$$\vec{J} = \vec{L} + \vec{S}$$

An electron in a central potential has orbital angular momentum $\vec{L}$ and spin $\vec{S}$. Its angular momentum states are

$$|\ell, m\rangle \otimes |\tfrac{1}{2}, \pm\rangle.$$

These are eigenstates of $\vec{L}^2$, L_z, $\vec{S}^2$, and S_z.

The electron's total angular momentum is

$$\vec{J} = \vec{L} + \vec{S}.$$

Its eigenstates of $\vec{J}^2$, J_z, $\vec{L}^2$, & $\vec{S}^2$ are

$$|J, M\rangle = |J, M, \ell, \tfrac{1}{2}\rangle.$$

withe eigenvalues

$$\vec{J}^2 |J, M\rangle = \hbar^2 J(J+1)\, |J, M\rangle$$
$$J_z |J, M\rangle = \hbar\, M\, |J, M\rangle$$
$$\vec{L}^2 |J, M, \ell, \tfrac{1}{2}\rangle = \hbar^2 \ell(\ell+1)\, |J, M, \ell, \tfrac{1}{2}\rangle$$
$$\vec{S}^2 |J, M, \ell, \tfrac{1}{2}\rangle = \hbar^2 \tfrac{1}{2}(\tfrac{3}{2})\, |J, M, \ell, \tfrac{1}{2}\rangle$$
$$= \tfrac{3}{4}\hbar^2\, |J, M, \ell, \tfrac{1}{2}\rangle.$$

The state of highest J and M is

$$|JM\rangle = |\ell+\tfrac{1}{2}, \ell+\tfrac{1}{2}\rangle = |\ell, \ell\rangle\, |\tfrac{1}{2}, +\rangle.$$

Recall $J_- = J_1 - iJ_2$ and

$$J_- |jm\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\, |j, m-1\rangle$$
$$= \hbar\sqrt{(j+m)(j+1-m)}\, |j, m-1\rangle.$$

We use J_- to find $|l+\frac{1}{2}, l-\frac{1}{2}\rangle$:

$$J_- |l+\tfrac{1}{2}, l+\tfrac{1}{2}\rangle = \hbar\sqrt{(2l+1)}\; |l+\tfrac{1}{2}, l-\tfrac{1}{2}\rangle$$

$$= (L_- + S_-)\, |l,l\rangle |\tfrac{1}{2},+\rangle$$

$$= (L_- |l,l\rangle) |\tfrac{1}{2},+\rangle + |l,l\rangle S_- |\tfrac{1}{2},+\rangle$$

$$= \hbar\sqrt{2l}\, |l,l-1\rangle |\tfrac{1}{2},+\rangle + |l,l\rangle \hbar |\tfrac{1}{2},-\rangle.$$

Thus

$$|l+\tfrac{1}{2}, l-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2l+1}}\left(\sqrt{2l}\, |l,l-1\rangle |\tfrac{1}{2},+\rangle + |l,l\rangle |\tfrac{1}{2},-\rangle\right).$$

Now we use J_- again to find $|l+\frac{1}{2}, l-\frac{3}{2}\rangle$:

$$J_- |l+\tfrac{1}{2}, l-\tfrac{1}{2}\rangle = \hbar\sqrt{2l\cdot 2}\; |l+\tfrac{1}{2}, l-\tfrac{3}{2}\rangle$$

$$= \frac{(L_- + S_-)}{\sqrt{2l+1}}\left(\sqrt{2l}\, |l,l-1\rangle |\tfrac{1}{2},+\rangle + |l,l\rangle |\tfrac{1}{2},-\rangle\right)$$

$$= \frac{1}{\sqrt{2l+1}}\Big(\sqrt{2l}\, L_- |l,l-1\rangle |\tfrac{1}{2},+\rangle + \sqrt{2l}\, |l,l-1\rangle S_- |\tfrac{1}{2},+\rangle$$
$$+ L_- |l,l\rangle |\tfrac{1}{2},-\rangle + |l,l\rangle S_- |\tfrac{1}{2},-\rangle\Big).$$

