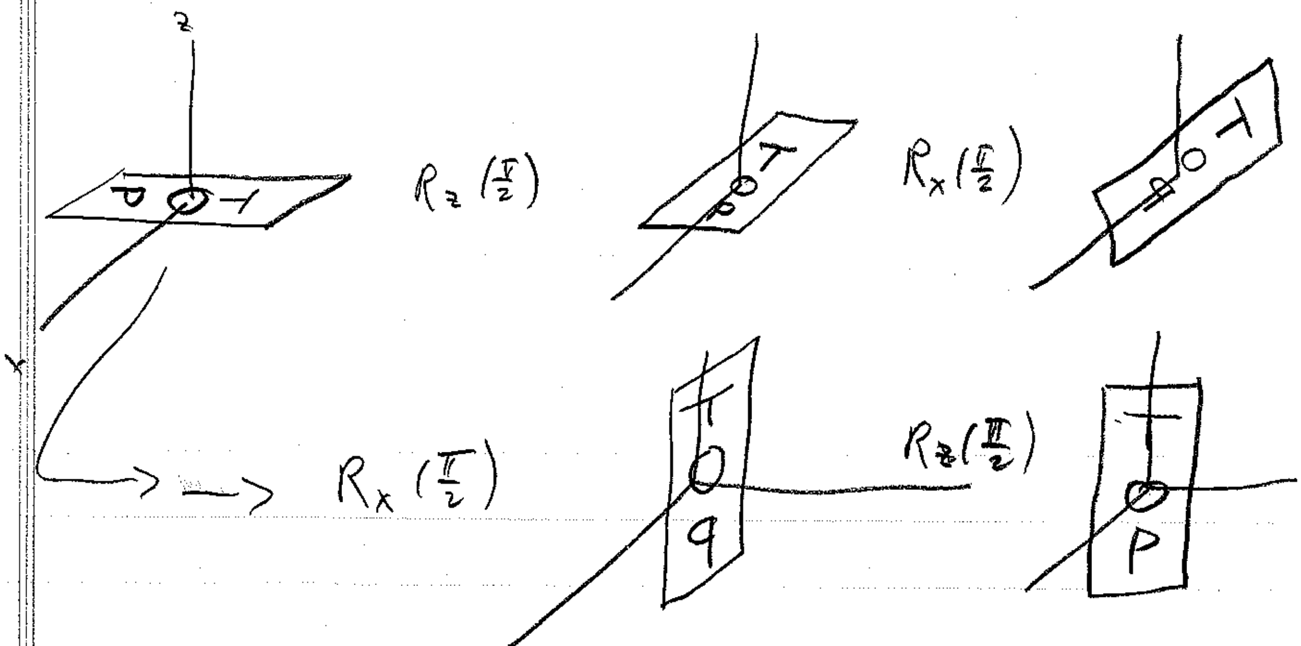
Chapter III

Angular Momentum and Rotations

$$\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = R \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \qquad R^T R = R R^T = 1 \quad \text{so}$$

$$\sum_{i=1}^{3} V_i'^2 = \sum_{ijk=1}^{3} \left(R_{ij} V_j\right)\left(R_{ik} V_k\right) = \sum_{jk=1}^{3} \left(R^T R\right)_{jk} V_j V_k$$

$$= \sum_{jk=1}^{3} \delta_{jk} V_j V_k = \sum_j V_j^2 .$$



$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_z(\epsilon) = \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & -\epsilon & 0 \\ \epsilon & 1 - \frac{\epsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_x(\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\epsilon^2}{2} & -\epsilon \\ 0 & \epsilon & 1-\frac{\epsilon^2}{2} \end{pmatrix}$$

$$R_y(\epsilon) = \begin{pmatrix} 1-\frac{\epsilon^2}{2} & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1-\frac{\epsilon^2}{2} \end{pmatrix}$$

$x \to y, \quad y \to z, \quad z \to x.$

$$R_x(\epsilon) R_y(\epsilon) = \begin{pmatrix} 1-\frac{\epsilon^2}{2} & 0 & \epsilon \\ \epsilon^2 & 1-\frac{\epsilon^2}{2} & -\epsilon \\ -\epsilon & \epsilon & 1-\epsilon^2 \end{pmatrix}$$

$$R_y(\epsilon) R_x(\epsilon) = \begin{pmatrix} 1-\frac{\epsilon^2}{2} & \epsilon^2 & \epsilon \\ 0 & 1-\frac{\epsilon^2}{2} & -\epsilon \\ -\epsilon & \epsilon & 1-\epsilon^2 \end{pmatrix}$$

So

$$R_x(\epsilon)R_y(\epsilon) - R_y(\epsilon) R_x(\epsilon) = \begin{pmatrix} 0 & -\epsilon^2 & 0 \\ \epsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= R_z(\epsilon^2) - 1 = R_z(\epsilon^2) - R_{\text{any}}(0).$$

Do demo.

Like translations: $U(a) = e^{-i\frac{a \cdot p}{\hbar}}$

and

time evolution $U(t) = e^{-itH/\hbar}$

so too

$$R_x(\theta) = e^{-i\theta J_x/\hbar}, \qquad R_y(\theta) = e^{-i\theta J_y/\hbar}, \qquad R_z(\theta) = e^{-i\theta \frac{J_z}{\hbar}}$$

So for small angles

$$R_y^{-1}(\epsilon)\, R_x^{-1}(\epsilon)\, R_y(\epsilon)\, R_x(\epsilon) = e^{iJ_y\epsilon/\hbar}\, e^{iJ_x\frac{\epsilon}{\hbar}}\, e^{-iJ_y\frac{\epsilon}{\hbar}}\, e^{-iJ_x\frac{\epsilon}{\hbar}}$$

$$\simeq \left(1 + i\frac{J_y\epsilon}{\hbar} - \frac{\epsilon^2}{2}\frac{J_y^2}{\hbar^2}\right)\left(1 + i\frac{J_x\epsilon}{\hbar} - \frac{\epsilon^2}{2}\frac{J_x^2}{\hbar^2}\right)$$

$$\times \left(1 - i\frac{J_y\epsilon}{\hbar} - \frac{J_y^2\epsilon^2}{2\hbar^2}\right)\left(1 - i\frac{J_x\epsilon}{\hbar} - \frac{J_x^2\epsilon^2}{2\hbar^2}\right)$$

