The integral over q^0 here yields a delta function in time, so this is equivalent to a correction to the interaction Hamiltonian $V(t)$, of the form

$$-\frac{1}{2}\int d^3x \int d^3y \, \frac{j^0(\mathbf{x},t)\, j^0(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|} .$$

This is just right to cancel the Coulomb interaction (8.4.25). Our result is that the photon propagator can be taken effectively as the covariant quantity

$$\Delta^{\text{eff}}_{\mu\nu}(x-y) = (2\pi)^{-4}\int d^4q \frac{\eta_{\mu\nu}}{q^2 - i\epsilon} e^{iq\cdot(x-y)} \tag{8.5.8}$$

with the Coulomb interaction dropped from now on. We see that the apparent violation of Lorentz invariance in the instantaneous Coulomb interaction is cancelled by another apparent violation of Lorentz invariance, that as noted in Section 5.9 the fields $a^\mu(x)$ are not four-vectors, and therefore have a non-covariant propagator. From a practical point of view, the important point is that in the momentum space Feynman rules, the contribution of an internal photon line is simply given by

$$\frac{-i}{(2\pi)^4}\frac{\eta_{\mu\nu}}{q^2 - i\epsilon} \tag{8.5.9}$$

and the Coulomb interaction is dropped.

8.6 Feynman Rules for Spinor Electrodynamics

We are now in a position to state the Feynman rules for calculating the S-matrix in quantum electrodynamics. For definiteness, we will consider the electrodynamics of a single species of spin $\frac{1}{2}$ particles of charge $q = -e$ and mass m. We will call these fermions electrons, but the same formalism applies to muons and other such particles. The simplest gauge- and Lorentz-invariant Lagrangian for this theory is*

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \bar{\Psi}\left(\gamma^\mu[\partial_\mu + ieA_\mu] + m\right)\Psi . \tag{8.6.1}$$

The electric current four-vector is then simply

$$J^\mu = \frac{\partial \mathscr{L}}{\partial A_\mu} = -ie\,\bar{\Psi}\gamma^\mu\Psi . \tag{8.6.2}$$

* In Chapter 12 we will discuss reasons why more complicated terms are excluded from the Lagrangian density.

The interaction (8.4.23) in the interaction picture is here

$$V(t) = +ie \int d^3x \ (\bar{\psi}(\mathbf{x}, t)\gamma^\mu \psi(\mathbf{x}, t)) \ a_\mu(\mathbf{x}, t) + V_{\text{Coul}}(t) \,. \tag{8.6.3}$$

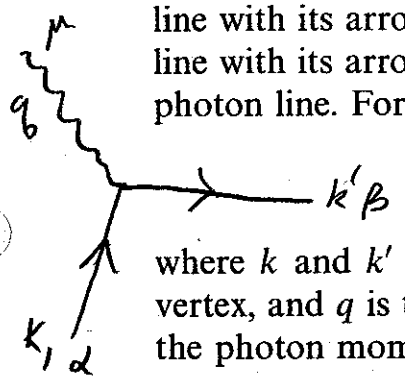
(There is no V_{matter} here.) As we have seen, the Coulomb term $V_{\text{Coul}}(t)$ just serves to cancel a part of the photon propagator that is non-covariant and local in time.

Following the general results of Section 6.3, we can state the momentum space Feynman rules for the connected part of the S-matrix in this theory as follows:

(i) Draw all Feynman diagrams with up to some given number of vertices. The diagrams consist of electron lines carrying arrows and photon lines without arrows, with the lines joined at vertices, at each of which there is one incoming and one outgoing electron line and one photon line. There is one external line coming into the diagram from below or going upwards out of the diagram for each particle in the initial or final states, respectively; electrons are represented by external lines carrying arrows pointing upwards into or out of the diagram, while positrons are represented by lines carrying arrows pointing downwards into or out of the diagram. There are also as many internal lines as are needed to give each vertex the required number of attached lines. Each internal line is labelled with an off-mass-shell four-momentum flowing in a definite direction along the line (taken conventionally to flow along the direction of the arrow for electron lines.) Each external line is labelled with the momentum and spin z-component or helicity of the electron or photon in the initial and final states.

