Compton Scattering

The process $p + k \to p' + k'$ in which a photon of momentum k scatters off an electron of momentum p is called Compton scattering. The amplitude to lowest order is

$$\langle p'k' | S | pk\rangle = \langle p'k' | \frac{e^2}{2} \mathcal{T} \int d^4x\, d^4y$$

$$\times\ \bar{\psi}(x) \gamma^\mu A_\mu(x) \psi(x)\ \bar{\psi}(y) \gamma^\nu A_\nu(y) \psi(y) | pk\rangle.$$

The variables x and y play symmetric roles. So if we single out x as the vertex that emits the final electron and let y be the vertex that absorbs the initial electron then we may drop the 2 and simplify the calculation.

We get

$$\langle p'k' | S | pk\rangle = \frac{e^2}{(2\pi)^3} \langle p'k' | \int d^4x\, d^4y$$

$$\times\ \theta(x^0 - y^0)\ \bar{u}(p',r')\, b^\dagger(p',r')\, e^{-ip'\cdot x} \gamma^\mu A_\mu(x) \psi(x))$$

$$\times\ \bar{\psi}(y) \gamma^\nu A_\nu(y)\ u(p,r)\, e^{ipy}\, b(p,r)$$

$$+\ \theta(y^0 - x^0)\ \bar{\psi}(y) \gamma^\nu A_\nu(y)\ u(p,r)\, e^{ipy}\, b(p,r)$$

$$\times\ \bar{u}(p',r')\, b^\dagger(p',r')\, e^{-ip'\cdot x} \gamma^\mu A_\mu(x)\, \psi(x) | pk\rangle.$$

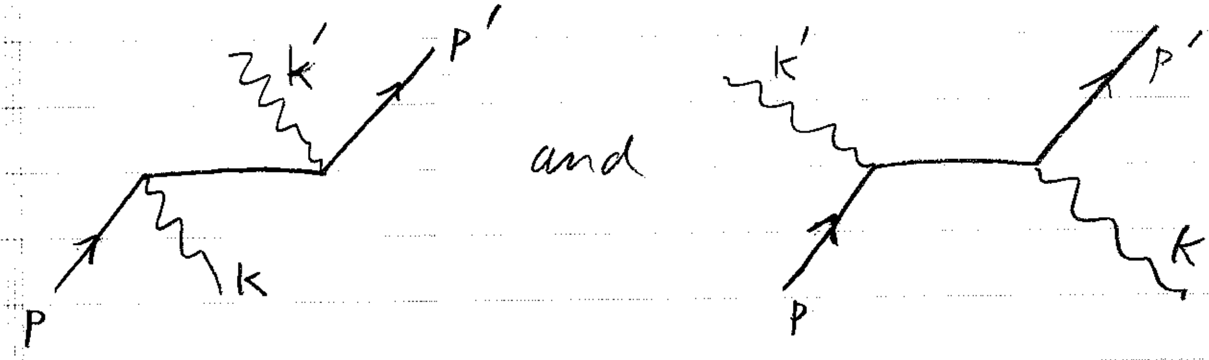
In the $\theta(x^0-y^0)$ term, both b'^{+} and b are adjacent to the final and initial states, respectively. But in the second term, b must cross two fermi operators to reach $|pk\rangle$. This double crossing changes no sign since $(-1)^2=1$. But then b'^{+} must cross $\bar{\psi}(y)$ to reach $\langle p'k'|$. That crossing does introduce a minus sign. So we have

$$\langle p'k'|S|pk\rangle = \frac{e^2}{(2\pi)^3}\int d^4x\, d^4y\, \langle k'|$$
$$\times\ \theta(x^0-y^0)\, \bar{u}_i(p',r')\, e^{-ip'\cdot x}\, \gamma^{\mu}_{i\ell} A_\mu(x)\, \psi_\ell(x)$$
$$\times\ \bar{\psi}_m(y)\, \gamma^{\nu}_{mn} A_\nu(y)\, u_n(p,r)\, e^{ipy}$$
$$-\ \theta(y^0-x^0)\, \bar{\psi}_m(y)\, \gamma^{\nu}_{mn} A_\nu(y)\, u_n(p,r)\, e^{ipy}$$
$$\times\ \bar{u}_i(p',r')\, e^{-ip'\cdot x}\, \gamma^{\mu}_{i\ell} A_\mu(x)\, \psi_\ell(x)\, |k\rangle .$$

The ψ and $\bar{\psi}$ terms now combine to form the electron propagator:

$$\langle p'k'|S|pk\rangle = e^2\int \frac{d^4x\, d^4y}{(2\pi)^3}\, \frac{d^4q}{(2\pi)^4}\, e^{-ip'x+ipy}\, \langle k'|$$
$$\times\ \bar{u}'_i \gamma^{\mu}_{i\ell} A_\mu(x)\, \frac{[-i(-i\gamma^\mu q_\mu + m)]_{\ell m}}{q^2+m^2-i\epsilon}\, e^{iq(x-y)}\, \gamma^{\nu}_{mn} A_\nu(y)\, u_n |k\rangle$$

