The Hamiltonian of Free Vacuum QED

We set

$$f_{kr} = \left(\frac{\hbar}{2\epsilon_0 V \omega_k}\right)^{\frac{1}{2}}$$

so that

$$\vec{A}(\vec{x},t) = \sum_{kr} f_{kr}\left[\epsilon_r a_r(k) e^{i(kx-\omega t)} + \epsilon_r^* a_r^\dagger(k) e^{-i(kx-\omega t)}\right].$$

The hamiltonian H_{0F} is

$$H_{0F} = \int d^3x \left[\frac{\epsilon_0}{2}\vec{E}(x,t)^2 + \frac{1}{2\mu_0}\vec{B}^2(x,t)\right]$$

in which

$$\vec{E} = -\dot{\vec{A}} \qquad \text{and} \qquad B = \nabla \times A.$$

So the E^2 part of H_{0F} is

$$H_{0FE} = \int d^3x \frac{\epsilon_0}{2}\dot{A}^2$$

$$= \frac{\epsilon_0}{2}\int d^3x \sum_{kr} f_{kr}\left[-i\omega\epsilon_r a_r e^{i(k\cdot x-\omega t)} + i\omega\epsilon^* a^\dagger e^{-i(k\cdot x-\omega t)}\right]$$

$$\times \sum_{k'r'} f_{k'r'}\left[-i\omega'\epsilon_{r'} a_{r'} e^{i(k'x-\omega' t)} + i\omega'\epsilon^* a^\dagger e^{-i(k'\cdot x-\omega' t)}\right].$$

Now $\int d^3x\, e^{ikx+ik'\cdot x} = \delta_{k,-k'} V$ and

$$\int d^3x\, e^{ik\cdot x - ik'\cdot x} = \delta_{kk'} V$$

So H_{0FE} is

$$H_{0FE} = \frac{\epsilon_0}{2} \sum_{k,r,r'} \frac{-\omega_k^2 \hbar V}{2\epsilon_0 V \omega_k} \vec{\epsilon}_r(k)\cdot\vec{\epsilon}_{r'}(-k)\, a_r(\vec{k})\, a_{r'}(-\vec{k}) e^{-2i\omega_k t}$$

$$+ \frac{\epsilon_0}{2} \sum_{krr'} \frac{-\omega_k^2 \hbar}{2\epsilon_0 \omega_k} \vec{\epsilon}^*_r(k)\cdot\vec{\epsilon}^*_{r'}(-\vec{k})\, a_r^\dagger(k)\, a_{r'}^\dagger(-k) e^{2i\omega_k t}$$

$$+ \frac{\epsilon_0}{2} \sum_{kr} \frac{\omega_k \hbar}{2\epsilon_0} a_r(k) a_r^\dagger(k) + \frac{\omega_k \hbar}{2\epsilon_0} a_r^\dagger(k) a_r(k)) .$$

Obviously, we want the complicated sums to cancel similar terms in H_{0FB} which involves

$$B = \nabla \times A = \sum_{kr} f_{kr} \left[ik \times \epsilon_r(k) a_r(k) e^{i(k\cdot x - \omega t)} - ik \times \epsilon_r^*(k) a_r^\dagger(k) e^{-i(k\cdot x - \omega t)} \right].$$

So

$$H_{0FB} = \frac{1}{2\mu_0} \int d^3x\, (\nabla \times A)^2$$

$$= \frac{1}{2\mu_0} \int d^3x \sum_{kr} f_{kr} \left[ik\times\epsilon a e^{i(k\cdot x - \omega t)} - ik \times \epsilon^* a^\dagger e^{-i(k\cdot x - \omega t)} \right]$$

$$\times \sum_{k'r'} f_{k'r'} \left[ik' \times \epsilon' a' e^{i(k'\cdot x - \omega' t)} - ik' \times \epsilon'^* a'^\dagger e^{-i(k'\cdot x - \omega' t)} \right].$$

So

$$H_{0FB} = \frac{1}{2\mu_0} \sum_{k r r'} \frac{\hbar}{2\epsilon_0 \omega_k} k \times \epsilon_r(k) \cdot k \times \epsilon_{r'}(-k) a_r(k) a_{r'}(-k) e^{-2i\omega_k t}$$

$$+ \frac{1}{2\mu_0} \sum_{k r r'} \frac{\hbar}{2\epsilon_0 \omega_k} k \times \epsilon^*(k) \cdot k \times \epsilon^*(-k) a_r^\dagger(k) a_{r'}^\dagger(-k) e^{2i\omega_k t}$$

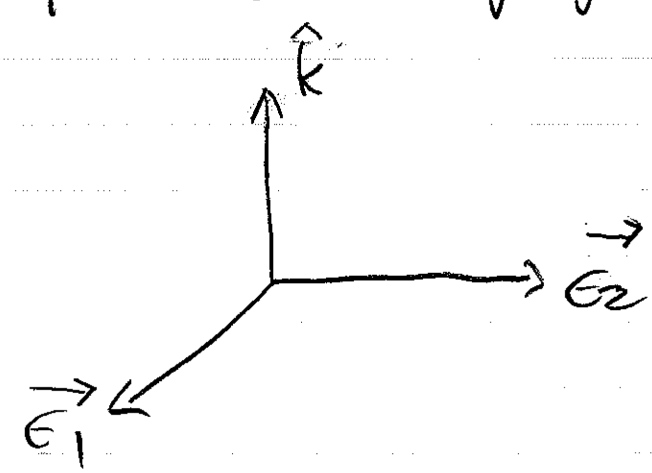
$$+ \frac{1}{2\mu_0} \sum_{k r} \frac{\hbar}{2\epsilon_0 \omega_k} k \times \epsilon \cdot k \times \epsilon^* \left[a_r(k) a_r^\dagger(k) + a_r^\dagger(k) a_r(k) \right].$$

Again, we want to keep the simple terms and cancel the complicated ones.

