The Interaction Picture

Suppose we have a hamiltonian $H = H_0 + V$ consisting of a simple part H_0 whose eigenstates $|n\rangle$ we know

$$H_0|n\rangle = E_n|n\rangle \tag{1}$$

and a potential V. In the usual Schrödinger picture, a state

$$|\psi, 0\rangle = \sum_n c_n|n\rangle \tag{2}$$

at time $t = 0$ will evolve into the state

$$|\psi, t\rangle = e^{-iHt/\hbar}\,|\psi, 0\rangle = e^{-iHt/\hbar}\,\sum_n c_n|n\rangle \tag{3}$$

at time t. We may write this state as

$$|\psi, t\rangle = \sum_n c_n(t)e^{-iH_0t/\hbar}|n\rangle = \sum_n c_n(t)e^{-iE_nt/\hbar}|n\rangle. \tag{4}$$

The coefficients $c_n(t)$ would be constants if the potential V were absent. We may highlight them by multiplying the state $|\psi, t\rangle$ by $\exp(+iH_0t/\hbar)$

$$|\psi, t\rangle_I = e^{iH_0t/\hbar}\,|\psi, t\rangle = \sum_n c_n(t)|n\rangle. \tag{5}$$

This state is said to be in the *interaction picture* because its time dependence arises entirely from the potential V.

By making more explicit the time dependence of the Schrödinger-picture state $|\psi, t\rangle$ in this last equation

$$|\psi, t\rangle_I = e^{iH_0t/\hbar}\,|\psi, t\rangle = e^{iH_0t/\hbar}\,e^{-iHt/\hbar}\,|\psi, 0\rangle \tag{6}$$

and differentiating, we find

$$\begin{aligned} i\hbar\frac{d|\psi, t\rangle_I}{dt} &= i\hbar\frac{d}{dt}\left(e^{iH_0t/\hbar}\,e^{-iHt/\hbar}\right)\,|\psi, 0\rangle \\ &= e^{iH_0t/\hbar}\,(H - H_0)\,e^{-iHt/\hbar}\,|\psi, 0\rangle \\ &= e^{iH_0t/\hbar}\,V\,e^{-iHt/\hbar}\,|\psi, 0\rangle \\ &= e^{iH_0t/\hbar}\,V\,e^{-iH_0t/\hbar}\,e^{iH_0t/\hbar}\,e^{-iHt/\hbar}\,|\psi, 0\rangle \\ &= e^{iH_0t/\hbar}\,V\,e^{-iH_0t/\hbar}\,|\psi, t\rangle_I. \end{aligned} \tag{7}$$

The potential V with the simple time dependence of H_0 is called the potential in the interaction picture

$$V_I(t) = e^{iH_0 t/\hbar}\, V\, e^{-iH_0 t/\hbar}. \tag{8}$$

In terms of it, the time dependence of the state $|\psi, t\rangle_I$ in the interaction picture is

$$i\hbar \frac{d|\psi, t\rangle_I}{dt} = V_I(t)\, |\psi, t\rangle_I. \tag{9}$$

When we integrate this differential equation, we must keep the potential $V_I(t')$ to the left of $V_I(t'')$ if $t' > t''$ and to its right if $t' < t''$. Using a capital T to denote this kind of "time ordering," we have

$$|\psi, t\rangle_I = T\left\{\exp\left[-\frac{i}{\hbar}\int_0^t dt'\, V_I(t')\right]\right\}|\psi, 0\rangle_I. \tag{10}$$

This time-ordered product of exponential factors is usually beyond the mental range of humans. Freeman Dyson expanded it as

$$\begin{aligned} T\left\{\exp\left[-\frac{i}{\hbar}\int_0^t dt'\, V_I(t')\right]\right\} &= 1 - \frac{i}{\hbar}\int_0^t dt'\, V_I(t') \\ &+ \frac{(-i)^2}{2\hbar^2}\int_0^t dt' \int_0^t dt''\, T\left\{V_I(t')V_I(t'')\right\} \\ &+ \frac{(-i)^3}{3!\hbar^3}\int_0^t dt' \int_0^t dt'' \int_0^t dt'''\, T\left\{V_I(t')V_I(t'')V_I(t''')\right\} \\ &+ \ldots \end{aligned} \tag{11}$$

which makes sense as an asymptotic series when the potential V is tiny. Dyson's expansion is widely used in quantum electrodynamics.