Either $A_\mu(x)$ or $A_\nu(y)$ can absorb photon k or emit photon k'. So there are two diagrams



They arise respectively from the top and bottom lines of

$$\langle p'k'|S|pk\rangle = \frac{e^2}{(2\pi)^{10}} \int d^4x\, d^4y\, d^4q \; e^{-ip'\cdot x + ip\cdot y + iq(x-y)}$$

$$\times \frac{\bar{u}'\gamma^\mu}{\sqrt{2k^0 2k^{0\prime}}} \begin{pmatrix} \epsilon'^*_\mu \, e^{-ik'\cdot x} \\ \epsilon_\mu \, e^{ik x} \end{pmatrix} \frac{[-i(-i\gamma^\mu q_\mu + m)]}{q^2 + m^2 - i\epsilon} \gamma^\nu \begin{pmatrix} \epsilon_\nu e^{ik\cdot y} \\ \epsilon'^*_\nu e^{-ik'\cdot y} \end{pmatrix} u .$$

We use
$$\int d^4x\, e^{\pm i s\cdot x} = (2\pi)^4 \delta^{(4)}(s)$$

to do the x- and y-integrals. The upper integrals are

$$\int d^4x\, e^{i(-p' + q - k')\cdot x} = (2\pi)^4 \delta^4(q - p' - k')$$

and
$$\int d^4y\, e^{i(p - q + k)\cdot y} = (2\pi)^4 \delta^4(p + k - q) .$$

The lower x- and y-integrals are

$$\int d^4x\, e^{i(-p'+q+k)\cdot x} = (2\pi)^4\, \delta^4(q-p'+k)$$

and

$$\int d^4y\, e^{i(p-q-k')\cdot y} = (2\pi)^4\, \delta^4(p-k'-q).$$

So the upper integrals give

$$(2\pi)^8\, \delta^4(p+k-p'-k')\, \delta^4(p+k-q)$$

and the lower ones

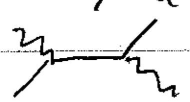
$$(2\pi)^8\, \delta^4(p+k-p'-k')\, \delta^4(p-k'-q).$$

So we find

$$\langle p'k'|S|pk\rangle = \frac{e^2}{(2\pi)^2}\frac{\delta^4(p+k-p'-k')}{\sqrt{2k^0\, 2k'^0}}$$

$$\times\left[\bar{u}'\gamma^\mu \epsilon'^*_\mu \frac{[-i(-i\gamma^\mu(p+k)_\mu + m)]}{(p+k)^2+m^2-i\epsilon}\gamma^\nu \epsilon_\nu u \right.$$

$$\left. +\ \bar{u}'\gamma^\mu \epsilon_\mu \frac{[-i(-i\gamma^\mu(p-k')_\mu + m)]}{(p-k')^2+m^2-i\epsilon}\gamma^\nu \epsilon'^*_\nu u \right].$$

The first term is for  and the second term is for .

Let's use Feynman's slash notation

$$\not{p} = \gamma^\mu p_\mu = \gamma_\mu p^\mu = \sum_{\mu=0}^{3} \gamma_\mu p^\mu .$$

$$\not{\epsilon} = \gamma^\nu \epsilon_\nu$$

$$\not{\epsilon}'^* = \gamma^\nu \epsilon'^*_\nu \qquad \text{etc.}$$

Note also that $p^2 = \vec{p}^{\,2} - p^{0\,2} = -m^2$ and $k^2 = \vec{k}^{\,2} - k^{0\,2} = 0$. So $(p+k)^2 = p^2 + 2p\cdot k + k^2 = 2p\cdot k - m^2$

$$\langle p'k'|S|pk\rangle = \frac{-ie^2}{(2\pi)^2} \frac{\delta^4(p'+k'-p-k)}{\sqrt{2k^0\, 2k^{0\prime}}}$$

$$\times \left[\frac{\bar{u}'\not{\epsilon}'^*[-i(\not{p}+\not{k})+m]\not{\epsilon}u}{2p\cdot k} + \frac{\bar{u}'\not{\epsilon}[-i(\not{p}-\not{k}')+m]\not{\epsilon}'^* u}{-2p\cdot k'} \right]$$

and we've dropped the $-i\epsilon$'s because $p\cdot k$ and $p\cdot k'$ aren't zero.

