1.1) $[AB, CD] = ABCD - CDAB \equiv LHS$

$$RHS \equiv -AC\{D,B\} + A\{C,B\}D - C\{D,A\}B + \{C,A\}DB$$
$$= -ACDB - ACBD + ACBD + ABCD - CDAB - CADB + CADB + ACDB$$
$$= -ACBD + ACBD + ABCD - CDAB - CADB + CADB$$
$$= ABCD - CDAB - CADB + CADB$$
$$= ACCD - CDAB = [AB, CD].$$

1.2) $X = a_0 + \vec{\sigma} \cdot \vec{a}$

a) $a_0 = \frac{1}{2} \operatorname{tr} X \qquad a_i = \frac{1}{2} \operatorname{tr}(X \sigma_i)$

b) $a_0 = \frac{1}{2}(X_{11} + X_{22}) \qquad a_1 = \frac{1}{2}(X_{12} + X_{21})$

$$a_2 = \frac{1}{2}\left(X_{12}\, \sigma^2_{21} + X_{21}\, \sigma^2_{12}\right) \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{i}{2}(X_{12} - X_{21})$$

$$a_3 = \frac{1}{2}(X_{11} - X_{22})$$

1.3) $\det(\sigma \cdot a') = \det\left(e^{i\hat{n}\cdot\frac{\vec{\sigma}}{2}\phi}\, \sigma \cdot a\, e^{-i\hat{n}\cdot\frac{\vec{\sigma}}{2}\phi}\right)$

$$= \det\left(e^{i\hat{n}\cdot\frac{\sigma}{2}\phi}\right) \det(\sigma \cdot a) \det\left(e^{-i\hat{n}\cdot\frac{\sigma}{2}\phi}\right)$$

Since $e^{i\hat{n}\cdot\frac{\sigma}{2}\phi}\, e^{-i\hat{n}\cdot\frac{\sigma}{2}\phi} = 1$, and since $\det X^{-1} = (\det X)^{-1}$,

$$\det(\sigma \cdot a') = \det(\vec{\sigma} \cdot a).$$

1.3.1 $|\sigma\cdot a'| = |\exp(\frac{i\sigma}{2}\cdot n\phi)||\sigma\cdot a||\exp(-\frac{i\sigma}{2}\cdot n\phi)|$

$= |\sigma\cdot a|$

because $\exp(i\frac{\sigma}{2}\cdot n\phi)\exp(-i\frac{\sigma}{2}\cdot n\phi) = 1$,

If $\hat{n} = (0,0,1)$, then

$$e^{i\frac{\sigma}{2}\cdot n\phi} = e^{i\frac{\sigma_3}{2}\phi} = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}.$$ So

$$\sigma\cdot a' = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}\begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}\begin{pmatrix} a_3 e^{-i\phi/2} & (a_1 - ia_2)e^{+i\phi/2} \\ (a_1 + ia_2)e^{-i\phi/2} & -a_3 e^{i\phi/2} \end{pmatrix}$$

$$= \begin{pmatrix} a_3 & (a_1 - ia_2)e^{i\phi} \\ (a_1 + ia_2)e^{-i\phi} & -a_3 \end{pmatrix}$$

So $a_3' = a_3$ $\qquad a_1' \pm i a_2' = (a_1 \pm i a_2)e^{\mp i\phi}$

So $\qquad a_1' - i a_2' = (a_1 - i a_2)e^{i\phi}$

$2a_1' = a_1(e^{i\phi} + e^{-i\phi}) + ia_2(e^{-i\phi} - e^{i\phi})$

$a_1' = \cos\phi\, a_1 + \sin\phi\, a_2$

$2ia_2' = a_1(e^{-i\phi} - e^{i\phi}) + ia_2(e^{-i\phi} + e^{i\phi})$

$a_2' = -\sin\phi\, a_1 + \cos\phi\, a_2$

3.2 Now $\vec{a}'$ is $\vec{a}$ rotated by ϕ about the $\hat{z}$ axis. If $\vec{a} = (1, 0, 0)$, then $a' = (\cos\phi, -\sin\phi, 0)$. Thus the rotation is clockwise, i.e., the axis of the right-handed rotation is $-\hat{z}$.

