Spontaneous Emission

Let's compute the lifetime of the 2p state of atomic hydrogen. The initial state is

$$|i\rangle = |2,1,m\rangle \tag{1}$$

and the final state is the ground state 1s

$$|f;k\rangle = |1,0,0,k,r\rangle \tag{2}$$

and a single photon of wave number k and polarization r.

As before

$$\langle f,k,r|i,t\rangle_I = -\frac{i}{\hbar}\int_0^t \langle 100\, k,r|e^{iH_0t'/\hbar}\; -\frac{e}{mc}\, p\cdot A(\vec{x},0)\, e^{-iH_0t'/\hbar}|21m\rangle\, dt' \tag{3}$$

in which

$$H_0 = H_0^{AT} + H_0^{EM}. \tag{4}$$

The field $A(x,0)$ is

$$\vec{A}(x,0) = \sum_{kr}\left(\frac{hc^2}{V\omega_k}\right)^{1/2}\left[e_r(k)\,a_r(k)\,e^{i\vec{k}\cdot\vec{x}} + e_r^*(k)\,a_r^\dagger(k)\,e^{-i\vec{k}\cdot\vec{x}}\right]. \tag{5}$$

So

$$\langle f,k,r|i,t\rangle_I = \frac{ie}{\hbar mc}\int_0^t e^{i(E_1+\hbar\omega-E_2)t'/\hbar}\sum_{kr}\langle 100kr|\,\vec{p}\cdot\vec{e}_r^{\,*}\,a_r^\dagger(k)\,e^{-i\vec{k}\cdot\vec{x}}|21m\rangle\, dt'\left(\frac{hc^2}{V\omega}\right)^{\frac{1}{2}}. \tag{6}$$

Now

$$\langle kr | a^{\dagger}_{r'}(k') | 0 \rangle = \sqrt{0+1}\, \delta_{rr'}\, \delta_{\vec{k}, \vec{k}'} \qquad (7)$$

so

$$\langle f k r | i, t \rangle_I = \frac{e}{\hbar m c} \left(\frac{h c^2}{V \omega} \right)^{\frac{1}{2}} \langle 100 | p \cdot e^* e^{-i\vec{k}_1 \cdot \vec{x}} | 21m \rangle$$

$$\times \left(\frac{e^{i(\omega_{12}+\omega)t} - 1}{\omega_{12}+\omega} \right) \qquad 8$$

So the transition rate is

$$\hat{w}_{i \to f} = \frac{2\pi}{\hbar} \left(\frac{e}{mc} \right)^2 \frac{h c^2}{\omega V} \left| \langle 100 | p \cdot e^* e^{-ik_1 \cdot x} | 21m \rangle \right|^2$$

$$\delta(E_1 - E_2 + \hbar\omega). \qquad (9)$$

Now since

$$|k \cdot x| \sim \frac{2\pi a_0}{\lambda} \approx \frac{6 \cdot \frac{1}{20}}{300} \approx 10^{-3} \qquad (10)$$

the dipole approximation is excellent

$$\langle 100 | p \cdot e^* e^{-ik_1 \cdot x} | 21m \rangle \approx \langle 100 | p \cdot e^* | 21m \rangle . \qquad 11$$

And as before

$$\langle 100 | \vec{p} | 21m \rangle = \frac{m}{i\hbar} \langle 100 | [\vec{x}, H_0^{AT}] | 21m \rangle$$

$$= i m \omega_{12} \langle 100 | \vec{x} | 21m \rangle . \qquad 12$$

Since the ground state $|100\rangle$ is spherically symmetric, we may choose any value of m. Let's take $m=0$. Then only the z component of $\vec{x}$ survives

$$\langle 100 | x_i | 210\rangle = \delta_{i3} \langle 100 | z | 210\rangle . \qquad 13$$

This matrix element is worked out in special problem 8.2:

$$\langle 100 | z | 210\rangle = \frac{2^7\sqrt{2}}{3^5} a_0 \qquad 14$$

where $a_0 = \hbar^2/me^2$ is the Bohr radius, about half an Ångstrom.

So

$$\hat{w}_{i\to f} = \frac{2\pi}{\hbar}\left(\frac{e}{mc}\right)^2 \frac{hc^2}{\omega V} \left| m\, \omega_{12}\, e^*(\vec{k},r)_3 \langle 100 | z | 210\rangle \right|^2 \times \delta(\hbar\omega_{12} - \hbar\omega) \qquad 15$$

$$= \frac{2\pi}{\hbar}\left(\frac{e}{mc}\right)^2 \frac{hc^2}{\omega V} m^2 \omega_{12}^2 \frac{2^{15}}{3^{10}} a_0^2 \left| e(k,r)_3 \right|^2 \delta(\hbar(\omega_{12}-\omega)) \qquad 16$$

$$= 4\pi^2 \frac{e^2}{V} \omega \frac{2^{15}}{3^{10}} a_0^2 \left| e(k,r)_3 \right|^2 \delta(\hbar(\omega_{12}-\omega)). \qquad 17$$

Now the polarization vectors $e(k,r)$ for $r=1,2$ span the subspace perpendicular to $\vec{k}$ so

$$\sum_{r=1}^{2} e(k,r)\, e^{+}(k,r) = 1 - \hat{k}\,\hat{k}^{T} \qquad 18$$

and so

$$\sum_{r=1}^{2} |e(k,r)_3|^2 = \left(\sum_{r=1}^{2} e(k,r) e^{+}(k,r)\right)_{33} \qquad 19$$

$$= \delta_{33} - \hat{k}_3 \hat{k}_3 = 1 - \hat{k}_3^2 , \qquad 20$$

So if we use coordinates in which $\hat{k}$ has polar angles θ and ϕ, then

$$\sum_{r=1}^{2} |e(k,r)_3|^2 = 1 - \cos^2\theta . \qquad 21$$

Now we must integrate over final states, which amounts to integrating over final photon momenta and summing over photon polarizations:

$$\vec{k} = \frac{2\pi}{L}\vec{n} \qquad 22$$

so

$$\vec{n} = \frac{L}{2\pi}\vec{k} \qquad 23$$

so

$$\sum_{\vec{n}} = \int \left(\frac{L}{2\pi}\right)^3 d^3k = V \int \frac{d^3k}{(2\pi)^3} \qquad 24$$