Let's follow Weinberg and write

$$S \equiv \langle p'k'|S|pk\rangle = -2\pi i\, \delta^4(p'+k'-p-k)\, M$$

so that M is

$$M = \frac{e^2}{4(2\pi)^3\sqrt{k^0 k'^0}}\,\bar{u}'\left\{ \not{\epsilon}'^{*} \frac{[-i(\not{p}+\not{k})+m]}{2p\cdot k}\not{\epsilon} - \not{\epsilon}\frac{[-i(\not{p}-\not{k}')+m]}{2p\cdot k'}\not{\epsilon}'^{*}\right\} u .$$

But our S has fields and states have continuum normalization. To get back to box normalization, we write

$$M_{box} = \left[\frac{(2\pi)^3}{V}\right]^{\frac{N_i+N_f}{2}} M$$

where N_i is the number of incoming particles and N_f is the number of final-state particles. Now

$$P = |S_{box}|^2 = (2\pi)^2 |M_{box}|^2 \left[\delta^4(p'+k'-p-k)\right]^2$$

to which we apply the rule

$$\int d^4x = (2\pi)^4\delta(0) = VT.$$

So

$$P = (2\pi)^2\left(\frac{(2\pi)^3}{V}\right)^4 |M|^2 \frac{VT}{(2\pi)^4}\delta^4(p'+k'-p-k).$$

The rate then is

$$\hat{w} = \frac{(2\pi)^{10}}{V^3}|M|^2\,\delta(p'+k'-p-k)$$

and the full rate is

$$dW = \frac{(2\pi)^{10}}{V^3} |M|^2 \delta(p'+k'-p-k) \left(\frac{V}{(2\pi)^3}\right)^2 d^3p' d^3k'$$

$$= \frac{(2\pi)^4}{V} |M|^2 \delta(p'+k'-p-k)\, d^3p' d^3k'.$$

We now divide by the flux in the rest frame of the electron

$$F = \frac{c}{V} = \frac{1}{V} \quad \text{in natural units.}$$

So

$$d\sigma = (2\pi)^4 |M|^2 \delta(p'+k'-p-k)\, d^3p' d^3k',$$

which is Eq. (8.7.7) of Weinberg in the rest frame of the electron where $u = 1$ as shown by his (8.7.10).

More generally

$$d\sigma = (2\pi)^4 u^{-1} |M|^2 \delta^4(p'+k'-p-k)\, d^3p' d^3k'$$

where the relative velocity of the initial-state particles is

$$u = \frac{\sqrt{(P_1 \cdot P_2)^2 - m_1^2 m_2^2}}{E_1 E_2},$$

Here $p_1 = p$ and $p_2 = k$, so

$$u = \frac{\sqrt{(p \cdot k)^2}}{p^0 k^0} .$$

In the rest frame of the electron, $p = (m, \vec{0})$ and so u is

$$u = \frac{\sqrt{(m k^0)^2}}{m k^0} = 1 .$$

We follow Weinberg now for the spin sums and the traces of the γ-matrices:

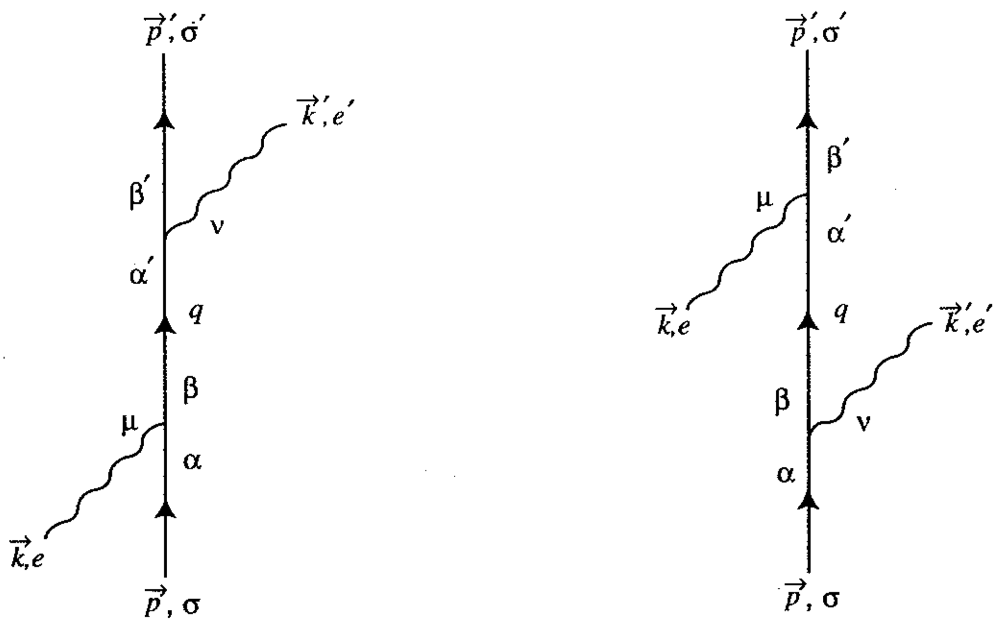


Figure 8.1. The two lowest-order Feynman diagrams for Compton scattering. Straight lines are electrons; wavy lines are photons.