The terms involving only J_x (or only J_y) are the same as if J_y (or J_x) were absent; thus these terms cancel. So

$$R_y^{-1} R_x^{-1} R_y R_x \simeq 1 - \frac{J_y J_x \epsilon^2}{\hbar^2} + \frac{J_y J_x \epsilon^2}{\hbar^2} + \frac{J_x J_y \epsilon^2}{\hbar^2} - \frac{J_y J_x \epsilon^2}{\hbar^2}$$

$$R_y^{-1}(\epsilon) R_x^{-1}(\epsilon)\, R_y(\epsilon)\, R_x(\epsilon) \simeq 1 + \frac{\epsilon^2}{\hbar^2}[J_x, J_y]$$

Do Demo. Demo shows

$$R_y^{-1}(\epsilon) R_x^{-1}(\epsilon) R_y(\epsilon) R_x(\epsilon) \simeq 1 + i\epsilon^2 \frac{J_z}{\hbar} \simeq R_z(\epsilon^2)$$

So $\quad [J_x, J_y] = i\hbar J_z$

$$[J_a, J_b] = i\hbar\, \epsilon_{abc} J_c$$

$$|\alpha\rangle_R = \mathcal{D}(R)|\alpha\rangle$$

$$U_\epsilon = 1 - iG\epsilon$$

UPS	time	rotation
$G = \dfrac{\vec{p}\cdot\hat{n}}{\hbar}$	$G = \dfrac{H}{\hbar}$	$G = \dfrac{\vec{J}\cdot\hat{n}}{\hbar}$

eg.

$$\mathcal{D}_x(\phi) = \lim_{N\to\infty}\left(1 - i\frac{J_x\phi}{N\hbar}\right)^N = e^{-i\frac{J_x\phi}{\hbar}}$$

$$\approx 1 - i\frac{J_x\phi}{\hbar} - \frac{J_x^2\phi^2}{2\hbar^2} + \cdots$$

Groups

$$R\cdot 1 = R = 1\cdot R \Rightarrow D(R)\cdot 1 = 1\cdot D(R) = D(R)$$

$$R_1R_2 = R_3 \Rightarrow D(R_1)D(R_2) = D(R_3)$$

$$RR^{-1} = R^{-1}R = 1 \Rightarrow D(R)D^{-1}(R) = D^{-1}(R)D(R) = 1.$$

$$R_1(R_2R_3) = (R_1R_2)R_3 \Rightarrow D(R_1)\big(D(R_2)D(R_3)\big)$$
$$= \big(D(R_1)D(R_2)\big)D(R_3).$$
$$= D(R_1)D(R_2)D(R_3).$$

$$\left(1 - \frac{iJ_x\epsilon}{\hbar} - \frac{J_x^2\epsilon^2}{2\hbar^2}\right)\left(1 - \frac{iJ_y\epsilon}{\hbar} - \frac{J_y^2\epsilon^2}{2\hbar^2}\right)$$

$$\left(1 - \frac{iJ_y\epsilon}{\hbar} - \frac{J_y^2\epsilon^2}{2\hbar^2}\right)\left(1 - \frac{iJ_x\epsilon}{\hbar} - \frac{J_x^2\epsilon^2}{2\hbar^2}\right)$$

$$= 1 - \frac{iJ_z\epsilon^2}{\hbar} - 1 = -\frac{iJ_z\epsilon^2}{\hbar}$$

$O(\epsilon)$-terms all cancel. $O(\epsilon^2)$ terms give

$$[J_x, J_y] = i\hbar J_z$$

$$[J_i, J_j] = \sum_{k=1}^{3} i\hbar\,\epsilon_{ijk} J_k$$

These are the (non-abelian) commutation relations of $SO(3)$.

SU(2)

$$S_x = \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+| \right) = S_1$$

$$S_y = \frac{\hbar}{2}\left(-i|+\rangle\langle-| + i|-\rangle\langle+| \right) = S_2$$

$$S_z = \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-| \right) = S_3$$

$$[S_i, S_j] = i\hbar \epsilon_{ijk} S_k$$

$$|\alpha\rangle_R = D_z(\phi)|\alpha\rangle$$

$$\langle S_1\rangle(\phi) = {}_R\langle\alpha| S_1 |\alpha\rangle_R = \langle\alpha| D_z^\dagger(\phi) S_1 D_z(\phi)|\alpha\rangle$$

$$\langle S_1\rangle(\phi) \equiv S_1(\phi) = \langle\alpha| e^{\frac{iS_z\phi}{\hbar}} S_1 e^{\frac{-iS_z\phi}{\hbar}} |\alpha\rangle$$

$$\frac{d S_1(\phi)}{d\phi} = \langle\alpha| e^{\frac{iS_z\phi}{\hbar}} \frac{i}{\hbar}[S_3, S_1] e^{-i\frac{S_z\phi}{\hbar}} |\alpha\rangle$$

$$= \frac{i}{\hbar}(ih)\epsilon_{312} \langle\alpha| e^{\frac{iS_z\phi}{\hbar}} S_2 e^{-i\frac{S_z\phi}{\hbar}} |\alpha\rangle$$

$$= -S_2(\phi)$$

$$\frac{d S_2(\phi)}{d\phi} = \langle\alpha| e^{i\frac{S_z(\phi)}{\hbar}} \frac{i}{\hbar}[S_z, S_2] e^{-i\frac{S_z\phi}{\hbar}} |\alpha)$$

$$= \langle\alpha| e^{\frac{iS_z\phi}{\hbar}} \left(-\frac{\hbar}{\hbar}\right) \epsilon_{321} S_1 e^{-i\frac{S_z\phi}{\hbar}} |\alpha\rangle$$

$$= S_1(\phi).$$

So

$$\frac{dS_1(\phi)}{d\phi} = -S_2(\phi)$$

$$\frac{dS_2(\phi)}{d\phi} = S_1(\phi)$$

$$\frac{d^2}{d\phi^2}\begin{pmatrix} S_1(\phi) \\ S_2(\phi) \end{pmatrix} = -\begin{pmatrix} S_1(\phi) \\ S_2(\phi) \end{pmatrix}$$

$$S_1(\phi) = S_1(0)\cos\phi - S_2(0)\sin\phi$$

$$S_2(\phi) = S_1(0)\sin\phi + S_2(0)\cos\phi$$

which are (3.2.8-9). Clearly

$$S_3(\phi) = S_3(0) = \langle\alpha| S_3 |\alpha\rangle.$$

$$\langle S_i\rangle' = R_{ij}\langle S_j\rangle.$$

In general $\langle J_i\rangle = R_{ij}\langle J_j\rangle$.

