis vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$. (b) Using the formulas of (a), derive a formula for the laplacian $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$.

(a) First, we compute the derivatives of $\hat{r}, \hat{\theta}$, and $\hat{\phi}$ with respect to r, θ, and ϕ.

$$\frac{\partial \hat{r}}{\partial r} = 0$$

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{x} \cos\theta \cos\phi + \hat{y} \cos\theta \sin\phi - \hat{z} \sin\theta = \hat{\theta}$$

$$\frac{\partial \hat{r}}{\partial \phi} = -\hat{x} \sin\theta \sin\phi + \hat{y} \sin\theta \cos\phi = \sin\theta \, \hat{\phi}$$

$$\frac{\partial \hat{\theta}}{\partial r} = 0$$

$$\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{x} \sin\theta \cos\phi - \hat{y} \sin\theta \sin\phi - \hat{z} \cos\theta = -\hat{r}$$

$$\frac{\partial \hat{\theta}}{\partial \phi} = -\hat{x} \cos\theta \sin\phi + \hat{y} \cos\theta \cos\phi = \cos\theta \, \hat{\phi}$$

$$\frac{\partial \hat{\phi}}{\partial r} = 0$$

$$\frac{\partial \hat{\phi}}{\partial \theta} = 0$$

$$\frac{\partial \hat{\phi}}{\partial \phi} = -\hat{x}\cos\phi - \hat{y}\sin\phi = -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}$$

(b) Now the laplacian Δ is

$$\nabla\cdot\nabla = \left(\hat{r}\,\partial_r + \frac{\hat{\theta}}{r}\partial_\theta + \frac{\hat{\phi}}{r\sin\theta}\partial_\phi\right)\cdot\left(\hat{r}\,\partial_r + \frac{\hat{\theta}}{r}\partial_\theta + \frac{\hat{\phi}}{r\sin\theta}\partial_\phi\right)$$

$$= \partial_r^2 + \frac{\hat{\theta}}{r}\cdot\frac{\partial\hat{r}}{\partial\theta}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{\hat{\phi}}{r\sin\theta}\cdot\frac{\partial\hat{r}}{\partial\phi}\partial_r$$

$$+ \frac{\hat{\phi}}{r\sin\theta}\cdot\frac{\partial\hat{\theta}}{\partial\phi}\partial_\theta + \frac{1}{r^2\sin^2\theta}\partial_\phi^2$$

$$= \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r}\partial_r + \frac{\cot\theta}{r}\partial_\theta + \frac{1}{r^2\sin^2\theta}\partial_\phi^2$$

$$= \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].$$

4. Consider the two-body hamiltonian

$$H = \frac{\boldsymbol{p}_1^2}{2m_1} + \frac{\boldsymbol{p}_2^2}{2m_2} + V(\boldsymbol{r}_1 - \boldsymbol{r}_2). \tag{1}$$

Define the total momentum as

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 \tag{2}$$

the relative momentum as

$$\boldsymbol{p} = \frac{m_2\, \boldsymbol{p}_1 - m_1\, \boldsymbol{p}_2}{m_1 + m_2}. \tag{3}$$

and the separation as

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2. \tag{4}$$

Now show that H can be written as

$$H = \frac{\boldsymbol{P}^2}{2M} + \frac{\boldsymbol{p}^2}{2\mu} + V(\boldsymbol{r}) \tag{5}$$

in which $M = m_1 + m_2$ is the total mass, and

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{6}$$

is the reduced mass.

$$\frac{P^2}{2M} + \frac{p^2}{2\mu} = \frac{(p_1 + p_2)^2}{2(m_1+m_2)} + \frac{(m_2 p_1 - m_1 p_2)^2 (m_1+m_2)}{2(m_1+m_2)^2 m_1 m_2}$$

$$= \frac{1}{2(m_1+m_2)}\left[p_1^2 + 2p_1 \cdot p_2 + p_2^2 + \frac{m_2^2 p_1^2 - 2m_1 m_2\, p_1 \cdot p_2 + m_1^2 p_2^2}{m_1 m_2}\right]$$

$$= \frac{1}{2(m_1+m_2)}\left[p_1^2\left(1 + \frac{m_2}{m_1}\right) + p_2^2\left(1 + \frac{m_1}{m_2}\right)\right]$$

$$= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}.$$ Thus

$$H = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} + V(\vec{r}_1 - \vec{r}_2) = \frac{P^2}{2M} + \frac{p^2}{2\mu} + V(\vec{r}).$$

5. Using the same notation as in the last problem and the definition

$$\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2} \tag{7}$$

of the center of mass, show that the total orbital angular momentum of the two bodies

$$\boldsymbol{L} = \boldsymbol{L}_1 + \boldsymbol{L}_2 = \boldsymbol{r}_1 \times \boldsymbol{p}_1 + \boldsymbol{r}_2 \times \boldsymbol{p}_2 \tag{8}$$

can be written as

$$\boldsymbol{L} = \boldsymbol{R} \times \boldsymbol{P} + \boldsymbol{r} \times \boldsymbol{p}. \tag{9}$$

$$R \times P = \left(\frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \right) \times (P_1 + P_2)$$

$$r \times p = (\vec{r}_1 - \vec{r}_2) \times \frac{(m_2 p_1 - m_1 p_2)}{(m_1 + m_2)}$$

So

$$\vec{R} \times \vec{P} + \vec{r} \times \vec{p} = r_1 \times p_1 \frac{m_1}{M} + r_2 \times p_2 \frac{m_2}{M}$$

$$+ \frac{m_1}{M} r_1 \times p_2 + \frac{m_2}{M} r_2 \times p_1 + \frac{m_2}{M} r_1 \times p_1 + \frac{m_1}{M} r_2 \times p_2$$

$$- \frac{m_2}{M} r_2 \times p_1 - \frac{m_1}{M} r_1 \times p_2$$

$$= \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$