8.7 Compton Scattering

As an example of the methods described in this chapter, we shall consider here the scattering of a photon by an electron (or other particle of spin $\frac{1}{2}$ and charge $-e$), to lowest order in e. We label the initial and final photon momenta and polarization vectors by k^μ, e^μ and k'^μ, e'^μ, where $k^0 = |\mathbf{k}|$ and $k^{0'} = |\mathbf{k}'|$. Also, the initial and final electron momenta and spin z-components are labelled p^μ, σ and p'^μ, σ', where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$ and $p'^0 = \sqrt{\mathbf{p}'^2 + m^2}$, with m the electron mass. The lowest order Feynman diagrams for this process are shown in Figure 8.1. Using the rules outlined in the previous section, the corresponding S-matrix element is

$$\begin{aligned}
&S(\mathbf{p},\sigma \;+\; \mathbf{k},e \;\to\; \mathbf{p}',\sigma' \;+\; \mathbf{k}',e') = \\
&\quad \frac{\bar{u}(\mathbf{p}',\sigma')_{\beta'}}{(2\pi)^{3/2}} \; \frac{e_\nu^{*'}}{(2\pi)^{3/2}\sqrt{2k^{0'}}} \; \frac{u(\mathbf{p},\sigma)_\alpha}{(2\pi)^{3/2}} \; \frac{e_\mu}{(2\pi)^{3/2}\sqrt{2k^0}} \\
&\quad \times \int d^4q \left[\frac{-i}{(2\pi)^4}\right] \left[\frac{-i\, \not{q} + m}{q^2 + m^2 - i\epsilon}\right]_{\alpha'\beta} \\
&\quad \times \Bigg\{ \left[e(2\pi)^4 \gamma^\nu_{\beta'\alpha'} \delta^4(q - p' - k')\right] \left[e(2\pi)^4 \gamma^\mu_{\beta\alpha} \delta^4(q - p - k)\right] \\
&\quad + \left[e(2\pi)^4 \gamma^\mu_{\beta'\alpha'} \delta^4(q + k - p')\right] \left[e(2\pi)^4 \gamma^\nu_{\beta\alpha} \delta^4(q + k' - p)\right] \Bigg\} . \qquad (8.7.1)
\end{aligned}$$

Performing the (trivial) q-integral, collecting factors of i and 2π, and rewriting the result in matrix notation, we have more simply

$$S = \frac{-ie^2\delta^4(p'+k'-p-k)}{(2\pi)^2\sqrt{2k^{0'}\cdot 2k^0}}\bar{u}(\mathbf{p}',\sigma')\left[\not{e}^{*'}\left(\frac{-i(\not{p}+\not{k})+m}{(p+k)^2+m^2}\right)\not{e}\right.$$
$$\left. + \not{e}\left(\frac{-i(\not{p}-\not{k}')+m}{(p-k')^2+m^2}\right)\not{e}^{*'}\right]u(\mathbf{p},\sigma)\,. \tag{8.7.2}$$

(Here $\not{e}^*$ means $e^*_\mu\gamma^\mu$, not $(\not{e})^*$. Also, we drop the $-i\epsilon$, because the denominators here do not vanish.) Because $p^2 = -m^2$ and $k^2 = k'^2 = 0$, the denominators can be simplified

$$(p+k)^2 + m^2 = 2p\cdot k\,, \tag{8.7.3}$$
$$(p-k')^2 + m^2 = -2p\cdot k'\,. \tag{8.7.4}$$

Also, the 'Feynman amplitude' M is defined in general by Eq. (3.3.2), which (because some scattering is assumed to take place) here reads

$$S = -2\pi i\delta^4(p'+k'-p-k)M\,, \tag{8.7.5}$$

so

$$M = \frac{e^2}{4(2\pi)^3\sqrt{k^0k^{0'}}}\bar{u}(\mathbf{p}'\sigma')\left\{\not{e}^{*'}\left[-i(\not{p}+\not{k})+m\right]\not{e}\big/p\cdot k\right.$$
$$\left. -\not{e}\left[-i(\not{p}-\not{k}')+m\right]\not{e}^{*'}\big/p\cdot k'\right\}u(\mathbf{p},\sigma)\,. \tag{8.7.6}$$

The differential cross-section is given in terms of M by Eq. (3.4.15), which here reads

$$d\sigma = (2\pi)^4 u^{-1}|M|^2\delta^4(p'+k'-p-k)d^3p'd^3k'\,. \tag{8.7.7}$$

Since one of the particles here is massless, Eq. (3.4.17) for the initial velocity gives

$$u = |p\cdot k|\big/p^0k^0\,. \tag{8.7.8}$$

To go further, it will be convenient to adopt a specific coordinate frame. Since electrons in atoms move non-relativistically, the laboratory frame for high-energy (X ray or gamma ray) photon–electron scattering experiments is usually (though not always) one in which the initial electron can be taken to be at rest. We will adopt this frame here, so that

$$\mathbf{p} = 0\,, \qquad p^0 = m\,. \tag{8.7.9}$$

The velocity (8.7.8) is then simply

$$u = 1\,. \tag{8.7.10}$$

To save writing, we denote the photon energies by

$$\omega = k^0 = |\mathbf{k}| = -p \cdot k \big/ m\,, \tag{8.7.11}$$

$$\omega' = k^{0'} = |\mathbf{k}'| = -p \cdot k' \big/ m\,. \tag{8.7.12}$$

The three-momentum delta function in Eq. (8.7.7) just serves to eliminate the differential d^3p', setting $\mathbf{p}' = \mathbf{k} - \mathbf{k}'$. This leaves the remaining energy delta function

$$\delta(p'^0 + k'^0 - p^0 - k^0) = \delta\left(\sqrt{(\mathbf{k} - \mathbf{k}')^2 + m^2} + \omega' - m - \omega\right)\,. \tag{8.7.13}$$

