Supersymmetric Quantum Mechanics

First, let's go one step beyond the simple harmonic oscillator

$$H_0 = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right) \quad \text{where}$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right).$$

We may replace x by $f(x)$ so that

$$H = \hbar\omega\left(b^\dagger b + c\right) \text{ where now}$$

$$b = \sqrt{\frac{m\omega}{2\hbar}}\left(f(x) + \frac{ip}{m\omega}\right).$$

Now

$$H = \hbar\omega\left[\left(\frac{m\omega}{2\hbar}\right)\left(f(x) - \frac{ip}{m\omega}\right)\left(f(x) + \frac{ip}{m\omega}\right) + c\right]$$

$$= \hbar\omega\left[\left(\frac{m\omega}{2\hbar}\right)\left(f^2(x) + \frac{p^2}{m^2\omega^2} + \frac{i}{m\omega}[f(x), p]\right) + c\right]$$

$$= \hbar\omega\left[\left(\frac{m\omega}{2\hbar}\right)\left(f^2(x) + \frac{p^2}{m^2\omega^2} - \frac{\hbar}{m\omega} f'(x)\right) + c\right]$$

$$= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 f^2(x) - \frac{\hbar\omega}{2} f'(x) + \hbar\omega c.$$

Once again, the ground state $|0\rangle$ satisfies

$$\langle x | b | 0\rangle = \sqrt{\frac{m\omega}{2\hbar}} \langle x | f(x) + \frac{ip}{m\omega} | 0\rangle = 0$$

or

$$f(x) \langle x|0\rangle = -\frac{i}{m\omega} \frac{\hbar}{i} \frac{d}{dx} \langle x|0\rangle$$ or

(or with prime meaning d/dx)

$$\frac{\langle x|0\rangle'}{\langle x|0\rangle} = -\frac{m\omega}{\hbar} f(x)$$

$$\log \langle x|0\rangle = -\frac{m\omega}{\hbar} \int_0^x dx' \, f(x') + c$$

$$\langle x|0\rangle = N \exp\left[-\frac{m\omega}{\hbar} \int_0^x dx' \, f(x')\right].$$

This solution makes sense if $f(x) \to +\infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. For example, if $f(x) = dx^3$, with $d > 0$, then

$$d \int_0^x dx' \, x'^3 = d \frac{x^4}{4}$$ and so

$$\langle x|0\rangle = N \exp\left(-\frac{m\omega d x^4}{4\hbar}\right)$$

is the normalizable ground state with energy

$$H|0\rangle = \hbar\omega(b^\dagger b + c)|0\rangle$$

$$= \frac{\hbar\omega c}{2}|0\rangle.$$

So the ground-state energy depends upon the value of c.

But if $f(x) = dx^3$ with $d<0$ or if $f(x)$ is an even function of x, then $b|0\rangle = 0$ will have no normalizable solution, and the ground state must be found by solving the second-order differential equation $\langle x|H|0\rangle = E_0 \langle x|0\rangle$.

Now supersymmetric (susy) quantum mechanics as invented by Edward Witten. Think of a spin-1/2 particle of mass m on a line. Consider the "charges"

$$Q_1 = \frac{1}{2}\left(\frac{\sigma_1 P}{\sqrt{m}} + \sigma_2 \sqrt{m}\, W(x)\right)$$

$$Q_2 = \frac{1}{2}\left(\frac{\sigma_2 P}{\sqrt{m}} - \sigma_1 \sqrt{m}\, W(x)\right).$$

Note that both are hermitian

$$Q_1^\dagger = Q_1 \quad \text{and} \quad Q_2^\dagger = Q_2.$$

Algebra plays a big role in susy.

Note that

$$Q_1^2 = \frac{1}{4}\,\frac{\sigma_1^2 p^3}{m} + \frac{1}{4}\,\sigma_2^2 m W^2(x)$$

$$+ \frac{1}{2}\,\sigma_1 p \sigma_2 W(x) + \frac{1}{2}\,\sigma_2 W(x) \sigma_1 p.$$

Now

$$\sigma_i \sigma_j = \delta_{ij} + \sum_{k=1}^{3} i \epsilon_{ijk} \sigma_k$$

So

$$\sigma_1 \sigma_2 = i\sigma_3 \quad \text{and} \quad \sigma_2 \sigma_1 = -i\sigma_3$$

while

$$\sigma_1^2 = \sigma_2^2 = 1.$$

So

$$Q_1^2 = \frac{1}{4}\,\frac{p^2}{m} + \frac{1}{4}\,m W^2(x) + \frac{i\sigma_3}{2}\,p W(x) - \frac{i\sigma_3}{2}\,W(x) p$$

$$= \frac{1}{4}\,\frac{p^2}{m} + \frac{1}{4}\,m W^2(x) - \frac{i\sigma_3}{2}\,[W(x), p]$$

$$= \frac{1}{4}\,\frac{p^2}{m} + \frac{1}{4}\,m W^2(x) + \frac{\hbar\sigma_3}{2}\,W'(x)$$

where $W'(x) = dW(x)/dx$. Similarly

$$Q_2^2 = \frac{1}{4}\,\frac{p^2}{m} + \frac{1}{4}\,m W^2(x) - \frac{1}{2}\,\sigma_2 p \sigma_1 W(x) - \frac{1}{2}\,\sigma_1 W(x) \sigma_2 p$$

$$= \frac{1}{4}\,\frac{p^2}{m} + \frac{1}{4}\,m W^2(x) - \frac{1}{2}\,i\sigma_3\,[W(x), p]$$

$$= \frac{1}{4}\,\frac{p^2}{m} + \frac{1}{4}\,m W^2(x) + \frac{\hbar}{2}\,\sigma_3 W'(x) = Q_1^2.$$