If one uses a classical field to describe an external field imposed upon the system, then the potential V has its own explicit time dependence. In this case, the hamiltonian H inherits the time dependence of $V(t)$, and the Schrödinger-picture state $|\psi, t\rangle$ evolves in time not as in (3) but rather as

$$|\psi, t\rangle = T\left\{\exp\left[-\frac{i}{\hbar}\int_0^t dt'\, \left(H_0 + V(t')\right)\right]\right\}|\psi, 0\rangle. \tag{12}$$

But the interaction-picture state $|\psi, t\rangle_I$

$$|\psi, t\rangle_I = e^{iH_0 t/\hbar}|\psi, t\rangle \tag{13}$$

still obeys the differential equation (9) but with an interaction-picture potential

$$V_I(t) = e^{iH_0 t/\hbar}\, V(t)\, e^{-iH_0 t/\hbar} \tag{14}$$

that has both its own explicit time dependence as well as that induced by H_0. To see why, we differentiate (13) with $|\psi, t\rangle$ given by (12) and with $H(t) = H_0 + V(t)$

$$\begin{aligned}
i\hbar \frac{d|\psi, t\rangle_I}{dt} &= i\hbar \frac{d}{dt} \left(e^{iH_0 t/\hbar}\, T \left\{ \exp\left[-\frac{i}{\hbar} \int_0^t dt' \; (H_0 + V(t')) \right] \right\} \right) |\psi, 0\rangle \\
&= e^{iH_0 t/\hbar} \left[H_0 + V(t) - H_0 \right] T \left\{ \exp\left[-\frac{i}{\hbar} \int_0^t dt' \, H(t') \right] \right\} |\psi, 0\rangle \\
&= e^{iH_0 t/\hbar}\, V(t)\, T \left\{ \exp\left[-\frac{i}{\hbar} \int_0^t dt' \, H(t') \right] \right\} |\psi, 0\rangle \\
&= e^{iH_0 t/\hbar}\, V(t)\, e^{-iH_0 t/\hbar}\, e^{iH_0 t/\hbar}\, T \left\{ \exp\left[-\frac{i}{\hbar} \int_0^t dt' \, H(t') \right] \right\} |\psi, 0\rangle \\
&= e^{iH_0 t/\hbar}\, V(t)\, e^{-iH_0 t/\hbar}\, |\psi, t\rangle_I \\
&= V_I(t)\, |\psi, t\rangle_I
\end{aligned} \tag{15}$$

which is (9) but with the time dependence of $V_I(t)$ given by (14). Thus, we may integrate this equation as we did (9) and again obtain (10) but with $V_I(t)$ given by (14).

The point of the interaction picture is that it isolates the time dependence of the coefficients $c_n(t)$ in the expansion (5) of the state $|\psi, t\rangle_I$

$$|\psi, t\rangle_I = e^{iH_0 t/\hbar}\, |\psi, t\rangle = \sum_n c_n(t) |n\rangle. \tag{16}$$

Thus the time derivative of $c_n(t)$ by (15) is

$$i\hbar\, \dot{c}_n(t) = \frac{d\langle n|\psi, t\rangle_I}{dt} = \langle n|V_I(t)|\psi, t\rangle_I. \tag{17}$$

By inserting a complete set of eigenstates $|m\rangle$ of H_0 and using (14 & 16), we find for it the expression

$$\begin{aligned}
i\hbar\, \dot{c}_n(t) &= \sum_m \langle n|V_I(t)|m\rangle \langle m|\psi, t\rangle_I = \sum_m \langle n|V_I(t)|m\rangle\, c_m(t) \\
&= \sum_m \langle n|e^{iH_0 t/\hbar}\, V(t)\, e^{-iH_0 t/\hbar}|m\rangle\, c_m(t) \\
&= \sum_m \langle n|V(t)|m\rangle\, e^{i(E_n - E_m)t/\hbar}\, c_m(t).
\end{aligned} \tag{18}$$

In terms of the Bohr frequencies

$$\omega_{nm} = \frac{E_n - E_m}{\hbar} \tag{19}$$

this set of coupled linear first-order differential equations for the $c_n(t)$ takes the form

$$i\hbar\,\dot{c}_n(t) = \sum_m \langle n|V(t)|m\rangle\, e^{i\omega_{nm}t}\, c_m(t). \tag{20}$$