Virial Theorem

$V(x,y,z)$ is homogeneous of degree n in $x\ y\ z$ if

$$V(\alpha x, \alpha y, \alpha z) = \alpha^n V(x,y,z).$$

It follows that

$$\left.\frac{d}{d\alpha} V(\alpha x, \alpha y, \alpha z)\right|_{\alpha=1} = x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = nV$$

that is

$$\vec{r}\cdot\vec{\nabla} V = nV,$$

Now

$$[H, r\cdot p] = \left[\frac{\vec{p}^2}{2m} + V(\vec{r}), r\cdot p\right]$$

$$= \sum_{i=1}^{3}\left[\frac{\vec{p}^2}{2m}, r_i\right]p_i + \sum_{i=1}^{3} r_i\,[V(\vec{r}), p_i]$$

$$= \sum_{i=1}^{3}\left(-\frac{i\hbar}{2m}\frac{\partial p^2}{\partial p_i}\,p_i + i\hbar r_i\frac{\partial V(\vec{r})}{\partial r_i}\right)$$

$$= -\frac{i\hbar}{m}\vec{p}^2 + i\hbar\,\vec{r}\cdot\vec{\nabla} V(\vec{r}),$$

Now take the mean value of $[H, r\cdot p]$

in an e-state of H

$$H|E\rangle = E|E\rangle.$$

We find

$$\langle E|[H, \vec{r}\cdot\vec{p}]|E\rangle = \langle E|H r\cdot p - r\cdot p H|E\rangle$$

$$= \langle E|E r\cdot p - r\cdot p E|E\rangle = 0$$

Thus

$$\langle E|\frac{\vec{p}^{\,2}}{m}|E\rangle = \langle E|\vec{r}\cdot\vec{\nabla}V(\vec{r})|E\rangle.$$

So if $V(\vec{r})$ is homogeneous of degree n, then

$$\langle E|\frac{\vec{p}^{\,2}}{2m}|E\rangle = \frac{n}{2}\langle E|V|E\rangle$$

or the mean value $\bar{K}$ of the kinetic energy is $\frac{n}{2}$ times the mean value $\bar{V}$ of the potential energy in any stationary state

$$\bar{K} = \frac{n}{2}\bar{V}.$$

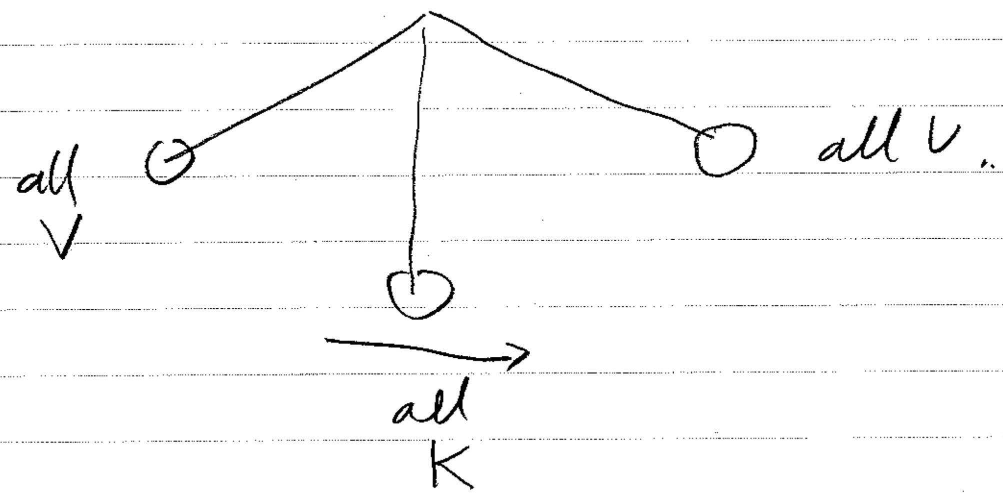
For the harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \quad \text{and} \quad V(\alpha x) = \alpha^2 V(x)$$

so $n = 2$ and

$$\overline{K} = \overline{V}$$

which makes sense since



For the hydrogen atom

$$V = -\frac{e^2}{r} = -\frac{e^2}{\sqrt{x^2+y^2+z^2}} \qquad \text{so}$$

$$V(\alpha x, \alpha y, \alpha z) = \alpha^{-1} V(x,y,z) \quad \text{and } n = -1.$$

So for the H-atom

$$\overline{K} = -\tfrac{1}{2}\overline{V}$$

in any e-state of H.