This fixes ω' to satisfy

$$\sqrt{\omega^2 - 2\omega\omega' \cos\theta + \omega'^2 + m^2} = \omega + m - \omega'\,,$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. Squaring both sides and cancelling ω'^2 terms gives*

$$\omega' = \omega\,\frac{m}{m + \omega(1 - \cos\theta)} \equiv \omega_c(\theta)\,. \tag{8.7.14}$$

The energy delta function (8.7.13) can be written

$$\begin{aligned}\delta(p'^0 + k'^0 - p^0 - k^0) &= \frac{\delta(\omega' - \omega_c(\theta))}{|\partial[\sqrt{\omega^2 - 2\omega\omega'\cos\theta + \omega'^2 + m^2} + \omega']/\partial\omega'|} \\ &= \frac{\delta(\omega' - \omega_c(\theta))}{|(\omega' - \omega\,\cos\theta)/p'^0 + 1|} \\ &= \frac{p'^0\omega'}{m\omega}\delta(\omega' - \omega_c(\theta))\,. \end{aligned} \tag{8.7.15}$$

Also, the differential d^3k' can be written

$$d^3k' = \omega'^2 d\omega' d\Omega\,, \tag{8.7.16}$$

where $d\Omega$ is the solid angle into which the final photon is scattered. The final delta function in Eq. (8.7.15) just serves to eliminate the differential $d\omega'$ in Eq. (8.7.16), leaving us with a differential cross-section

$$d\sigma = (2\pi)^4 |M|^2\, \frac{p'^0 \omega'^3}{m\omega}\, d\Omega \tag{8.7.17}$$

with $p'^0 = m + \omega - \omega'$, and ω' given by Eq. (8.7.14).

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* Equivalently, there is an increase in wavelength

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1 - \cos\theta}{m}\,.$$

The verification of this formula in the scattering of X rays by electrons by A.H. Compton in 1922–3 played a key role in confirming Einstein's 1905 proposal of a quantum of light, which soon after Compton's experiments came to be known as the photon.

Usually we do not measure the spin z-component of the initial or final electron. In such cases, we must sum over σ' and average over σ, or in other words take half the sum over σ and σ':

$$d\bar{\sigma}(\mathbf{p} + \mathbf{k}, e \to \mathbf{p}' + \mathbf{k}', e') \equiv \tfrac{1}{2}\sum_{\sigma',\sigma} d\sigma(\mathbf{p},\sigma + \mathbf{k}, e \to \mathbf{p}',\sigma' + \mathbf{k}', e')\,. \tag{8.7.18}$$

To calculate this, we use the standard formula

$$\sum_{\sigma} u_\alpha(\mathbf{p},\sigma)\bar{u}_\beta(\mathbf{p},\sigma) = \frac{(-i\,\not{p} + m)_{\alpha\beta}}{2p^0} \tag{8.7.19}$$

and likewise for the sum over σ'. It follows that for an arbitrary 4×4 matrix A

$$\begin{aligned}\sum_{\sigma,\sigma'} |\bar{u}(\mathbf{p}',\sigma')Au(\mathbf{p},\sigma)|^2 &= \sum_{\sigma,\sigma'} (\bar{u}(\mathbf{p}',\sigma')Au(\mathbf{p},\sigma))(\bar{u}(\mathbf{p},\sigma)\beta A^\dagger \beta u(\mathbf{p}',\sigma'))\\ &= \sum_{\sigma,\sigma'} A_{\beta\alpha}u_\alpha(\mathbf{p},\sigma)\bar{u}_\gamma(\mathbf{p},\sigma)(\beta A^\dagger\beta)_{\gamma\delta}u_\delta(\mathbf{p}',\sigma')\bar{u}_\beta(\mathbf{p}',\sigma')\\ &= \mathrm{Tr}\left\{A\left(\frac{-i\,\not{p}+m}{2p^0}\right)\beta A^\dagger\beta\left(\frac{-i\,\not{p}'+m}{2p'^0}\right)\right\}.\end{aligned} \tag{8.7.20}$$