(ii) Associate factors with the components of the diagram as follows:

Vertices



Label each vertex with a four-component Dirac index α at the electron line with its arrow coming into the vertex, a Dirac index β at the electron line with its arrow going out of the vertex, and a spacetime index μ at the photon line. For each such vertex, include a factor

$$(2\pi)^4 e (\gamma^\mu)_{\beta\alpha} \delta^4(k - k' + q) \,, \tag{8.6.4}$$

where k and k' are the electron four-momenta entering and leaving the vertex, and q is the photon four-momentum entering the vertex (or minus the photon momentum leaving the vertex).

External lines:

Label each external line with the three-momentum **p** and spin z-component or helicity σ of the particle in the initial or final state. For each line for an electron in the final state running out of a vertex carrying a Dirac label β on this line, include a factor*

$$\frac{\bar{u}_\beta(\mathbf{p},\sigma)}{(2\pi)^{3/2}} \cdot \tag{8.6.5}$$

For each line for a positron in the final state running into a vertex carrying a Dirac label α on this line, include a factor

$$\frac{v_\alpha(\mathbf{p},\sigma)}{(2\pi)^{3/2}} \cdot \tag{8.6.6}$$

For each line for an electron in the initial state running into a vertex carrying a Dirac label α on this line, include a factor

$$\frac{u_\alpha(\mathbf{p},\sigma)}{(2\pi)^{3/2}} \cdot \tag{8.6.7}$$

For each line for a positron in the initial state running out of a vertex carrying a Dirac label β on this line, include a factor

$$\frac{\bar{v}_\beta(\mathbf{p},\sigma)}{(2\pi)^{3/2}} \cdot \tag{8.6.8}$$

The us and vs are the four-component spinors discussed in Section 5.5. For each line for a photon in the final state connected to a vertex carrying a spacetime label μ on this line, include a factor

$$\frac{e^*_\mu(\mathbf{p},\sigma)}{(2\pi)^{3/2}\sqrt{2p^0}} \cdot \tag{8.6.9}$$

For each line for a photon in the initial state connected to a vertex carrying a spacetime label μ on this line, include a factor

$$\frac{e_\mu(\mathbf{p},\sigma)}{(2\pi)^{3/2}\sqrt{2p^0}} \cdot \tag{8.6.10}$$

The e_μ are the photon polarization four-vectors described in the previous section.

Internal lines:

For each internal electron line carrying a four-momentum k and running from a vertex carrying a Dirac label β to another vertex carrying a Dirac

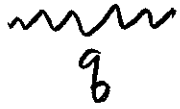
* A matrix β has been extracted from the interaction in (8.6.4), so that $\bar{u}$ and $\bar{v}$ appear instead of $u^\dagger$ and $v^\dagger$.

label α, include a factor

$$\frac{-i}{(2\pi)^4}\,\frac{[-i\not{k}+m]_{\alpha\beta}}{k^2+m^2-i\epsilon}\,. \tag{8.6.11}$$

(We are here using the very convenient 'Dirac slash' notation; for any four-vector v^μ, $\not{v}$ denotes $\gamma_\mu v^\mu$.) For each internal photon line carrying a four-momentum q that runs between two vertices carrying spacetime labels μ and ν include a factor

$$\frac{-i}{(2\pi)^4}\,\frac{\eta_{\mu\nu}}{q^2-i\epsilon}\,. \tag{8.6.12}$$



(iii) Integrate the product of all these factors over the four-momenta carried by the internal lines, and sum over all Dirac and spacetime indices.

(iv) Add up the results obtained in this way from each Feynman diagram.

Additional combinatoric factors and fermionic signs may need to be included, as described in parts (v) and (vi) of Section 6.1.

The difficulty of evaluating Feynman diagrams increases rapidly with the number of internal lines and vertices, so it is important to have some idea of what numerical factors tend to suppress the contributions of the more complicated diagrams. We shall estimate these numerical factors including not only the factors of the electronic charge e associated with vertices, but also the factors of 2 and π from vertices, propagators, and momentum space integrals.