But say $\phi = 2\pi$. Then

$$e^{-iS_3\frac{2\pi}{\hbar}}\left(|+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle\right) = e^{-i\frac{\hbar}{2}\frac{2\pi}{\hbar}}|+\rangle\langle+|\alpha\rangle$$

$$+ e^{i\frac{\hbar}{2}\frac{2\pi}{\hbar}}|-\rangle\langle-|\alpha\rangle = e^{-i\pi}|+\rangle\langle+|\alpha\rangle + e^{i\pi}|-\rangle\langle-|\alpha\rangle = -|\alpha\rangle!$$

spin precession

$e<0 \quad H = -\frac{e}{mc} S \cdot B = \omega S_z$

$\omega = \frac{|e|B}{m_e c}$.

$U(t,0) = e^{-\frac{iHt}{\hbar}} = e^{-i\frac{\omega t}{\hbar} S_z}$

$\langle S_x \rangle_t = \langle S_x \rangle_0 \cos\omega t - \langle S_y \rangle_0 \sin\omega t$

$\langle S_y \rangle_t = \langle S_x \rangle_0 \sin\omega t + \langle S_y \rangle_0 \cos\omega t$

$\langle S_z \rangle_t = \langle S_z \rangle_0$

$m_\mu \approx 105\ MeV \qquad \mu \to e + \nu_\mu + \bar{\nu}_e$

$\vec{P}_e$'s are opposite to $\vec{S}_\mu$

$|\alpha, t=0; t\rangle = e^{-itH/\hbar} |\alpha, t=0; 0\rangle$

$= e^{-\frac{i\omega t S_z}{\hbar}} \left(|+\rangle\langle +|\alpha\rangle + |-\rangle\langle -|\alpha\rangle \right)$

$= e^{-\frac{i\omega t}{2}} |+\rangle\langle +|\alpha\rangle + e^{\frac{i\omega t}{2}} |-\rangle\langle -|\alpha\rangle$

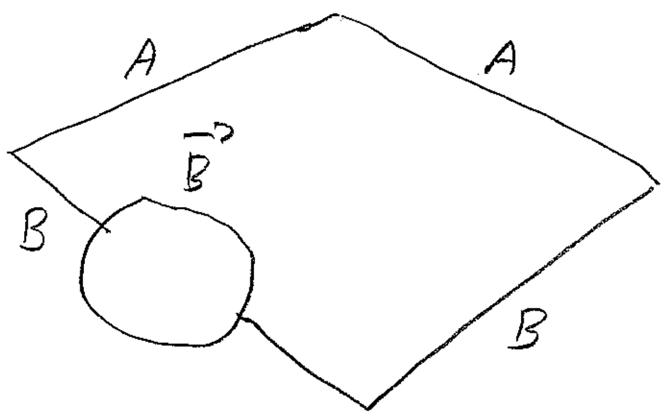
So when $\omega t = 4\pi$, $|\alpha, t=0; t\rangle = |\alpha, t=0; 0\rangle$

$T_{prec} = \frac{2\pi}{\omega}, \qquad T_{state} = \frac{4\pi}{\omega}$.

Neutron interferometry

due to charged quarks inside $n = \begin{matrix} u \\ d\ d \end{matrix}$

$$H = \omega S_z = \frac{g_n e B}{m_p c} S_z \qquad g_n \cong -1.91$$



$$c_2 = c_2(B=0)\, e^{\mp i\omega T/2}$$

where T is the time in $\vec{B}$

$$T = \frac{\ell}{v} \qquad mv = p = \frac{h}{\lambda} \qquad v = \frac{\hbar}{m\lambda\!\!\!^{-}}$$

$$T = \ell \frac{m_p \lambda\!\!\!^{-}}{\hbar} = \frac{4\pi}{\omega} = \frac{4\pi m_p c}{g_n e B}$$

successive maxima separated by $\Delta B = \dfrac{4\pi m_p c}{g_n e T}$ or

$$\Delta B = \frac{4\pi c \hbar}{e g_n \lambda\!\!\!^{-} \ell}.$$

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$$\sigma_i \sigma_j = \delta_{ij} + i \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k$$

$$\mathrm{tr}\, \sigma_i = 0$$

$$\sigma \cdot a\ \sigma \cdot b = a_i b_j \sigma_i \sigma_j = a_i b_j (\delta_{ij} + i \epsilon_{ijk} \sigma_k)$$

$$= a \cdot b + i (a \times b) \cdot \vec{\sigma}$$

$$e^{-i \frac{\vec{S} \cdot \hat{n} \phi}{\hbar}} \doteq e^{-i \frac{\vec{\sigma}\hbar}{2} \cdot \frac{\hat{n}\phi}{\hbar}} = e^{-i \frac{\vec{\sigma} \cdot \hat{n} \phi}{2}}$$

$$(\sigma \cdot \hat{n})^2 = \hat{n} \cdot \hat{n} = 1 \qquad (\sigma \cdot \hat{n})^{2m} = 1$$

$$(\sigma \cdot \hat{n})^{2m+1} = \sigma \cdot \hat{n}$$

$$e^{-i \frac{\sigma \cdot \hat{n} \phi}{2}} = \sum_{m=0}^{\infty} \left(\frac{-i\phi}{2}\right)^m \frac{1}{m!} (\sigma \cdot \hat{n})^m$$

$$= \sum_{k=0}^{\infty} \left(\frac{-i\phi}{2}\right)^{2k} \frac{1}{(2k)!} + \sum_{k=0}^{\infty} \left(\frac{-i\phi}{2}\right)^{2k+1} \frac{\sigma \cdot \hat{n}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\phi}{2}\right)^{2k} - i \vec{\sigma} \cdot \hat{n} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{\phi}{2}\right)^{2k+1}$$

$$= \cos \frac{\phi}{2} - i \vec{\sigma} \cdot \hat{n} \sin \frac{\phi}{2}$$

$$e^{-i\frac{\vec{\sigma}}{2}\cdot\hat{n}\phi} = \begin{pmatrix} \cos\frac{\phi}{2} - i n_3 \sin\frac{\phi}{2} & (-in_1 - n_2)\sin\frac{\phi}{2} \\ (-in_1 + n_2)\sin\frac{\phi}{2} & \cos\frac{\phi}{2} + i n_3 \sin\frac{\phi}{2} \end{pmatrix}$$

$$\chi' = e^{-i\frac{\sigma\cdot\hat{n}\phi}{2}}\chi$$

$$(\chi^\dagger \sigma_k \chi)' = \chi^\dagger e^{i\frac{\sigma\cdot\hat{n}\phi}{2}} \sigma_k e^{-i\frac{\sigma\cdot\hat{n}\phi}{2}} \chi$$

$$= R_{k\ell}\, \chi^\dagger \sigma_\ell \chi$$

e.g. $$e^{i\sigma_3\frac{\phi}{2}} \sigma_2\, e^{-i\sigma_3\frac{\phi}{2}} = \sigma_2 \cos\phi + \sigma_1 \sin\phi$$

