Creation and Annihilation Operators for Identical Particles

First, box-quantized bosons: the 3-momentum $\boldsymbol{p} = 2\pi\hbar\boldsymbol{n}/L$ in which $\boldsymbol{n}$ is a 3-vector of integers and L is the length of the edge of the box.

A boson creation operator $a^\dagger(p)$ of type p, which may label momentum and spin, takes the vacuum $|0\rangle$ into the state $|p\rangle$ of one boson of type p

$$a^\dagger(p)\,|0\rangle = |p\rangle. \tag{1}$$

A state of two identical bosons, one of type p and one of type p', is

$$|p,p'\rangle = \frac{1}{\sqrt{2}}\left(|1:p,2:p'\rangle + |1:p',2:p\rangle\right). \tag{2}$$

The creation operator $a^\dagger(p)$ takes the state $|p'\rangle$ into this state $|p,p'\rangle$

$$a^\dagger(p)\,|p'\rangle = |p,p'\rangle. \tag{3}$$

The state $|p,p'\rangle$ is normalized when $p \neq p'$. But when $p = p'$, the state $|p,p'\rangle$ has norm 2

$$|p,p\rangle = \frac{1}{\sqrt{2}}\left(|1:p,2:p\rangle + |1:p,2:p\rangle\right) = \sqrt{2}\,|1:p,2:p\rangle. \tag{4}$$

The normalized state is $|2p\rangle$

$$|2p\rangle = |1:p,2:p\rangle. \tag{5}$$

So by (3–5), $a^\dagger(p)$ takes the state $|p\rangle$ into the state $\sqrt{2}\,|2p\rangle$

$$a^\dagger(p)\,|p\rangle = |p,p\rangle = \sqrt{2}\,|2p\rangle. \tag{6}$$

Incidentally, in continuum quantization the case $p = p'$ never happens.

The creation operator $a^\dagger(p)$ takes the state $|p',p''\rangle$ into the state $|p,p',p''\rangle$

$$a^\dagger(p)\,|p',p''\rangle = |p,p',p''\rangle \tag{7}$$

which is

$$\begin{aligned} |p,p',p''\rangle \;=\; & \frac{1}{\sqrt{3!}}\left(|1:p,2:p',3:p''\rangle + |1:p,2:p'',3:p'\rangle\right. \\ & + |1:p',2:p,3:p''\rangle + |1:p',2:p'',3:p\rangle \\ & \left. + |1:p'',2:p,3:p'\rangle + |1:p'',2:p',3:p\rangle\right). \end{aligned} \tag{8}$$

This state is normalized when the p's are all different. But when they all are the same, then its norm is $3! = 6$

$$|p, p, p\rangle = \sqrt{3!}\, |1 : p, 2 : p, 3 : p\rangle. \tag{9}$$

The normalized state is

$$|3p\rangle = |1 : p, 2 : p, 3 : p\rangle. \tag{10}$$

Equation (7) for the case $p = p' = p''$ is

$$a^\dagger(p)\, |p, p\rangle = |p, p, p\rangle. \tag{11}$$

We can use (4, 5, 9, & 10) to write this as

$$a^\dagger(p)\, \sqrt{2}\, |2p\rangle = \sqrt{3!}\, |3p\rangle \tag{12}$$

or more simply as

$$a^\dagger(p)\, |2p\rangle = \sqrt{3}\, |3p\rangle. \tag{13}$$

The creation operator $a^\dagger(p_{n+1})$ takes the state $|p_n, p_{n-1}, \ldots, p_1\rangle$ into the state $|p_{n+1}, p_n, \ldots, p_1\rangle$

$$a^\dagger(p_{n+1})\, |p_n, p_{n-1}, \ldots, p_1\rangle = |p_{n+1}, p_n, \ldots, p_1\rangle \tag{14}$$

which is a sum over all $(n+1)!$ different permutations π of the sequence $1, 2, \ldots, n+1$ into the sequence $i_1, i_2, \ldots, i_{n+1}$ of the states $|1 : p_{i_{(n+1)}}, 2 : p_{i_n}, \ldots, n+1 : p_{i_1}\rangle$

$$|p_{n+1}, p_n, \ldots, p_1\rangle = \frac{1}{\sqrt{(n+1)!}} \sum_{\pi} |1 : p_{i_{(n+1)}}, 2 : p_{i_n}, \ldots, n+1 : p_{i_1}\rangle. \tag{15}$$