But $S_- |\frac{1}{2},-\rangle = 0$, and the other terms give

$$J_-|l+\tfrac{1}{2},l-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2l+1}}\Big(\sqrt{2l}\,\hbar\sqrt{(2l-1)\cdot 2}\,|l,l-2\rangle|\tfrac{1}{2},+\rangle$$

$$+\sqrt{2l}\,|l,l-1\rangle\hbar|\tfrac{1}{2},-\rangle \;+\hbar\sqrt{2l}\,|l,l-1\rangle|\tfrac{1}{2}-\rangle\Big).$$

Thus

$$|l+\tfrac{1}{2},l-\tfrac{3}{2}\rangle = \frac{1}{\sqrt{4l}}\frac{1}{\sqrt{2l+1}}\Big(\sqrt{2l}\sqrt{2(2l-1)}\,|l,l-2\rangle|\tfrac{1}{2},+\rangle$$

$$+2\sqrt{2l}\,|l,l-1\rangle|\tfrac{1}{2},-\rangle\Big)$$

$$=\frac{1}{\sqrt{2l+1}}\Big(\sqrt{2l-1}\,|l,l-2\rangle|\tfrac{1}{2},+\rangle+\sqrt{2}\,|l,l-1\rangle|\tfrac{1}{2},-\rangle\Big).$$

The other $|l+\frac{1}{2},M\rangle$ states are

$$|l+\tfrac{1}{2},M\rangle = \frac{1}{\sqrt{2l+1}}\Big(\sqrt{l+M+\tfrac{1}{2}}\,|l,M-\tfrac{1}{2}\rangle|\tfrac{1}{2}+\rangle$$

$$+\sqrt{l-M+\tfrac{1}{2}}\,|l,M+\tfrac{1}{2}\rangle|\tfrac{1}{2},-\rangle\Big)$$

where

$$M = l+\tfrac{1}{2},\ l-\tfrac{1}{2},\ l-\tfrac{3}{2},\ \ldots\ -l+\tfrac{1}{2},\ -(l+\tfrac{1}{2}).$$

These are all the states with $J=l+\frac{1}{2}$.

The other states have $J=l-\frac{1}{2}$.

To find them, we know that the state

$|l-\frac{1}{2}, l-\frac{1}{2}\rangle$ must be the linear combination of the states $|l,l\rangle|\frac{1}{2},-\rangle$ and $|l,l-1\rangle|\frac{1}{2},+\rangle$ that is orthogonal to the state

$$|l+\tfrac{1}{2}, l-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2l+1}}\left(\sqrt{2l}\,|l,l-1\rangle|\tfrac{1}{2}+\rangle + |l,l\rangle|\tfrac{1}{2},-\rangle\right).$$

By convention, we choose

$$|l-\tfrac{1}{2}, l-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2l+1}}\left(\sqrt{2l}\,|l,l\rangle|\tfrac{1}{2}-\rangle - |l,l-1\rangle|\tfrac{1}{2},+\rangle\right).$$

Similarly,

$$|l-\tfrac{1}{2}, l-\tfrac{3}{2}\rangle = \frac{1}{\sqrt{2l+1}}\left(\sqrt{2l-1}\,|l,l-1\rangle|\tfrac{1}{2},-\rangle - \sqrt{2}\,|l,l-2\rangle|\tfrac{1}{2}+\rangle\right).$$

More generally,

$$|l-\tfrac{1}{2}, M\rangle = \frac{1}{\sqrt{2l+1}}\left(\sqrt{l+M+\tfrac{1}{2}}\,|l,M+\tfrac{1}{2}\rangle|\tfrac{1}{2}-\rangle - \sqrt{l-M+\tfrac{1}{2}}\,|l,M-\tfrac{1}{2}\rangle|\tfrac{1}{2}+\rangle\right).$$

for

$$M = l-\tfrac{1}{2},\ l-\tfrac{3}{2},\ \ldots,\ -l+\tfrac{3}{2},\ -(l-\tfrac{1}{2}).$$