NB $$e^{-i\frac{\sigma\cdot\hat{n}}{2}2\pi} = \cos\pi - i\vec{\sigma}\cdot\hat{n}\sin\pi = -1.$$

$$\vec{S}\cdot\hat{n}\,|\hat{n},+\rangle = \frac{\hbar}{2}|\hat{n},+\rangle$$

$$|\hat{n},+\rangle \doteq e^{-i\frac{\sigma}{2}\cdot\hat{m}\theta}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\hat{m} = \frac{\hat{z}\times\hat{n}}{\sin\beta} \qquad \theta = \beta \qquad \phi = \alpha$$

$$\hat{n} = (\sin\beta\cos\alpha,\ \sin\beta\sin\alpha,\ \cos\beta)$$

$$\hat{z} \times \hat{n} = (-\sin\beta\sin\alpha,\ +\sin\beta\cos\alpha,\ 0)$$

$$\frac{\hat{z}\times\hat{n}}{\sin\beta} = (\sin\alpha,\ -\cos\alpha,\ 0)$$

$$|\hat{n}+\rangle \doteq e^{-i\frac{\vec{\sigma}}{2}\cdot\beta(\sin\alpha,\ +\cos\alpha,\ 0)}$$

$$= \left[\cos\frac{\beta}{2} - i(\sigma_1\sin\alpha + \sigma_2\cos\alpha)\sin\frac{\beta}{2}\right]\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}\cos(\beta/2)\\ (+i\sin\alpha + \cos\alpha)\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}\cos\beta/2\\ +e^{i\alpha}\sin\beta/2\end{pmatrix}$$

$$= e^{\frac{i\alpha}{2}}\begin{pmatrix}\cos\beta/2\, e^{-i\alpha/2}\\ \sin\beta/2\, e^{i\alpha/2}\end{pmatrix}$$

$$|\hat{n}-\rangle = \left[\cos\frac{\beta}{2} - i(-\sigma_1\sin\alpha + \sigma_2\cos\alpha)\sin\frac{\beta}{2}\right]\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}(i\sin\alpha - \cos\alpha)\sin\frac{\beta}{2}\\ \cos\beta/2\end{pmatrix} = \begin{pmatrix}-e^{-i\alpha}\sin\frac{\beta}{2}\\ \cos\beta/2\end{pmatrix}$$

$$= e^{-i\alpha/2}\begin{pmatrix}-\sin\frac{\beta}{2}e^{-i\alpha/2}\\ \cos\beta/2\, e^{i\alpha/2}\end{pmatrix}$$

For spin 1 it is convenient to set

$$(J_k)_{ij} = i\hbar\, \epsilon_{ikj}$$

$$J_1 = i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = i\hbar \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

and

$$J_3 = i\hbar \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_1 J_2 = -\hbar^2 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad J_2 J_1 = -\hbar^2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[J_1, J_2] = -\hbar^2 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = i\hbar J_3$$

In general

$$[J_i, J_j] = i\hbar\, \epsilon_{ijk} J_k,$$

So for spin 1 we may take

$$(J_k)_{ij} = i\hbar\epsilon_{ikj} \quad \text{which obey} \quad [J_i,J_j] = i\hbar\epsilon_{ijk}J_k$$

in which none of the three generators is diagonal.
This is the x, y, z representation.

$$D(\vec\theta) = e^{-i\vec\theta\cdot\vec J/\hbar}$$

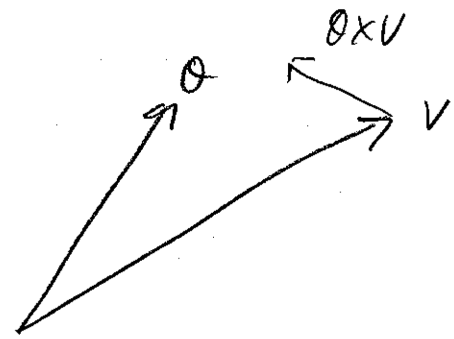
For small $|\vec\theta|$, $D(\vec\theta) \approx 1 - i\frac{\theta\cdot J}{\hbar}$ or

$$D(\vec\theta)_{ij} \approx \left(1 - i\frac{\theta_k J_k}{\hbar}\right)_{ij} = \delta_{ij} - i\theta_k i\epsilon_{ikj}$$

$$\approx \delta_{ij} + \epsilon_{ikj}\theta_k ,$$

So under D, $V_i' = D(\vec\theta)_{ij}V_j = V_i + \epsilon_{ikj}\theta_k V_j$

$$\vec V' = \vec V + \vec\theta\times\vec V$$



This is a right-handed rotation of V about $\vec\theta$ by $|\vec\theta|$.

Finite $\vec{\theta}$ now.

$$A \equiv \frac{-i\theta \cdot J}{\hbar} \qquad A_{ij} = \epsilon_{ikj}\,\theta_k$$

$$A^2_{il} = A_{ij}A_{jl} = \epsilon_{ikj}\,\theta_k\,\epsilon_{jml}\,\theta_m$$

$$= +\,\epsilon_{jik}\,\epsilon_{jml}\,\theta_k\theta_m$$

$$= (\delta_{im}\delta_{kl} - \delta_{il}\delta_{km})\,\theta_k\theta_m$$

$$= \theta_i\theta_l - \delta_{il}\theta^2 \qquad \text{So}$$

$$A^3_{im} = A^2_{il}A_{lm} = (\theta_i\theta_l - \delta_{il}\theta^2)\,\epsilon_{lkm}\theta_k$$

$$= -\theta^2\epsilon_{ikm}\theta_k = -\theta^2 A_{im} \qquad \text{So}$$

$$A^3 = -\theta^2 A \qquad \text{So} \quad e^A = \sum_{n=0}^{\infty}\frac{A^n}{n!} \quad \text{or}$$

$$e^A = 1 + A + \frac{A^2}{2} - \frac{\theta^2 A}{3!} - \frac{\theta^2 A^3}{4!} + \frac{\theta^4 A}{5!} + \frac{\theta^4 A^3}{6!} - \frac{\theta^6 A}{7!} \cdots$$

