Lie Algebra of Rotation Group

Group theory is hopeless, but the algebra of the generators of a group is fairly understanable. The most important of these Lie algebras is that of the 3 generators of rotations J_i

$$[J_i, J_j] = i\hbar \sum_k \epsilon_{ijk} J_k. \tag{1}$$

We have seen that $\vec{J}^2$ is a scalar

$$[J_i, \vec{J}^2] = 0. \tag{2}$$

So we may simultaneously diagonalize any one of the 3 J_i's and $\vec{J}^2$. By convention, we choose J_3 and $\vec{J}^2$.

Define

$$J_+ = J_1 + i J_2 \tag{3}$$

$$J_- = J_1 - i J_2 , \tag{4}$$

Note that
$$[J_3, J_+] = [J_3, J_1 + i J_2] = i\hbar \left(\epsilon_{312} J_2 + i \epsilon_{321} J_1 \right)$$

$$= i\hbar (J_2 - i J_1) = \hbar (J_1 + i J_2) = \hbar J_+. \tag{5}$$

Also

$$[J_3, J_-] = [J_3, J_1 - iJ_2]$$

$$= i\hbar \left(\epsilon_{312} J_2 - i \epsilon_{321} J_1 \right)$$

$$= \hbar \left(-J_1 + i J_2 \right) = -\hbar \left(J_1 - i J_2 \right)$$

$$= -\hbar J_-. \qquad (6)$$

And

$$[J_+, J_-] = [J_1 + iJ_2, J_1 - iJ_2]$$

$$= -i [J_1, J_2] + i [J_2, J_1]$$

$$= -2i [J_1, J_2] = (-2i) i\hbar \epsilon_{123} J_3$$

$$= 2\hbar J_3. \qquad (7)$$

Surely

$$[\vec{J}^2, J_+] = [\vec{J}^2, J_-] = [\vec{J}^2, J_3] = 0. \quad (8)$$

Now

$$[J_+ J_- = (J_1 + iJ_2)(J_1 - iJ_2)$$

$$= J_1^2 + J_2^2 - i[J_1, J_2]$$

$$= J_1^2 + J_2^2 - i(i\hbar)J_3$$

$$= J_1^2 + J_2^2 + \hbar J_3 \qquad (9)$$

Also

$$J_- J_+ = (J_1 - i J_2)(J_1 + i J_2)$$
$$= J_1^2 + J_2^2 + i [J_1, J_2]$$
$$= J_1^2 + J_2^2 + i (i\hbar) J_3$$
$$= J_1^2 + J_2^2 - \hbar J_3. \tag{10}$$

Equivalently

$$J_+ J_- = \vec{J}^2 - J_3^2 + \hbar J_3 \tag{11}$$

$$J_- J_+ = \vec{J}^2 - J_3^2 - \hbar J_3. \tag{12}$$

So $\quad \frac{1}{2}(J_+ J_- + J_- J_+) = \vec{J}^2 - J_3^2 \quad$ or

$$\vec{J}^2 = \frac{1}{2}(J_+ J_- + J_- J_+) + J_3^2. \tag{13}$$

Let $|\psi\rangle$ be any normalized state. Then

$$0 \le \langle \psi | \vec{J} \cdot \vec{J} | \psi \rangle = \sum_{k=1}^{3} \langle \psi | J_k^2 | \psi \rangle$$

$$= \sum_{k=1}^{3} \| J_k |\psi\rangle \|^2 \ge 0. \tag{14}$$

So the e-vals of $\vec{J}^2$ must be non-negative. We set them to

$$j(j+1)\hbar^2 \quad \text{for} \quad j \ge 0. \tag{15}$$

Note that if

$$j(j+1)\hbar^2 = \lambda \geq 0 \tag{16}$$

then

$$j(j+1) = \alpha = \lambda/\hbar^2 \geq 0 \tag{17}$$

has roots

$$j^2 + j - \alpha = 0$$

$$j = \frac{-1 \pm \sqrt{1 + 4\alpha}}{2}$$

only one of which

$$j = \frac{\sqrt{1 + 4\alpha} - 1}{2} \tag{18}$$

is nonnegative.