Consider a connected Feynman diagram with V vertices, I internal lines, E external lines, and L loops. These quantities are not independent, but are subject to relations already used in Section 6.3:

$$L = I - V + 1\,, \qquad 2I + E = 3V\,.$$

There is a factor $e(2\pi)^4$ from each vertex, a factor $(2\pi)^{-4}$ from each internal line, and a four-dimensional momentum space integral for each loop. The volume element in four-dimensional Euclidean space in terms of a radius parameter κ is $\pi^2\kappa^2 d\kappa^2$, so each loop contributes a factor π^2. Thus the diagram will contain a factor

$$(2\pi)^{4V} e^V (2\pi)^{-4I} \pi^{2L} = (2\pi)^4 e^{E-2} \left(\frac{e^2}{16\pi^2}\right)^L\,.$$

The number E of external lines is fixed for a given process, so we see that the expansion parameter that governs the suppression of Feynman graphs for each additional loop is

$$\frac{e^2}{16\pi^2} = \frac{\alpha}{4\pi} = 5.81 \times 10^{-4}\,.$$

Fortunately this is small enough that good accuracy can usually be obtained from Feynman diagrams with at most a few loops.

* * *

We must say a little more about the spin states of photons and electrons in realistic experiments, where not every particle in the initial and final states has a definite known helicity or spin z-component. This consideration is especially important for photons, which in practice are often characterized by a state of transverse or elliptical polarization rather than helicity. As we saw in the previous section, for photons of helicity ± 1, the polarization vectors are

$$e(\mathbf{p}, \pm 1) = R(\hat{\mathbf{p}}) \begin{bmatrix} 1/\sqrt{2} \\ \pm i/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} ,$$

where $R(\hat{\mathbf{p}})$ is the standard rotation that takes the z-axis to the $\mathbf{p}$ direction. These are not the only possible photon states; in general, a photon state can be a linear combination of helicity states $\Psi_{\mathbf{p},\pm 1}$

$$\alpha_+ \Psi_{\mathbf{p},+1} + \alpha_- \Psi_{\mathbf{p},-1} \tag{8.6.13}$$

which is properly normalized if

$$|\alpha_+|^2 + |\alpha_-|^2 = 1 . \tag{8.6.14}$$

To calculate the S-matrix element for absorbing or emitting such a photon, we simply replace $e_\mu(\mathbf{p}, \pm 1)$ in the Feynman rules with

$$e_\mu(\mathbf{p}) = \alpha_+ \, e_\mu(\mathbf{p}, +1) + \alpha_- \, e_\mu(\mathbf{p}, -1) . \tag{8.6.15}$$

The polarization vectors for definite helicity satisfy the normalization condition

$$e^*_\mu(\mathbf{p}, \lambda') \, e^\mu(\mathbf{p}, \lambda) = \delta_{\lambda' \lambda} \tag{8.6.16}$$

and therefore in general

$$e^*_\mu(\mathbf{p}) \, e^\mu(\mathbf{p}) = 1 . \tag{8.6.17}$$

The two extreme cases are *circular polarization*, for which $\alpha_- = 0$ or $\alpha_+ = 0$, and *linear polarization*, for which $|\alpha_+| = |\alpha_-| = 1/\sqrt{2}$. For linear polarization, by an adjustment of the overall phase of the state (8.6.13), we can make α_+ and α_- complex conjugates, so that they can be expressed as

$$\alpha_\pm = \exp(\mp i\phi)/\sqrt{2} . \tag{8.6.18}$$

Then in the Feynman rules we should use a polarization vector

$$e_\mu(\mathbf{p}) = R(\hat{\mathbf{p}}) \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \\ 0 \end{bmatrix} . \tag{8.6.19}$$

That is, ϕ is the azimuthal angle of the photon polarization in the plane perpendicular to $\mathbf{p}$. Note that the photon polarization vector here is *real*, which is only possible for linear polarization. In between the extremes of circular and linear polarization are the states of *elliptic* polarization, for which $|\alpha_+|$ and $|\alpha_-|$ are non-zero and unequal.

More generally, an initial photon may be prepared in a statistical mixture of spin states. In the most general case, an initial photon may have any number of possible polarization vectors $e_\mu^{(r)}(\mathbf{p})$, each with probability P_r. The rate for absorbing such a photon in a given process will then be of the form

$$\Gamma = \sum_r P_r |e_\mu^{(r)}(\mathbf{p})\, M^\mu|^2 = M^{\mu *} M^\nu \rho_{\nu\mu} , \tag{8.6.20}$$

where ρ is the *density matrix*

$$\rho_{\nu\mu} \equiv \sum_r P_r\, e_\nu^{(r)}(\mathbf{p})\, e_\mu^{(r)*}(\mathbf{p}) . \tag{8.6.21}$$