The even terms are

$$1 + \frac{A^2}{2!} - \frac{\theta^2 A^2}{4!} + \frac{\theta^4 A^2}{6!} - \frac{\theta^6 A^2}{8!} + \cdots$$

$$= 1 - \frac{A^2}{\theta^2}\left(-\frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} \cdots\right)$$

$$= 1 - \frac{A^2}{\theta^2}(-1 + \cos\theta).$$

The odd terms are

$$A - \theta^2\frac{A}{3!} + \frac{\theta^4 A}{5!} - \frac{\theta^6 A}{7!} + \frac{\theta^8 A}{9!} \cdots$$

$$= \frac{A}{\theta}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!}\cdots\right) = \frac{A}{\theta}\sin\theta.$$

So $$e^A = 1 - \frac{A^2}{\theta^2}(-1+\cos\theta) + \frac{A}{\theta}\sin\theta.$$

Now $\left(\frac{A^2}{\theta^2}\right)_{ij} = \hat{\theta}_i\hat{\theta}_j - \delta_{ij}$ so

$$\left(e^{-i\theta\cdot J/\hbar}\right)_{ij} = \left(e^A\right)_{ij} = \delta_{ij} - (\hat{\theta}_i\hat{\theta}_j - \delta_{ij})(-1+\cos\theta) + \frac{A_{ij}}{\theta}\sin\theta$$

$$= \hat{\theta}_i\hat{\theta}_j - \cos\theta\,(\hat{\theta}_i\hat{\theta}_j - \delta_{ij}) + \epsilon_{ikj}\,\hat{\theta}_k\sin\theta$$

$$= \delta_{ij}\cos\theta + \epsilon_{ikj}\,\hat{\theta}_k\sin\theta + \hat{\theta}_i\hat{\theta}_j(1-\cos\theta)$$

where $\theta = \sqrt{\vec{\theta}\cdot\vec{\theta}}$. Now $\hat{\theta}_i\hat{\theta}_j = I_{ij} + \left(\frac{-i\hat{\theta}\cdot J}{\hbar}\right)^2_{ij}$, so

$$e^{-i\theta\cdot J/\hbar} = I\cos\theta - i\frac{\hat{\theta}\cdot\vec{J}}{\hbar}\sin\theta + \left(I - \frac{(\hat{\theta}\cdot J)^2}{\hbar^2}\right)(1-\cos\theta)$$

$$= I - i\frac{\hat{\theta}\cdot J}{\hbar}\sin\theta - (1-\cos\theta)\frac{(\hat{\theta}\cdot J)^2}{\hbar^2}$$ with $\epsilon_{ikj}\theta_k = -i\left(\frac{\theta\cdot J}{\hbar}\right)_{ij}$

The general element of SU(2) is

$$U(a,b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \text{with} \quad |a|^2 + |b|^2 = 1.$$

Both $U(a,b)$ and $U(-a,-b)$ correspond to a single 3×3 matrix in SO(3).

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Euler rotations

$$R(\alpha,\beta,\gamma) = R_{z'}(\gamma)\, R_{y'}(\beta)\, R_z(\alpha) \qquad \text{But}$$

$$R_{y'}(\beta) = R_z(\alpha)\, R_y(\beta)\, R_z^{-1}(\alpha) \qquad \text{and}$$

$$R_{z'}(\gamma) = R_{y'}(\beta)\, R_z(\gamma)\, R_{y'}^{-1}(\beta), \quad \text{and so}$$

$$R(\alpha,\beta,\gamma) = R_{y'}(\beta)\, R_z(\gamma)\, R_{y'}^{-1}(\beta)\, R_z(\alpha)\, R_y(\beta)\, R_z^{-1}(\alpha)\, R_z(\alpha)$$

$$= R_z(\alpha)\, R_y(\beta)\, R_z^{-1}(\alpha)\, R_z(\gamma)\, R_z(\alpha)\, R_y^{-1}(\beta)\, R_z^{-1}(\alpha)\, R_z(\alpha)\, R_y(\beta)$$

$$= R_z(\alpha)\, R_y(\beta)\, R_z(\gamma)\, R_y^{-1}(\beta)\, R_y(\beta)$$

$$= R_z(\alpha)\, R_y(\beta)\, R_z(\gamma) \qquad \text{So}$$

$$D(\alpha\beta\gamma) = D_z(\alpha)\, D_y(\beta)\, D_z(\gamma).$$

SU(2) case

$$D^{1/2}(\alpha,\beta,\gamma) \doteq e^{-i\frac{\sigma_3\alpha}{2}}\, e^{-i\sigma_2\frac{\beta}{2}}\, e^{-i\sigma_3\frac{\gamma}{2}}$$

$$= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \left(\cos\frac{\beta}{2} - i\sigma_2 \sin\frac{\beta}{2} \right) \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos\beta/2 & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\beta/2 \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{+i\alpha/2} \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2}\cos\beta/2 & -e^{i\gamma/2}\sin\beta/2 \\ e^{-i\gamma/2}\sin\beta/2 & e^{i\gamma/2}\cos\beta/2 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\beta/2 & -e^{i(\gamma-\alpha)/2}\sin\beta/2 \\ e^{i(\alpha-\gamma)/2}\sin\beta/2 & e^{i(\alpha+\gamma)/2}\cos\beta/2 \end{pmatrix}$$

$$D^{j}_{m'm}(\alpha\beta\gamma) = \langle jm'|\, e^{-i\frac{J_z}{\hbar}\alpha}\, e^{-i\frac{J_y}{\hbar}\beta}\, e^{-i\frac{J_z}{\hbar}\gamma}\, |jm\rangle$$

$$= e^{-i(m'\alpha+m\gamma)} \langle jm'|\, e^{-i\frac{J_y}{\hbar}\beta}\, |jm\rangle.$$

ρ random spins from oven.

Can't be $\frac{1}{2}(|+\rangle + |-\rangle)$.