The total rate w then is

$$w = \frac{2^{17}}{3^{10}} \pi^2 e^2 \int \frac{d^3k}{(2\pi)^3} a_0^2 \, \omega \sum_{r=1}^{2} |e(k,\lambda)_3|^2 \, \delta(\hbar(\omega_{12}-\omega)) \quad 25$$

$$= \pi^2 \frac{2^{17}}{3^{10}} e^2 a_0^2 \; 2\pi \int_{-1}^{1} \frac{d\cos\theta}{(2\pi)^3} \int_0^\infty dk\,\omega(1-\cos^2\theta)k^2 \, \delta(\hbar c(k_{12}-k)) \quad 26$$

$$= \frac{\pi^3}{3^{10}} \frac{2^{18}\, e^2 a_0^2}{(2\pi)^3} \int_{-1}^{1} dx\,(1-x^2) \; \frac{\omega_{21}}{\hbar c} \left(\frac{\omega_{21}}{c}\right)^2 \quad 27$$

$$= \frac{2^{15}}{3^{10}} e^2 a_0^2 \left[x - \frac{x^3}{3}\right]_{-1}^{1} \frac{\omega_{21}^3}{\hbar c^3} \quad 28$$

$$= \frac{2^{15}}{3^{10}} e^2 a_0^2 \left(2 - \frac{2}{3}\right) \frac{\omega_{21}^3}{\hbar c^3} \quad 29$$

$$= \frac{2^{17}}{3^{11}} e^2 a_0^2 \frac{\omega_{21}^3}{\hbar c^3} . \quad 30$$

Now $\hbar\omega_{21} = \frac{1}{2} m c^2 \alpha^2 \left(\frac{1}{1^2} - \frac{1}{2^2}\right)$

$$= \frac{3}{8} m c^2 \alpha^2 \quad 31$$

So since $a_0 = \hbar^2 / m e^2$,

$$W = \frac{2^{17}}{3^{11}} \left(\frac{3}{8}\right)^3 \left(\frac{mc^2\alpha^2}{\hbar}\right)^3 \frac{e^2 a_0^2}{\hbar c^3} \tag{32}$$

$$= \frac{2^8}{3^8} \left(\frac{mc^2\alpha^2}{\hbar}\right)^3 \frac{e^2}{\hbar c^3} \left(\frac{\hbar^2}{me^2}\right)^2 \tag{33}$$

$$= \left(\frac{2}{3}\right)^8 \frac{m c^3 \alpha^6}{\hbar^4} \frac{e^2 \hbar^4}{e^4} = \left(\frac{2}{3}\right)^8 m \alpha^6 \frac{c^3}{e^2} \tag{34}$$

$$= \left(\frac{2}{3}\right)^8 \frac{mc^2 \alpha^6}{\hbar} \frac{\hbar c}{e^2} = \left(\frac{2}{3}\right)^8 \frac{mc^2}{\hbar} \alpha^5 . \tag{35}$$

The lifetime τ is W^{-1}

$$\tau = \frac{1}{W} = \left(\frac{3}{2}\right)^8 \frac{\hbar}{mc^2} \alpha^{-5} . \tag{36}$$

Numerically

$$\left.\begin{aligned} mc^2 &= 0.511 \text{ MeV} \\ \hbar &= 6.582 \times 10^{-22} \text{ MeV s} \\ \alpha &= 1/137.036 \end{aligned}\right\} \tag{37}$$

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$$\tau = 25.63 \cdot \frac{6.582 \times 10^{-22} \text{ MeV s}}{0.511 \text{ MeV}} (137.036)^5 = 1.60 \times 10^{-9} \text{ s} \tag{38}$$

is the life-time of the 2p state of H.

The $2p \to 1s$ transition is said to be an "allowed" transition because it proceeds thru the 1 part of

$$e^{-ik\cdot x} = 1 - ik\cdot x + \frac{(-ik\cdot x)^2}{2!} + \cdots \qquad 39$$

But the $2s \to 1s$ transition must go thru a higher term in this expansion. It is therefore suppressed by (at least) a factor of

$$|k\cdot x|^2 \sim \left(\frac{\omega}{c} a_0\right)^2$$

$$= \left(\frac{\hbar\omega}{\hbar c} \frac{\hbar^2}{me^2}\right)^2 = \left(\frac{\hbar\omega}{mc^2} \frac{\hbar c}{e^2}\right)^2$$

$$= \left(\frac{1}{2}\frac{3}{4} \frac{mc^2\alpha^2}{mc^2} \frac{1}{\alpha}\right)^2 = \left(\frac{3}{8}\alpha\right)^2$$

$$= \left(\frac{3}{8\cdot 137}\right)^2 = 7.5\times 10^{-6}$$

since $a_0 = \hbar^2/me^2$ and $\alpha = e^2/\hbar c = 1/137$. Transitions that proceed via this $k\cdot x$ term are called "magnetic-dipole" or "electric-quadrupole" transitions, depending upon whether the anti-symmetric or symmetric part of $e\cdot p\, k\cdot x$ is effective.