Since ρ is obviously a Hermitian positive matrix of unit trace (because $\sum_r P_r = 1$) with $\rho_{\nu 0} = \rho_{0\mu} = 0$ and $\rho_{\nu\mu} p^\nu = \rho_{\nu\mu} p^\mu = 0$, it may be written as

$$\rho_{\nu\mu} = \sum_{s=1,2} \lambda_s\, e_\nu(\mathbf{p}; s)\, e_\mu^*(\mathbf{p}; s) , \tag{8.6.22}$$

where $e_\mu(\mathbf{p}; s)$ are the two orthonormal eigenvectors of ρ with

$$e_0(\mathbf{p}; s) = e_\mu(\mathbf{p}; s) p^\mu = 0 \tag{8.6.23}$$

and λ_s are the corresponding eigenvalues, with

$$\lambda_s \geq 0 , \qquad \sum_{s=1,2} \lambda_s = 1 .$$

We may then write the rate for the photon absorption process as

$$\Gamma = \sum_{s=1,2} \lambda_s |e_\nu(\mathbf{p}; s) M^\nu|^2 . \tag{8.6.24}$$

Thus any statistical mixture of initial photon states is always equivalent to having just two orthonormal polarizations $e_\nu(\mathbf{p}; s)$ with probabilities λ_s.

In particular, if we know nothing whatever about the initial photon polarization, then the two probabilities λ_s for the polarization vectors

$e_v(\mathbf{p};s)$ are equal, so that $\lambda_1 = \lambda_2 = \frac{1}{2}$, and the density matrix (and hence the absorption rate) is an average over initial polarizations

$$\rho_{ij} = \tfrac{1}{2} \sum_{s=1,2} e_i(\mathbf{p};s)\, e_j^*(\mathbf{p};s) = \tfrac{1}{2}\,(\delta_{ij} - \hat{p}_i \hat{p}_j)\ . \tag{8.6.25}$$

Fortunately, this result does not depend on the particular pair of polarization vectors $e_i(\mathbf{p};s)$ over which we average; for unpolarized photons we can average the absorption rate over any pair of orthonormal polarization vectors. Similarly, if we make no attempt to measure the polarization of a photon in the final state, then the rate may be calculated by summing over any pair of orthonormal final photon polarization vectors.

The same remarks apply to electrons and positrons; if (as is usually the case) we make no attempt to prepare an electron or positron so that some spin states are more likely than others, then the rate is to be calculated by *averaging* over any two orthonormal initial spin states, such as those with spin z-component $\sigma = \pm\frac{1}{2}$; if we make no attempt to measure a final electron's or positron's spin state, then we must *sum* the rate over any two orthonormal final spin states, such as those with spin z-component $\sigma = \pm\frac{1}{2}$. Such sums may be performed using the relations (5.5.37) and (5.5.38):

$$\sum_\sigma u_\alpha(\mathbf{p},\sigma)\bar{u}_\beta(\mathbf{p},\sigma) = \left(\frac{-i\not{p} + m}{2p^0}\right)_{\alpha\beta}\ , \tag{8.6.26}$$

$$\sum_\sigma v_\alpha(\mathbf{p},\sigma)\bar{v}_\beta(\mathbf{p},\sigma) = \left(\frac{-i\not{p} - m}{2p^0}\right)_{\alpha\beta}\ , \tag{8.6.27}$$

where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$. For instance, if the initial state contains an electron with momentum $\mathbf{p}$ and spin z-component σ, and a positron with momentum $\mathbf{p}'$ and spin z-component σ', then the S-matrix element for the process will be of the form $(\bar{v}_\alpha(\mathbf{p}',\sigma')\,\mathcal{M}_{\alpha\beta}\,u_\beta(\mathbf{p},\sigma))$. Hence if neither electron nor positron spins are observed, the rate will be proportional to

$$\begin{aligned}&\tfrac{1}{4}\sum_{\sigma',\sigma} \left|(\bar{v}_\alpha(\mathbf{p}',\sigma')\,\mathcal{M}_{\alpha\beta}\,u_\beta(\mathbf{p},\sigma))\right|^2 \\ &\quad = \tfrac{1}{4}\mathrm{Tr}\left\{\beta\,\mathcal{M}^\dagger\,\beta\left(\frac{-i\not{p}' - m}{2p^{0\prime}}\right)\mathcal{M}\left(\frac{-i\not{p} + m}{2p^0}\right)\right\}\ .\end{aligned}$$

Techniques for the calculation of such traces are described in the Appendix to this chapter.