Instead $\sum_j w_j = 1$

$$[A] = \sum_i w_i \langle \alpha^i | A | \alpha^i \rangle$$

$$= \sum_{ij} w_i \langle \alpha^i | A | a_j \rangle \langle a_j | \alpha^i \rangle$$

$$= \sum_{ij} w_i |\langle \alpha_i | a_j \rangle|^2 a_j$$

$$= \sum_{ijk} w_i \langle \alpha^i | b_j \rangle \langle b_j | A | b_k \rangle \langle b_k | \alpha^i \rangle$$

$$= \sum_{jk} \sum_i w_i \langle b_k | \alpha^i \rangle \langle \alpha^i | b_j \rangle \langle b_j | A | b_k \rangle$$

Let $\rho = \sum_i w_i |\alpha^i\rangle\langle \alpha^i|$

$$[A] = \sum_{jk} \langle b_k | \rho | b_j \rangle \langle b_j | A | b_k \rangle$$

$$= \sum_k \langle b_k | \rho A | b_k \rangle = \mathrm{tr}\, \rho A .$$

$$1 = [1] = \mathrm{tr}\, \rho = \sum_k \sum_i w_i \langle b_k | \alpha^i \rangle \langle \alpha^i | b_k \rangle$$

$$= \sum_i w_i \langle \alpha^i | \alpha^i \rangle = \sum_i w_i = 1.$$

$$\rho^\dagger = \sum_i w_i^* |\alpha^i\rangle\langle\alpha^i| = \rho$$

since the w_i's are probabilities

But if $|\psi\rangle$ is any state, then the prob of finding $|\psi\rangle$ is

$$[\,|\psi\rangle\langle\psi|\,] = \mathrm{tr}\,\rho\,|\psi\rangle\langle\psi|\,] = \langle\psi|\rho|\psi\rangle \geq 0$$

So $\rho > 0$. Thus $\rho^\dagger = \rho$, $\mathrm{tr}\,\rho = 1$, $\rho > 0$ are the three properties of all density operators.

If $\rho = |\phi\rangle\langle\phi|$, then

$$\rho^2 = |\phi\rangle\langle\phi|\phi\rangle\langle\phi| = |\phi\rangle\langle\phi| = \rho.$$

$$\rho(\rho - 1) = 0 \qquad \mathrm{tr}\,\rho^2 = \mathrm{tr}\,\rho = 1.$$

$$\rho_{diag} \doteq \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}.$$

Nice examples in pp 179–181.

$S_z -$: $\quad \rho = |-\rangle\langle-|$

$$\doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \;\; 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$S_y \pm$: $\quad \rho = \frac{1}{\sqrt{2}}\left(|+\rangle \pm i|-\rangle\right)\frac{1}{\sqrt{2}}\left(\langle+| \mp i\langle-|\right)$

$$\doteq \frac{1}{2}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}(1, \mp i) = \frac{1}{2}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}$$

unpolarized: $\quad \rho = \frac{1}{2} \qquad \mathrm{tr}\,\rho \vec{S} = \vec{0}$

$\frac{1}{3} S_x, \;\; \frac{1}{3} S_y, \;\; \frac{1}{3} S_z$:

$$\rho = \frac{1}{3}\frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)\frac{1}{\sqrt{2}}\left(\langle+| + \langle-|\right)$$

$$+ \frac{1}{3}\frac{1}{\sqrt{2}}\left(|+\rangle + i|-\rangle\right)\frac{1}{\sqrt{2}}\left(\langle+| - i\langle-|\right)$$

$$+ \frac{1}{3} \quad |+\rangle\langle+|$$

$$= \frac{2}{3}|+\rangle\langle+| + \frac{1}{3}|-\rangle\langle-| + \frac{(1-i)}{6}|+\rangle\langle-| + \frac{(1+i)}{6}|-\rangle\langle+| .$$

$\mathrm{tr}\,\rho = 1$.

$$\mathrm{tr}\,\rho S_x = \frac{\hbar}{2}\left(\frac{1-i}{6} + \frac{1+i}{6}\right) = \frac{\hbar}{6}$$

$$\mathrm{tr}\,\rho S_y = \frac{\hbar}{2}\,\mathrm{tr}\,\rho\left(-i|+\rangle\langle-| + i|-\rangle\langle+|\right) = \frac{\hbar}{2}\left[-i\frac{(1+i)}{6} + i\frac{(1-i)}{6}\right]$$

$$\mathrm{tr}\rho S_y = \frac{\hbar}{6}$$

$$\mathrm{tr}\rho S_z = \frac{\hbar}{2}\mathrm{tr}\rho\left(|+\rangle\langle+| - |-\rangle\langle-|\right) = \frac{\hbar}{2}\left(\frac{2}{3} - \frac{1}{3}\right) = \frac{\hbar}{6}.$$

$$\rho(t) = \sum_i w_i \, |\alpha^{(i)}, t_0; t\rangle\langle\alpha^{(i)}, t_0; t|$$

$$= e^{-i\frac{H(t-t_0)}{\hbar}} \rho(0)\, e^{iH(t-t_0)/\hbar}$$

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho]. \longleftrightarrow \frac{\partial\rho_{cl}}{\partial\tau} = -\{\rho_{cl}, H\}_{PB}.$$

Liouville

Continuum ρ's

$$[A] = \int d^3x' d^3x'' \langle x''|\rho|x'\rangle\langle x'|A|x''\rangle$$

$$\langle x''|\rho|x'\rangle = \langle x''|\sum_i w_i |\alpha_i\rangle\langle\alpha_i|x'\rangle$$

$$= \sum_i w_i \psi_i(x'')\psi_i^*(x').$$

Stat. Mech.

chaos $\rho \doteq \frac{1}{N}$ order $\rho^2 = \rho$; $\rho = |\psi\rangle\langle\psi|$.

$$\sigma = -\mathrm{tr}(\rho\ln\rho)$$

$$= -\sum_k \rho_{kk}\ln(\rho_{kk}) \geqslant 0 .$$

Chaos $\sigma = - \operatorname{tr} \frac{1}{N} \ln \frac{1}{N}$

$$= - \sum_{k=1}^{N} \frac{1}{N} \ln \frac{1}{N} = - \ln \frac{1}{N} = \ln N.$$

Order $\sigma = - \operatorname{tr} |\psi\rangle\langle\psi| \ln |\psi\rangle\langle\psi|$

$$= - \sum_{k} \rho_{ii} \ln \rho_{ii} \; ; \; \rho_{ii} = \begin{cases} 0 \\ 1 \end{cases}$$

$\sigma = 0$.

Entropy

$$S = k\sigma = -k \operatorname{tr}(\rho \ln \rho).$$

Equilibrium

$\frac{\partial \rho}{\partial t} = 0$ so $[\rho, H] = 0$

$U = \operatorname{tr} \rho H$ constant.

Let's maximize S subject to the constraints

$[\rho, H] = 0$, $\operatorname{tr} \rho = 1$, and $\operatorname{tr} \rho H = U$.