Recalling that $\beta\gamma_\mu^\dagger\beta = -\gamma_\mu$, Eq. (8.7.6) gives now

$$\begin{aligned}\sum_{\sigma,\sigma'}|M|^2 &= \frac{e^4}{64(2\pi)^6\omega\omega' p^0 p'^0}\\ &\times \mathrm{Tr}\left[\left\{\not{e}'^*\frac{[-i(\not{p}+\not{k})+m]}{p\cdot k}\not{e} - \not{e}\frac{[-i(\not{p}-\not{k}')+m]}{p\cdot k'}\not{e}'^*\right\}(-i\,\not{p}+m)\right.\\ &\times\left.\left\{\not{e}^*\frac{[-i(\not{p}+\not{k})+m]}{p\cdot k}\not{e}' - \not{e}'\frac{[-i(\not{p}-\not{k}')+m]}{p\cdot k'}\not{e}^*\right\}(-i\,\not{p}'+m)\right].\end{aligned} \tag{8.7.21}$$

(Recall again that $\not{e}^*$ means $e^*_\mu\gamma^\mu$, not $(e_\mu\gamma^\mu)^*$, and likewise for $\not{e}'^*$.) We work in a 'gauge' in which

$$e\cdot p = e^*\cdot p = e'\cdot p = e'^*\cdot p = 0 \tag{8.7.22}$$

such as for instance Coulomb gauge in the laboratory frame, where $e^0 = e'^0 = 0$ and $\mathbf{p} = 0$. This implies that

$$\begin{aligned}[-i\,\not{p}+m]\,\not{e}[-i\,\not{p}+m] &= \not{e}[i\,\not{p}+m]\,[-i\,\not{p}+m]\\ &= \not{e}(\not{p}^2+m^2) = \not{e}(p_\mu p^\mu + m^2) = 0\end{aligned}$$

and likewise for $\not{e}'^*$, $\not{e}'$, and $\not{e}^*$. Eq. (8.7.21) can therefore be written in the greatly simplified form

$$\sum_{\sigma,\sigma'} |M|^2 = \frac{-e^4}{64(2\pi)^6 \omega\omega' p^0 p'^0} \mathrm{Tr}\left[\left\{\frac{\not{e}'^* \not{k} \not{e}}{p\cdot k} + \frac{\not{e} \not{k}' \not{e}'^*}{p\cdot k'}\right\}(-i\not{p} + m)\right.$$
$$\left. \times \left\{\frac{\not{e}^* \not{k} \not{e}'}{p\cdot k} + \frac{\not{e}' \not{k}' \not{e}^*}{p\cdot k'}\right\}(-i\not{p}' + m)\right] . \qquad (8.7.23)$$

The trace of any product of an odd number of gamma matrices vanishes, so this breaks up into terms of zeroth and second order in m:

$$\sum_{\sigma,\sigma'} |M|^2 = \frac{e^4}{64(2\pi)^6 \omega\omega' p^0 p'^0} \left(\frac{T_1}{(p\cdot k)^2} + \frac{T_2}{(p\cdot k)(p\cdot k')} + \frac{T_3}{(p\cdot k)(p\cdot k')}\right.$$
$$\left. + \frac{T_4}{(p\cdot k')^2} - \frac{m^2 t_1}{(p\cdot k)^2} - \frac{m^2 t_2}{(p\cdot k)(p\cdot k')} - \frac{m^2 t_3}{(p\cdot k)(p\cdot k')} - \frac{m^2 t_4}{(p\cdot k')^2}\right) \qquad (8.7.24)$$

where

$$T_1 = \mathrm{Tr}\left\{\not{e}'^* \not{k} \not{e} \not{p} \not{e}^* \not{k} \not{e}' \not{p}'\right\} , \qquad (8.7.25)$$

$$T_2 = \mathrm{Tr}\left\{\not{e}'^* \not{k} \not{e} \not{p} \not{e}' \not{k}' \not{e}^* \not{p}'\right\} , \qquad (8.7.26)$$

$$T_3 = \mathrm{Tr}\left\{\not{e} \not{k}' \not{e}'^* \not{p} \not{e}^* \not{k} \not{e}' \not{p}'\right\} , \qquad (8.7.27)$$

$$T_4 = \mathrm{Tr}\left\{\not{e} \not{k}' \not{e}'^* \not{p} \not{e}' \not{k}' \not{e}^* \not{p}'\right\} , \qquad (8.7.28)$$

$$t_1 = \mathrm{Tr}\left\{\not{e}'^* \not{k} \not{e} \not{e}^* \not{k} \not{e}'\right\} , \qquad (8.7.29)$$

$$t_2 = \mathrm{Tr}\left\{\not{e}'^* \not{k} \not{e} \not{e}' \not{k}' \not{e}^*\right\} , \qquad (8.7.30)$$

$$t_3 = \mathrm{Tr}\left\{\not{e} \not{k}' \not{e}'^* \not{e}^* \not{k} \not{e}'\right\} , \qquad (8.7.31)$$

$$t_4 = \mathrm{Tr}\left\{\not{e} \not{k}' \not{e}'^* \not{e}' \not{k}' \not{e}^*\right\} . \qquad (8.7.32)$$