We will write the evals of J_3 as

$$m\hbar. \tag{19}$$

What are j and m? We must solve

$$\vec{J}^2 |j, m\rangle = j(j+1)\hbar^2 |j, m\rangle \tag{20}$$

and

$$J_3 |j, m\rangle = m\hbar |j, m\rangle. \tag{21}$$

Let's first see why

$$-j \le m \le j. \tag{22}$$

Surely

$$0 \le \| J_+ |jm\rangle \|^2 = \langle jm | \hat{J}_- J_+ |jm\rangle$$

$$= \langle jm | \vec{J}^2 - J_3^2 - \hbar J_3 |jm\rangle$$

$$= j(j+1)\hbar^2 - (m\hbar)^2 - m\hbar^2$$

$$= \hbar^2 \left[j(j+1) - m(m+1) \right]$$

$$= \hbar^2 (j-m)(j+m+1) \ge 0 \tag{23}$$

as well as

$$0 \le \| J_- |jm\rangle \|^2 = \langle jm | J_+ J_- |jm\rangle$$

$$= \langle jm | \vec{J}^2 - J_3^2 + \hbar J_3 |jm\rangle$$

$$= \hbar^2 \left[j(j+1) - m(m-1) \right]$$

$$= \hbar^2 (j-m+1)(j+m) \ge 0. \tag{24}$$

Well

$$(j-m)(j+m+1) \ge 0 \implies -(j+1) \le m \le j \tag{25}$$

and

$$(j-m+1)(j+m) \ge 0 \implies -j \le m \le j+1 \tag{26}$$

Thus

$$-j \le m \le j. \tag{27}$$

Next we want to see that if

$$J^2 |jm\rangle = \hbar^2 j(j+1) |jm\rangle \tag{28}$$

$$J_3 |jm\rangle = \hbar m |jm\rangle \tag{29}$$

then for $m = -j$

$$J_- |j,-j\rangle = 0 \tag{30}$$

and if $m > -j$

$$J^2 J_- |jm\rangle = \hbar^2 j(j+1) J_- |jm\rangle \tag{31}$$

and

$$J_3 J_- |jm\rangle = (m-1)\hbar J_- |jm\rangle, \tag{32}$$

Our result (24)

$$\| J_- |jm\rangle \|^2 = \hbar^2 (j-m+1)(j+m) \tag{33}$$

shows that $J_- |j,-j\rangle = 0 \tag{34}$

which is (30). (Incidentally, since $j \geq 0$

$$0 = (j-m+1)(j+m) = j(j+1) - m(m-1) \tag{35}$$

implies $m = -j$ or $m = j+1$ which by (27) is impossible. So if $J_- |jm\rangle = 0$, then $m = -j$. $\tag{36}$

Now assume $m > -j$. By (8)

$$[J^2, J_-] = 0 \qquad (37)$$

and so

$$[J^2, J_-] |j\,m\rangle = 0 \qquad (38)$$

or

$$J^2 J_- |j\,m\rangle = J_- J^2 |j\,m\rangle$$
$$= J_- \hbar^2 j(j+1) |j\,m\rangle \qquad (39)$$
$$= \hbar^2 j(j+1) J_- |j\,m\rangle \qquad (39)$$

which is (31).

We continue to assume $m > -j$. By (6)

$$[J_3, J_-] = -\hbar J_- \qquad (40)$$

so

$$[J_3, J_-] |j\,m\rangle = -\hbar J_- |j\,m\rangle \qquad (41)$$

or

$$J_3 J_- |j\,m\rangle = J_- J_3 |j\,m\rangle - \hbar J_- |j\,m\rangle$$

$$= J_- \hbar m |j\,m\rangle - \hbar J_- |j\,m\rangle$$

$$= \hbar(m-1) J_- |j\,m\rangle \qquad (42)$$

which is (32). So if $m > -j$, then $J_- |j\,m\rangle$ is an e-vec of both J^2 and J_3 with e-vals $\hbar^2 j(j+1)$ and $\hbar(m-1)$.

What about $J_+ |j\,m\rangle$?

Assume again that $|j, m\rangle$ is an evec of J^2 and J_3 as in (28-29). Then we want to see that if $m = j$

$$J_+ |j, j\rangle = 0 \tag{43}$$

and if $m < j$

$$J^2 J_+ |j m\rangle = \hbar^2 j(j+1) J_+ |j m\rangle \tag{44}$$

and

$$J_3 J_+ |j m\rangle = \hbar (m+1) J_+ |j m\rangle. \tag{45}$$

To prove (43), we use (23)

$$\| J_+ |j m\rangle \|^2 = \hbar^2 (j - m)(j + m + 1) \tag{46}$$

which says that $J_+ |j j\rangle = 0$. It's also true that $J_+ |j m\rangle = 0$ implies $m = j$.