Use energy basis $H|i\rangle = E_i |i\rangle$ $\rho = \sum \rho_i |i\rangle\langle i|$

$$Q = -k\,\mathrm{tr}\rho\ln\rho + \lambda(\mathrm{tr}\rho - 1) + \mu(\mathrm{tr}\rho H - u)$$

$$Q = -k\sum_i \rho_i \ln\rho_i + \lambda\left(\sum \rho_i - 1\right) + \mu\left(\sum \rho_i E_i - u\right)$$

$$0 = -k\ln\rho_i - k + \lambda + E_i\mu$$

$$0 = \sum \rho_i - 1 \qquad\qquad 0 = \sum \rho_i E_i - u$$

$$k\ln\rho_i = \lambda + \mu E_i - k$$

$$\ln\rho_i = \frac{1}{k}(\lambda + \mu E_i - k)$$

$$\rho_i = e^{(\lambda + \mu E_i - k)/k}$$

$$1 = \sum \rho_i = \sum_i e^{(\lambda + \mu E_i - k)/k}$$

So $e^{\lambda} = \left[\sum_i e^{(\mu E_i - k)/k}\right]^{-1}$ and

$$\rho_i = \frac{e^{(\mu E_i - k)/k}}{\sum_i e^{(\mu E_i - k)/k}} = \frac{e^{\mu E_i/k}}{\sum_i e^{\mu E_i/k}}$$

We let $\beta = -\frac{\mu}{k} = \frac{1}{kT}$ and then

$$\rho = \frac{e^{-\beta H}}{\mathrm{tr}\, e^{-\beta H}}.$$

Let $Z = \mathrm{tr}\, \rho e^{-\beta H}$. Then

$$\rho = \frac{e^{-\beta H}}{Z(\beta)}.$$

$$[A] = \frac{\mathrm{tr}\, e^{-\beta H} A}{\mathrm{tr}\, e^{-\beta H}}$$

$$[E] = U = \frac{\mathrm{tr}\, e^{-\beta H} H}{\mathrm{tr}\, e^{-\beta H}} = -\frac{\partial}{\partial \beta} \ln\left(\mathrm{tr}\, e^{-\beta H}\right)$$

$$U = -\frac{\partial}{\partial \beta} \ln Z(\beta).$$

$$\beta = \frac{1}{kT}.$$

As $T \to \infty$ $\quad \rho \to \frac{1}{N}$ chaos.

As $T \to 0$ $\quad \rho \to |0\rangle\langle 0|$ ground state.

If $H = +\omega S_z$

$$\rho \doteq \frac{e^{-\beta\omega S_z}}{\mathrm{tr}\, e^{-\beta\omega S_z}} \doteq \frac{e^{-\beta\frac{\hbar\omega}{2}\sigma_z}}{\mathrm{tr}\, e^{-\beta\frac{\hbar\omega}{2}\sigma_z}}$$

$$= \frac{\begin{pmatrix} e^{-\beta\hbar\omega/2} & 0 \\ 0 & e^{\beta\hbar\omega/2} \end{pmatrix}}{e^{\beta\frac{\hbar\omega}{2}} + e^{-\beta\hbar\omega/2}}$$

So $\mathrm{tr}\,\rho S_x = \mathrm{tr}\,\rho S_y = 0$ but

$$\mathrm{tr}\,\rho S_z = \frac{\hbar}{2}\left(\frac{e^{-\beta\hbar\omega/2} - e^{\beta\hbar\omega/2}}{e^{-\beta\hbar\omega/2} + e^{\beta\hbar\omega/2}}\right) = -\frac{\hbar}{2}\tanh\beta\frac{\hbar\omega}{2}$$

If

$$\frac{e}{m_e c}[S_z] = \chi B, \quad \text{then}$$

$$\chi = \frac{|e|\hbar}{2 m_e c B}\tanh\left(\beta\frac{\hbar\omega}{2}\right).$$

Evals and Evecs of $\vec{J}^2$

$$J^2 = J_i^2$$

$$[J^2, J_j] = J_i [J_i, J_j] + [J_i, J_j] J_i$$

$$= \sum_{ik} \left(J_i \, i\hbar \epsilon_{ijk} J_k + i\hbar \epsilon_{ijk} J_k J_i \right)$$

$$= i\hbar \sum_{ik} J_i J_k \left(\epsilon_{ijk} + \epsilon_{kji} \right)$$

$$= i\hbar \sum_{ik} J_i J_k \left(\epsilon_{ijk} - \epsilon_{ijk} \right) = 0.$$

So we diagonalize J^2 and J_3.

$$J^2 |ab\rangle = a |ab\rangle$$

$$J_3 |ab\rangle = b |ab\rangle.$$

Let $J_\pm = J_1 \pm i J_2$ ladders

$$[J_+, J_-] = [J_1 + iJ_2, J_1 - iJ_2]$$

$$= -i [J_1, J_2] + i [J_2, J_1] = -2i [J_1, J_2]$$

$$= -2i \, i\hbar \epsilon_{123} J_3 = 2\hbar J_3$$

$$[J_3, J_\pm] = [J_3, J_1 \pm i J_2] = i\hbar \epsilon_{312} J_2 \pm i \, i\hbar \epsilon_{321} J_1$$

$$[J_3, J_\pm] = i\hbar J_2 \mp \hbar \epsilon_{321} J_1 = i\hbar J_2 \pm \hbar J_1$$
$$= \pm\hbar (J_1 \pm i J_2) = \pm\hbar J_\pm$$
$$[J^2, J_\pm] = 0.$$

$$J_3 (J_\pm |ab\rangle) = ([J_3, J_\pm] + J_\pm J_3)|ab\rangle$$
$$= (\pm\hbar J_\pm + J_\pm J_3)|ab\rangle$$
$$= (\pm\hbar J_\pm + J_\pm b)|ab\rangle$$
$$= (b \pm \hbar)(J_\pm |ab\rangle) \qquad !$$

Digression:

$[J_3, J_\pm] = \pm\hbar J_\pm$ is like

$[x_i, \mathcal{J}(\ell)] = [x_i, e^{-i p\cdot\ell/\hbar}] = \ell_i e^{-ip\cdot\ell/\hbar}$ (which is just $e^{i\vec{p}\cdot\vec{\ell}/\hbar}\, \vec{x}\, e^{-i\vec{p}\cdot\vec{\ell}/\hbar} = \vec{x} + \vec{\ell}$).