The Appendix to this chapter shows how to calculate any trace $\mathrm{Tr}\{\not{a}\,\not{b}\,\not{c}\,\not{d}\ldots\}$ as a sum of products of scalar products of the four-vectors $a, b, c, d, \ldots$. In general, traces of products of 6 or 8 gamma matrices like the t_k or T_k would be given by a sum of 15 or 105 terms, respectively, but fortunately here most scalar products vanish; in addition to Eq. (8.7.22), we also have $k\cdot k = k'\cdot k' = 0$. (Furthermore, $e\cdot e^* = e'\cdot e'^* = 1$.) To simplify the calculation further, let us specialize to the case of *linear* polarization, where e^μ and e'^μ are real. Dropping the asterisks in Eqs. (8.7.25)–(8.7.32), we have then

$$T_1 = \mathrm{Tr}\left\{\not{e}' \not{k} \not{e} \not{p} \not{e} \not{k} \not{e}' \not{p}'\right\} .$$

Since $e^\mu p_\mu = 0$ and $e^\mu e_\mu = 1$, we have

$$\not{e}\,\not{p}\,\not{e} = -\not{p}\,\not{e}\,\not{e} = -\not{p}$$

so

$$T_1 = -\mathrm{Tr}\left\{\not{e}'\,\not{k}\,\not{p}\,\not{k}\,\not{e}'\,\not{p}'\right\}.$$

Also, $k^\mu k_\mu = 0$, so

$$\not{k}\,\not{p}\,\not{k} = -\not{k}\,\not{k}\,\not{p} + 2\,\not{k}p\cdot k = 2\,\not{k}p\cdot k$$

and hence

$$T_1 = -2p\cdot k\,\mathrm{Tr}\left\{\not{e}'\,\not{k}\,\not{e}'\,\not{p}'\right\}.$$

Using Eq. (8.A.6), this is

$$T_1 = -8p\cdot k\,[2e'\cdot k\;e'\cdot p' - k\cdot p'].$$

It is convenient to make the substitutions

$$e'\cdot p' = e'\cdot[p+k-k'] = e'\cdot k$$
$$k\cdot p' = -\tfrac{1}{2}(p'-k)^2 - \tfrac{1}{2}m^2 = -\tfrac{1}{2}(p-k')^2 - \tfrac{1}{2}m^2 = p\cdot k'$$

so

$$T_1 = -16\,p\cdot k(e'\cdot k)^2 + 8\,p\cdot k\,p\cdot k'. \tag{8.7.33}$$

A similar (though more lengthy) calculation gives

$$\begin{aligned}
T_2 = T_3 = &-8(e\cdot k')^2(p\cdot k) + 16\,(e\cdot e')^2 p\cdot k' p\cdot k + 8(e\cdot e')^2 k\cdot k' m^2 \\
&-8(e\cdot e')m^2(k\cdot e')(k'\cdot e) + 8(e'\cdot k)^2 p\cdot k' \\
&-4(k\cdot p)^2 + 4(k\cdot k')(p\cdot p') - 4(k\cdot p')(p\cdot k'),
\end{aligned} \tag{8.7.34}$$

$$T_4 = 16\,p\cdot k'(e\cdot k')^2 + 8(p\cdot k)(p\cdot k'), \tag{8.7.35}$$

$$t_1 = t_4 = 0, \tag{8.7.36}$$

$$t_2 = t_3 = -8\,e\cdot e'\,k\cdot e'\,k'\cdot e + 8(k\cdot k')(e\cdot e')^2 - 4(k\cdot k'). \tag{8.7.37}$$

Combining all these terms in Eq. (8.7.24) gives

$$\sum_{\sigma,\sigma'}|M|^2 = \frac{e^4}{64(2\pi)^6\omega\omega' p^0 p'^0}\left[\frac{8(k\cdot k')^2}{(k\cdot p)(k'\cdot p)} + 32(e\cdot e')^2\right]. \tag{8.7.38}$$

All this applies in any Lorentz frame. In the *laboratory* frame, we have the special results

$$k\cdot k' = \omega\omega'(cos\theta - 1) = m\omega\omega'\left(\frac{1}{\omega} - \frac{1}{\omega'}\right),$$
$$p\cdot k = -m\omega \qquad p\cdot k' = -m\omega'.$$

Combining Eq. (8.7.38) with Eq. (8.7.17), the laboratory frame cross-section is

$$\tfrac{1}{2}\sum_{\sigma,\sigma'} d\sigma(\mathbf{p},\sigma + \mathbf{k},e \to \mathbf{p}',\sigma' + \mathbf{k}',e') = \frac{e^4\omega'^2 d\Omega}{64\pi^2 m^2\omega^2} \times \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2 + 4(e\cdot e')^2\right]. \tag{8.7.39}$$

This is the celebrated formula derived (using old-fashioned perturbation theory) by O. Klein and Y. Nishina[4] in 1929.