To prove (44), we use (8)

$$0 = [J^2, J_+] |j m\rangle \tag{47}$$

so

$$= J^2 J_+ |j m\rangle = J_+ J^2 |j m\rangle$$
$$= J_+ \hbar^2 j(j+1) |j m\rangle \tag{48}$$

or

$$J^2 J_+ |j m\rangle = \hbar^2 j(j+1) J_+ |j m\rangle. \tag{49}$$

To prove (45), we use (5)

$$[J_3, J_+] = \hbar J_+ \tag{50}$$

to show that

$$[J_3, J_+]|jm\rangle = \hbar J_+|jm\rangle \tag{51}$$

$$J_3 J_+|jm\rangle = J_+ J_3|jm\rangle + \hbar J_+|jm\rangle$$

$$= \hbar m J_+|jm\rangle + \hbar J_+|jm\rangle$$

$$= \hbar(m+1) J_+|jm\rangle \tag{52}$$

which is (45).

Now we find what j and m can be. Again we assume

$$J^2|jm\rangle = \hbar^2 j(j+1)|jm\rangle \tag{53}$$

$$J_3|jm\rangle = \hbar m|jm\rangle. \tag{54}$$

By (27)

$$-j \le m \le j. \tag{55}$$

Let's write

$$m = -j - x \tag{56}$$

and

$$m = -j + y. \tag{57}$$

What can x and y be?

By (55) both x and y must be non-negative

$$x \geq 0 \qquad (58)$$
$$y \geq 0. \qquad (59)$$

But if x were not an integer, then by repeatedly using (52), we could raise the value of m beyond j. And if y were not an integer, by using (42) over and over, we could lower m below -j.

For instance, if $x = \frac{1}{2}$, then by applying (52) once we'd have

$$J_3 \, J + |j, j - \tfrac{1}{2}\rangle = \hbar \left(j + \tfrac{1}{2}\right) J + |j, j - \tfrac{1}{2}\rangle \qquad (60)$$

which would violate (55).

So both x and y must be non-negative <u>integers</u>. That is,

$$j = m + k \qquad (61)$$

$$j = -m + \ell \qquad (62)$$

with $x = k$ and $y = \ell$, where k & ℓ are integers.

the sum of (61) & (62) is

$$2j = k + l \tag{63}$$

So $2j$ is a non-negative integer,

$$2j = 0, 1, 2, 3, 4 \cdots \tag{64}$$

which means that

$$j = \frac{\text{non-negative integer}}{2} \tag{65}$$

So the possible values of j are $0, \frac{1}{2},$

$1, \frac{3}{2}, 2, \frac{5}{2}, 3$ etc.

The values of m run from

$-j$ to j in steps of $\underline{1}$. So for

a given j, there are $2j+1$ possible

values of m. A given system will

have one state $|jm\rangle$ for each m

in the interval $-j \leq m \leq j$, $m-j$ integral.

We have seen (52) that

$$J_+ |j,m\rangle = c |j, m+1\rangle \qquad (66)$$

as long as $m < j$. The constant c is restricted by the relation

$$\langle jm | J_- J_+ | j'm\rangle = |c|^2 \langle j,m+1 | j,m+1\rangle$$
$$= |c|^2$$
$$= \hbar^2 [j(j+1) - m(m+1)] \quad (67)$$

since these states are normalized to unity. The conventional choice is

$$c = \hbar \sqrt{j(j+1) - m(m+1)} \qquad (68)$$

so that

$$J_+ |jm\rangle = \hbar \sqrt{j(j+1) - m(m+1)} \; |j, m+1\rangle$$
$$= \hbar \sqrt{(j-m)(j+m+1)} \; |j, m+1\rangle. \qquad (69)$$

Because the states are normalized, we have

$$\langle j, m+1 | J_+ | jm\rangle = \hbar \sqrt{(j-m)(j+m+1)} \qquad (70)$$

whose complex conjugate is

$$\langle jm | J_- | j, m+1\rangle = \hbar \sqrt{(j-m)(j+m+1)} \qquad \text{or}$$

$$\langle j, m-1 | J_- | j, m\rangle = \hbar \sqrt{(j-m+1)(j+m)}. \qquad (71)$$

And since $J_-|jm\rangle$ is an e-vec of $\underline{J_3}$ with e-val $(m-1)\hbar$, it is orthonal to all e-vecs with different e-vals. Thus

$$J_-|jm\rangle = \hbar\sqrt{(j-m+1)(j+m)}\ |j, m-1\rangle. \quad (72)$$