When the p's are all different this state is normalized. But when they all are the same, then its norm is $(n+1)!$

$$|p, p, \ldots, p\rangle = \sqrt{(n+1)!}\, |1 : p, 2 : p, \ldots, n+1 : p\rangle. \tag{16}$$

The normalized state is

$$|(n+1)p\rangle = |1 : p, 2 : p, \ldots, n+1 : p\rangle. \tag{17}$$

By an inductive argument, one can show that

$$a^\dagger(p)\,|np\rangle = \sqrt{n+1}\,|(n+1)p\rangle. \tag{18}$$

The adjoint of this last equation is

$$\langle np|\,a(p) = \sqrt{n+1}\,\langle (n+1)p|. \tag{19}$$

Since

$$\langle np|mp\rangle = \delta_{nm} \tag{20}$$

it follows that

$$\langle np|\,a(p)|mp\rangle = \sqrt{n+1}\,\langle (n+1)p|mp\rangle = \sqrt{n+1}\,\delta_{n+1,m} = \sqrt{n+1}\,\delta_{n,m-1}. \tag{21}$$

Thus

$$a(p)|mp\rangle = \sqrt{m}\,|(m-1)p\rangle. \tag{22}$$

Finally, then

$$[a(p), a^\dagger(p')] = \delta_{p,p'}. \tag{23}$$

Apart from minus signs, fermions are much simpler. The creation operator $a^\dagger(p)$ turns the vacuum into the state $|p\rangle$

$$a^\dagger(p)|0\rangle = |p\rangle \tag{24}$$

and the state $|p'\rangle$ into the state $|p,p'\rangle$

$$a^\dagger(p)|p'\rangle = |p,p'\rangle = \frac{1}{\sqrt{2}}\left(|1:p,2:p'\rangle - |1:p',2:p\rangle\right). \tag{25}$$

This state is normalized if $p \neq p'$. But if $p = p'$, then it is zero

$$a^\dagger(p)|p\rangle = |p,p\rangle = \frac{1}{\sqrt{2}}\left(|1:p,2:p\rangle - |1:p,2:p\rangle\right) = 0. \tag{26}$$

This is Pauli's exclusion principle.

By combining (24 & 25), we have

$$a^\dagger(p)|p'\rangle = a^\dagger(p)a^\dagger(p')|0\rangle = |p,p'\rangle \tag{27}$$

as well as

$$a^\dagger(p')|p\rangle = a^\dagger(p')a^\dagger(p)|0\rangle = |p',p\rangle = -|p,p'\rangle \tag{28}$$

in which we used the explicit form (25) of the state $|p, p'\rangle$. It follows that

$$a^\dagger(p')a^\dagger(p)|0\rangle = -a^\dagger(p)a^\dagger(p')|0\rangle. \tag{29}$$

One may further show that for any state $|p'', p''', \ldots\rangle$

$$a^\dagger(p')a^\dagger(p)|p'', p''', \ldots\rangle = -a^\dagger(p)a^\dagger(p')|p'', p''', \ldots\rangle \tag{30}$$

and so conclude that

$$a^\dagger(p')\, a^\dagger(p) = -a^\dagger(p)\, a^\dagger(p'). \tag{31}$$

The adjoint relation is

$$a(p')\, a(p) = -a(p)\, a(p'). \tag{32}$$

One may also show that

$$\left\{a(p), a^\dagger(p')\right\} = [a(p), a^\dagger(p')]_+ \equiv a(p)\, a^\dagger(p') + a^\dagger(p')\, a(p) = \delta_{p,p'}. \tag{33}$$