As discussed in Section 8.6, if the incoming photon is (as usual) not prepared in a state with any particular polarization, then we must average over two orthonormal values of $\mathbf{e}$. This average gives

$$\tfrac{1}{2}\sum_e e_i e_j = \tfrac{1}{2}(\delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j)$$

and the differential cross-section is then

$$\tfrac{1}{4}\sum_{e,\sigma,\sigma'} d\sigma(\mathbf{p},\sigma + \mathbf{k},e \to \mathbf{p}',\sigma' + \mathbf{k},e') = \frac{e^4\omega'^2 d\Omega}{64\pi^2 m^2\omega^2}\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2(\hat{\mathbf{k}}\cdot e')^2\right]. \tag{8.7.40}$$

We see that the scattered photon is preferentially polarized in a direction perpendicular to the incident as well as the final photon direction, i.e., perpendicular to the plane in which the scattering takes place. This is a well-known result, responsible among other things for the polarization of light from eclipsing binary stars.**

To calculate the cross-section for experiments in which the final photon polarization is not measured, we must sum Eq. (8.7.40) over e', using

$$\sum_{e'} e'_i e'_j = \delta_{ij} - \hat{\mathbf{k}}'_i\hat{\mathbf{k}}'_j\,.$$

This gives

$$\tfrac{1}{4}\sum_{e,e',\sigma,\sigma'} d\sigma(\mathbf{p},\sigma + \mathbf{k},e \to \mathbf{p}',\sigma' + \mathbf{k}',e') = \frac{e^4\omega'^2 d\Omega}{32\pi^2 m^2\omega^2}\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 1 + \cos^2\theta\right], \tag{8.7.41}$$

where θ is the angle between $\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}'$. In the non-relativistic case, $\omega << m$,

** The light from one of the stars is polarized when it is scattered by free electrons in the outer atmosphere of the other, cooler, star when both are along the same line of sight. This polarization is normally undetectable because it cancels when the astronomer adds up light from all parts of the star's disk. The polarization has been observed in eclipsing binary stars at times when the cooler star blocks the light from just one side of the hotter star.

Eq. (8.7.41) gives

$$\tfrac{1}{4}\sum_{e,e',\sigma,\sigma'} d\sigma = \frac{e^4\,d\Omega}{32\pi^2 m^2}(1+\cos^2\theta)\,. \tag{8.7.42}$$

The solid angle integral is

$$\int [1+\cos^2\theta]d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} [1+\cos^2\theta]\ \sin\theta\ d\theta = \frac{16\pi}{3}\,,$$

giving a total cross-section for $\omega \ll m$:

$$\sigma_{\mathrm{T}} = \frac{e^4}{6\pi m^2}\,. \tag{8.7.43}$$

This is often written $\sigma_{\mathrm{T}} = 8\pi r_0^2/3$, where $r_0 = e^2/4\pi m = 2.818 \times 10^{-13}$ cm is known as the *classical electron radius*. Expression (8.7.43) is called the *Thomson cross-section*, after J. J. Thomson, the discoverer of the electron. Eqs. (8.7.42) and (8.7.43) were originally derived using classical mechanics and electrodynamics, by calculating the reradiation of light by a non-relativistic point charge in a plane wave electromagnetic field.

8.8 Generalization : p-form Gauge Fields*

The antisymmetric field strength tensor $F_{\mu\nu}$ of electromagnetism is a special case of a general class of tensors of special importance in physics and mathematics. A *p-form* is an antisymmetric covariant tensor of rank p. From a p-form $t_{\mu_1,\mu_2,\cdots\mu_p}$ one may construct a $(p+1)$-form called the *exterior derivative*** dt by taking the derivative and then antisymmetrizing with respect to all indices:

$$\begin{aligned}(dt)_{\mu_1\mu_2\cdots\mu_{p+1}} &\equiv \partial_{[\mu_1} t_{\mu_2\mu_3\cdots\mu_{p+1}]}\\ &\equiv \partial_{\mu_1} t_{\mu_2\mu_3\cdots\mu_{p+1}} - \partial_{\mu_2} t_{\mu_1\mu_3\cdots\mu_{p+1}} + \cdots + (-1)^p \partial_{\mu_{p+1}} t_{\mu_1\mu_2\cdots\mu_p}\end{aligned} \tag{8.8.1}$$

with square brackets indicating antisymmetrization with respect to the indices within the brackets. Because derivatives commute, repeated exterior derivatives vanish

$$d(dt) = 0\,. \tag{8.8.2}$$

A p-form whose exterior derivative vanishes is called *closed*, while a p-form that is itself an exterior derivative is called *exact*. From Eq. (8.8.2)

* This section lies somewhat out of the book's main line of development, and may be omitted in a first reading.

** Exterior derivatives and p-forms play a special role in general relativity, in part because the exterior derivative of a tensor transforms like a tensor even though it is calculated using ordinary rather than covariant derivatives.[5]