If $\vec{a}$ is a vector operator, then $\vec{\sigma} \cdot \vec{a}$ should be a scalar — i.e., invariant under a rotation

$$e^{-i J \cdot \hat{n} \phi}$$

where $\vec{J} = \vec{L} + \vec{S}$. Such a rotation would rotate $\vec{a}$ by ϕ about $\hat{z}$ in a right-handed sense. Thus $\vec{\sigma} \cdot \vec{a}$ would be invariant.

1.4 (a) $X \to F \quad Y \to G$ for clarity

$$\text{tr}\, FG = \sum_{a'} \langle a'|F\,|G|a'\rangle = \sum_{a'b'} \langle a'|F|b'\rangle\langle b'|G|a'\rangle$$

$$= \sum_{a'b'} \langle b'|G|a'\rangle\langle a'|F|b'\rangle = \sum_{b'} \langle b'|GF|b'\rangle$$

$$= \text{tr}\, GF$$

(b) $$(FG)^{\dagger} = \left(\sum_{a'b'c'} |a'\rangle\langle a'|F|b'\rangle\langle b'|G|c'\rangle\langle c'| \right)^{\dagger}$$

$$= \sum_{a'b'c'} |c'\rangle\langle c'|G^{\dagger}|b'\rangle\langle b'|F^{\dagger}|a'\rangle\langle a'|$$

$$= G^{\dagger} F^{\dagger}$$

in which (1.2.35), (1.2.38), and the rule

$$(|\alpha\rangle c \langle\beta|)^{\dagger} = |\beta\rangle c^{*} \langle\alpha|,$$

where c is a number, were used.

(c) if $A = \sum_{a'} a' |a'\rangle\langle a'|$, then

$$\exp[i f(A)] = \sum_{a'} e^{i f(a')} |a'\rangle\langle a'|$$

(d) $\sum_{a'} \langle a'|x'\rangle\langle x''|a'\rangle = \langle x''|x'\rangle = \delta(x' - x'')$ by (1.7.2).

5.1 (a) $\langle a' | \alpha \rangle \langle \beta | a'' \rangle$

(b) $|\alpha\rangle = |z+\rangle$, $|\beta\rangle = |x+\rangle$.

$$\langle +z | z+ \rangle \langle x+ | z+ \rangle = \langle x+ | z+ \rangle = \frac{1}{\sqrt{2}}$$ since

we use
$$|x+\rangle = \frac{1}{\sqrt{2}} \left(|z+\rangle + |z-\rangle \right).$$

$$\langle +z | z+ \rangle \langle x+ | z- \rangle = \frac{1}{\sqrt{2}}$$

$$\langle -z | z+ \rangle \langle x+ | z\pm \rangle = 0.$$

So
$$|z+\rangle \langle x+| \doteq \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}.$$

1.6 $A|i\rangle = a_i|i\rangle$ and $A|j\rangle = a_j|j\rangle$,

Either $a_i = a_j$ or $a_i \neq a_j$.

Assume $\langle i|i\rangle = \langle j|j\rangle = 1$.

If $a_i \neq a_j$, then $\langle j|i\rangle = 0$ and by assumption

$$A(|i\rangle + |j\rangle) = \lambda(|i\rangle + |j\rangle) = a_i|i\rangle + a_j|j\rangle$$

Then

$$\lambda\langle i|(|i\rangle + |j\rangle) = \lambda = a_i$$

and $\lambda\langle j|(|i\rangle + |j\rangle) = \lambda = a_j$

which is a contradiction.

Thus $a_i = a_j$.