The easy way to proceed is to choose polarization vectors $\vec{\epsilon}_r(\vec{k})$ that are real. They must be orthogonal to $\vec{k}$

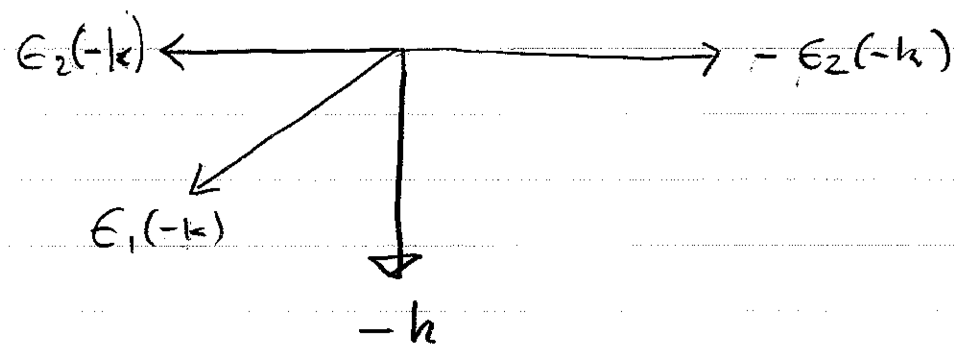
$$\vec{\epsilon}_r(\vec{k}) \cdot \vec{k} = 0$$

because of the Coulomb-gauge condition $\nabla \cdot A = 0$.



So
$$\hat{k} \times \epsilon_1 = \epsilon_2$$
$$\hat{k} \times \epsilon_2 = -\epsilon_1 .$$

Also by rotating the previous drawing by π about the ϵ_1-axis, we get



So that
$$-\hat{k} \times \epsilon_1(-k) = \epsilon_2(-k) = -\epsilon_2(k)$$
$$-\hat{k} \times \epsilon_2(-k) = -\epsilon_1(-k) = -\epsilon_1(k)$$

With these choices of polarization vectors,

$$\epsilon_1(k) \cdot \epsilon_1(-k) = 1$$
$$\epsilon_1(k) \cdot \epsilon_2(-k) = 0$$
$$\epsilon_2(k) \cdot \epsilon_1(-k) = 0$$
$$\epsilon_2(k) \cdot \epsilon_2(-k) = -1 .$$

Now the electric field energy is

$$H_{OFE} = \frac{1}{4}\sum_k -\hbar\omega_k \left[a_1(k)a_1(-k) - a_2(k)a_2(-k)\right] e^{-2i\omega_k t}$$

$$+ \frac{1}{4}\sum_k -\hbar\omega_k \left[a_1(k)a_1(-k) - a_2(k)a_2(-k)\right] e^{2i\omega_k t}$$

$$+ \frac{1}{4}\sum_{k,r} \hbar\omega_k \left[a_r(k)a_r^\dagger(k) + a_r^\dagger(k)a_r(k)\right].$$

The magnetic field energy involves

$$\hat{k} \times \epsilon_1(k) \cdot \hat{k} \times \epsilon_1(-k) = \epsilon_2(k)\cdot(-\epsilon_2(-k)) = \epsilon_2(k)\cdot\epsilon_2(k) = 1$$

$$\hat{k} \times \epsilon_1(k) \cdot \hat{k} \times \epsilon_2(-k) = \epsilon_2(k)\cdot \epsilon_1(-k) = \epsilon_2(k)\cdot\epsilon_1(k) = 0$$

$$\hat{k} \times \epsilon_2(k) \cdot \hat{k} \times \epsilon_1(-k) = -\epsilon_1(k)\cdot(-\epsilon_2(-k)) = 0$$

$$\hat{k} \times \epsilon_2(k) \cdot \hat{k} \times \epsilon_2(-k) = -\epsilon_1(k)\cdot\epsilon_1(-k) = -\epsilon_1(k)\cdot\epsilon_1(k) = -1.$$

So since $\epsilon_0 \mu_0 = 1/c^2$ and $\omega_k = kc$,

$$H_{OFB} = \frac{c^2}{4}\sum_k \frac{\hbar k^2}{\omega_k}\left[a_1(k)a_1(-k) - a_2(k)a_2(-k)\right] e^{-2i\omega_k t}$$

$$+ \frac{1}{4}\sum_k \hbar\omega_k \left[a_1^\dagger(k)a_1^\dagger(-k) - a_2^\dagger(k)a_2^\dagger(-k)\right] e^{2i\omega_k t}$$

$$+ \frac{1}{4}\sum_{k,r} \hbar\omega_k \left[a_r(k)a_r^\dagger(k) + a_r^\dagger(k)a_r(k)\right].$$

We see that the unwanted terms exactly cancel

$$H = H_{OFE} + H_{OFB}$$

$$= \frac{1}{2} \sum_{k\nu} \hbar\omega_k \left[a_\nu(k) a_\nu^\dagger(k) + a_\nu^\dagger(k) a_\nu(k) \right]$$

$$= \sum_{k\nu} \hbar\omega_k \left[a_\nu^\dagger(k) a_\nu(k) + \tfrac{1}{2} \right]$$

since

$$[a_\nu(k), a_{\nu'}^\dagger(k')] = \delta_{\nu\nu'} \delta_{kk'} .$$