The anticommutator $\{Q_1, Q_2\} = 0$.

$$Q_1 Q_2 = \frac{1}{4}\left(\frac{\sigma_1 p}{\sqrt{m}} + \sigma_2 \sqrt{m}\, W(x)\right)\left(\frac{\sigma_2 p}{\sqrt{m}} - \sigma_1 \sqrt{m}\, W(x)\right)$$

$$= \frac{1}{4}\ \frac{p^2}{m}\sigma_1\sigma_2 - m W^2(x)\ \sigma_2\sigma_1 + W(x) p - p W(x)$$

$$Q_2 Q_1 = \frac{1}{4}\frac{p^2}{m}\sigma_2\sigma_1 - m W^2(x)\sigma_1\sigma_2 - W(x) p + p W(x)$$

So $\sigma_i \sigma_j = \delta_{ij} + \sum_k i\epsilon_{ijk}\sigma_k \Rightarrow \sigma_1\sigma_2 + \sigma_2\sigma_1 = 0$

Thus

$$Q_1 Q_2 + Q_2 Q_1 = 0$$

The susy algebra here is

$$[Q_i, Q_j]_+ = \delta_{ij} H = \{Q_i, Q_j\}.$$

and

$$[Q_i, H] = 0. \qquad \text{So}$$

$$H = 2Q_1^2 = 2Q_2^2$$

$$= \frac{p^2}{2m} + \frac{m}{2} W^2(x) - \hbar \sigma_3 W'(x).$$

We say "susy is exact" if Q_1 or Q_2 annihilates the ground state $|0\rangle$.

$$Q_1 |0\rangle = 0 \text{ or } Q_2 |0\rangle = 0.$$

In both cases

$$H|0\rangle = 2Q_1^2 |0\rangle = 0 = 2Q_2^2 |0\rangle.$$

And so both

$$\langle 0 | Q_1^2 | 0\rangle = 0 = \langle 0 | Q_2^2 | 0\rangle,$$

which means both

$$Q_1 |0\rangle = 0 \quad \text{and} \quad Q_2 |0\rangle = 0.$$

Let's solve

$$0 = \langle x | Q_1 | 0\rangle = \frac{1}{2}\left(\frac{\sigma_1}{\sqrt{m}} \langle x | p | 0\rangle + \sigma_2 \sqrt{m} W(x) \langle x | 0\rangle \right)$$

$$\frac{\hbar}{i} \frac{\sigma_1}{\sqrt{m}} \langle x | 0\rangle' + \sigma_2 \sqrt{m}\, W(x) \langle x | 0\rangle = 0 \qquad \text{or}$$

$$\frac{\hbar}{i} \langle x | 0\rangle' = -i \sigma_3\, m\, W(x) \langle x | 0\rangle$$

in which $\langle x | 0\rangle$ has two components.

So $\langle x|0\rangle' = \frac{m}{\hbar}\sigma_3 W(x)\langle x|0\rangle$

which we integrate (as before) to

$$\langle x|0\rangle = N\exp\left[\int_0^x dx' \frac{m}{\hbar}\sigma_3 W(x)\right]\langle 0|0\rangle.$$

As for the case in which $H=\hbar\omega(b^\dagger b+c)$, we can find a solution if the leading power of $W(x)$ is odd — but not if it is even. For instance, if

$$W(x) = c_3x^3 + c_2x^2 + c_1x + c_0$$

then

$$\int_0^x dx'\, W(x') = \frac{c_3}{4}x^4 + \frac{c_2}{3}x^3 + \frac{c_1}{2}x^2 + c_0x$$

and

$$\langle x|0\rangle = N\exp\left[\frac{m}{\hbar}\sigma_3\left(\frac{c_3}{4}x^4 + \frac{c_2}{3}x^3 + \frac{c_1}{2}x^2 + c_0x\right)\right]$$

$$\times\langle 0|0\rangle.$$

This may look bad if $c_3>0$, but recall that $\langle 0|0\rangle$ is a spinor. We choose if $c_3>0$

$\langle 0|0\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ in which case

$$\langle x|0\rangle = N\exp\left[-\frac{m}{\hbar}\left(\frac{c_3}{4}x^4 + \frac{c_2}{3}x^3 + c_1\frac{x^2}{2} + c_0x\right)\right]\begin{pmatrix}0\\1\end{pmatrix}.$$

5. If $W(x) = c_4 x^4$, for instance, then

$$Q_1 |0\rangle = 0$$

has no solution, and we must instead solve

$$H|0\rangle = 2Q_1^2 |0\rangle = E_0 |0\rangle$$

which is much harder to do. Also, the energy E_0 must be positive, since $E_0 \geq 0$ and $E_0 = 0$ would imply

$$\langle 0| Q_1^2 |0\rangle = \frac{E_0}{2} \langle 0|0\rangle = 0$$

which implies

$$Q_1 |0\rangle = 0,$$

which we saw was impossible when the leading power of $W(x)$ is even.

Q_1 and Q_2 generate susy transformations. When

$$Q_1 |0\rangle = Q_2 |0\rangle = 0,$$

the ground state is invariant under susy transformations and we say susy is exact. In this case,

$$\langle 0| e^{-\alpha Q_1 - \beta Q_2} A e^{\alpha Q_1 + \beta Q_2} \cdots e^{-\alpha Q_1 - \beta Q_2} B \cdots Z e^{\alpha Q_1 + \beta Q_2} |0\rangle$$

$$= \langle 0| e^{\alpha Q_1 + \beta Q_2} A B \cdots Z e^{-\alpha Q_1 - \beta Q_2} |0\rangle$$

$$= \langle 0| A B \cdots Z |0\rangle.$$