Path Integrals and the WKB Approximation

When a classical trajectory

$$\delta S = 0$$

is available, the leading term is proportional to $e^{iS/\hbar}$,

Assume a particle of mass m under the influence of a potential $V(x)$ is in a stationary state of energy E

$$H|E\rangle = E|E\rangle$$

$$H = \frac{p^2}{2m} + V(x)$$

$$L = \frac{m}{2}\dot{x}^2 - V(x).$$

The action S is

$$S = \int dt \left[\frac{m\dot{x}^2}{2} - V(x(t))\right].$$

Since the particle has energy E,

$$\frac{m}{2}\dot{x}^2 + V(x) = E.$$

So

$$-V(x) = \frac{m}{2}\dot{x}^2 - E$$

and the action is

$$S = \int dt \left(m\dot{x}^2 - E\right)$$

$$= \int \dot{x}\,dt\, m\dot{x} \quad - \int E\,dt$$

$$= \int dx\, p - \int E\,dt = \int p\,dx - Et.$$

Now p itself is related to E and to $V(x)$ since

$$\frac{p^2}{2m} + V(x) = E$$

so

$$\frac{p^2}{2m} = E - V(x)$$

$$p = \sqrt{2m(E - V(x))}.$$

Thus the action S is

$$S = \int p\,dx - Et$$

$$= \int \sqrt{2m(E - V(x))}\,dx - Et,$$

which is called Hamilton's principal function.

So we expect to be able to approximate the wave function as

$$\langle x,t|E\rangle \propto e^{iS/\hbar} = c\, e^{\frac{i}{\hbar}\left(\int^{t}\sqrt{2m(E-V(x))}\,dx - Et\right)}$$

where c is some factor.

The function $c(x)$ is less important than the exponential; in section (2.4) $c(x)$ is shown to be proportional to

$$c(x) \propto \left[E-V(x)\right]^{-1/4}.$$

The full WKB approximation then is

$$\langle x,t|E\rangle = \frac{N}{\left[E-V(x)\right]^{1/4}}\, e^{\frac{i}{\hbar}\left(\int^{x}dx'\sqrt{2m(E-V(x'))} - Et\right)}.$$

It works best in the limit $\hbar \to 0$ and when the potential $V(x)$ changes little over a wavelength λ

$$\lambda = \frac{h}{\sqrt{2m(E-V(x))}} \ll \frac{4\pi(E-V(x))}{|dV/dx|} \qquad (2.4.37)$$

compared to the length scale of the problem.