

The Two-Body Problem

The hamiltonian for two non-relativistic particles of masses m_1 and m_2 interacting thru a potential

$$V = V(\vec{r}_1 - \vec{r}_2) \quad (1)$$

is

$$H = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + V(\vec{r}_1 - \vec{r}_2). \quad (2)$$

The total momentum \vec{P} is

$$\vec{P} = \vec{p}_1 + \vec{p}_2. \quad (3)$$

\vec{P} generates translations and commutes with H

$$[\vec{P}, H] = 0 \quad (4)$$

as we'll see.

The center of mass R is

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \quad (5)$$

and the separation \vec{r} is

$$\vec{r} = \vec{r}_1 - \vec{r}_2. \quad (6)$$

The relative momentum \vec{p} is

$$\vec{p} = \frac{m_2 \vec{p}_1 - m_1 \vec{p}_2}{m_1 + m_2} \quad (7)$$

Its summing form ensures that it is canonically conjugate to the separation \vec{r}

$$\begin{aligned} [r_i, p_j] &= [r_{1i} - r_{2i}, m_2 p_{1j} - m_1 p_{2j}] / (m_1 + m_2) \\ &= (i\hbar m_2 + i\hbar m_1) \delta_{ij} / (m_1 + m_2) \\ &= i\hbar \delta_{ij} \end{aligned} \quad (8)$$

and that it commutes with the center of mass \vec{R}

$$\begin{aligned} [R_i, p_j] &= [m_1 r_{1i} + m_2 r_{2i}, m_2 p_{1j} - m_1 p_{2j}] / (m_1 + m_2)^2 \\ &= i\hbar \delta_{ij} (m_1 m_2 - m_1 m_2) / (m_1 + m_2)^2 \\ &= 0. \end{aligned} \quad (9)$$

The total momentum \vec{P} and the center of mass \vec{R} also are canonically conjugate

$$\begin{aligned} [R_i, P_j] &= [m_1 r_{1i} + m_2 r_{2i}, p_{1j} + p_{2j}] / (m_1 + m_2) \\ &= i\hbar \delta_{ij}. \end{aligned} \quad (10)$$

Finally, the separation \vec{E} commutes with the total momentum \vec{P}

$$\begin{aligned} [v_i, P_j] &= [v_i - r_{zi}, p_{ij} + p_{zj}] \\ &= i\hbar \delta_{ij} (1-1) = 0 \end{aligned} \quad (11)$$

which is why $[H, \vec{P}] = 0$ as in (4).

One may show that these definitions imply that the energy operator H is

$$H = \frac{\vec{P}^2}{2M} + \frac{\vec{p}^2}{2\mu} + V(\vec{r}) \quad (12)$$

in which

$$M = m_1 + m_2 \quad (13)$$

is the total mass, and

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad (14)$$

is the reduced mass. For two particles of equal mass $m_1 = m_2 = m$, the reduced mass μ is $m/2$

$$\mu = \frac{m^2}{2m} = \frac{m}{2} \quad (15)$$

Since the center of mass \vec{R} does not appear in H , it follows from (11) and from

$$[P_i, P_j] = 0 \tag{16}$$

that

$$[\vec{P}_i, H] = 0 \tag{17}$$

which is (4) again. But

$$i\hbar \dot{R}_i = [R_i, H] = \frac{1}{2M} [R_i, \vec{P}^2] = i\hbar \frac{P_i}{M} \tag{18}$$

and so the time evolution of the center of mass is

$$R_i(t) = R_i(0) + \frac{P_i(0)}{M} t \tag{19}$$

in which $P_i(t) = P_i(0)$ in view of (17).

The total angular momentum \vec{L} is

$$\vec{L} = \vec{L}_1 + \vec{L}_2 = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 \tag{20}$$

$$= \vec{R} \times \vec{P} + \vec{r} \times \vec{p} \tag{21}$$

as you may show as a homework problem.