These are like $[a^\dagger a, a^\dagger] = a^\dagger$, $[a^\dagger a, a] = -a$.

$$J^2 (J_\pm |ab\rangle) = J_\pm J^2 |ab\rangle = a (J_\pm |a, b\rangle)$$

So $J_\pm$ changes b to $b \pm \hbar$ but leaves a alone.

$$J_\pm |a,b\rangle = c_\pm |a, b\pm\hbar\rangle.$$

We'll show that $a \geq b^2$.

$$\vec{J}^2 - J_3^2 = J_1^2 + J_2^2 = \frac{1}{2}\{J_1 \pm iJ_2, J_1 \mp iJ_2\}$$

$$= \frac{1}{2}\{J_+, J_+^\dagger\}$$

Now $(\langle\psi|J_+^\dagger)(J_+|\psi\rangle) \geq 0$ and $(\langle\psi|J_+)(J_+^\dagger|\psi\rangle) \geq 0$.

So

$$\langle ab|\, J^2 - J_3^2 \,|ab\rangle \geq 0$$

$$\parallel$$

$$a - b^2 \geq 0$$

So given a, there must be a b_{max} and a b_{min} such that

$$J_+ |a\ b_{max}\rangle = 0$$

and

$$J_- |a\ b_{min}\rangle = 0$$

Also $\quad J_- J_+ |a, b_{max}\rangle = 0$ or

$$(J_1 - iJ_2)(J_1 + iJ_2)|a, b_{max}\rangle = 0$$

$$= \left(J_1^2 + J_2^2 + i[J_1, J_2]\right)|a\,b_{max}\rangle$$

$$= \left(J_1^2 + J_2^2 - \hbar J_3\right)|a, b_{max}\rangle$$

$$= \left(\vec{J}^2 - J_3^2 - \hbar J_3\right)|a, b_{max}\rangle = \left(a - b^2 - \hbar b\right)|a, b_{max}\rangle = 0$$

So $a - b_m^2 - \hbar b_m = 0$ or

$$a = b_{max}(b_{max} + \hbar),$$

Also

$$0 = J_+ J_- |a, b'\rangle \qquad b' = b_{min}$$

$$= (J_1 + iJ_2)(J_1 - iJ_2)|a, b'\rangle$$

$$= \left(J_1^2 + J_2^2 - i[J_1, J_2]\right)|a, b'\rangle$$

$$= \left(J^2 - J_3^2 + \hbar J_3\right)|a, b'\rangle$$

$$= (a - b'^2 + \hbar b')|a, b'\rangle. \qquad \text{So}$$

$$a = b_{min}(b_{min} - \hbar).$$

$$b_{max}(b_{max} + \hbar) = b_{min}(b_{min} - \hbar)$$

$$b^2 + \hbar b - b'(b' - \hbar) = 0$$

$$b = \frac{-\hbar \pm \sqrt{\hbar^2 + 4b'(b' - \hbar)}}{2} = \frac{-\hbar \pm \sqrt{(2b' - \hbar)^2}}{2}$$

$$b = \frac{-\hbar \pm (2b' - \hbar)}{2} = \begin{cases} b' - \hbar \\ -b' \end{cases}$$

So either $b_{max} = b_{min} - \hbar$ (absurd) or

$$b_{max} = -b_{min} \geq 0$$

and $\quad -b_{max} \le b \le b_{max}$.

Ladders $\Rightarrow \quad b_{max} = b_{min} + n\hbar$

So $\quad b_{max} = \frac{n\hbar}{2} \equiv j\hbar$

So $\quad j = \frac{n}{2}$ is $\quad 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$

$b_{max} = j\hbar$

$a = b_{max}(b_{max} + \hbar) = \hbar j(\hbar j + \hbar)$

$= \hbar^2 j(j+1)$.

Define $\quad b = m\hbar$

m can be $\quad -j, -j+1, \cdots j-1, j \qquad (2j+1)$

$\vec{J}^2 |j,m\rangle = j(j+1)\hbar^2 |j,m\rangle$

$J_z |j,m\rangle = m\hbar |jm\rangle$.

$j = 0, \frac{1}{2}, 1, \cdots$

$m = -j, -j+1, \cdots, j-1, j \quad : \quad 2j+1$ states matrix elements

$$\langle j'm'|\vec{J}^2|jm\rangle = j'(j'+1)\hbar^2\delta_{j'j}\,\delta_{m'm}$$

$$\langle j'm'|J_z|jm\rangle = m\hbar\,\delta_{j'j}\,\delta_{m'm}$$

$$\langle jm|J_+^\dagger J_+|jm\rangle = \langle jm|J_-J_+|jm\rangle$$

$$= \langle jm|\vec{J}^2 - J_3^2 - \hbar J_3|jm\rangle = \hbar^2[j(j+1)-m^2-m]$$

$$= \hbar^2[j(j+1)-m(m+1)] = \hbar^2(j-m)(j+m+1)$$

Clearly

$$J_+|jm\rangle = c^+_{jm}|j,m+1\rangle$$

$$|c^+_{jm}|^2 = \hbar^2(j-m)(j+m+1)$$

So we may choose phases such that

$$J_+|jm\rangle = \hbar\sqrt{(j-m)(j+m+1)}\;|j,m+1\rangle.$$

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It follows that

$$\langle jm|J_- = \langle j\,m+1|\left(\hbar\sqrt{(j-m)(j+m+1)}\right)$$

So

$$\langle j'm'|J_-|jm\rangle = \langle j'\,m'+1|jm\rangle\,\hbar\sqrt{(j'-m')(j'+m'+1)}$$

$$= \delta_{j'j}\,\delta_{m'+1,m}\,\hbar\sqrt{(j'-m')(j'+m'+1)}$$

$$= \delta_{j'j}\,\delta_{m',m-1}\,\hbar\sqrt{(j-m+1)(j+m)}$$

$$J_-|jm\rangle = \hbar\sqrt{(j+m)(j-m+1)}\;|j,m-1\rangle.$$

reps. of rots.

$$D^{(j)}_{m'm}(R) = \langle jm'|D(R)|jm\rangle = \langle jm'|e^{-i\frac{\mathbf{J}\cdot\hat{n}\phi}{\hbar}}|jm\rangle$$

called Wigner functions.

$$J^2 D(R) = J^2 e^{-i\frac{\mathbf{J}\cdot\hat{n}\phi}{\hbar}} = e^{-i\frac{\mathbf{J}\cdot\hat{n}\phi}{\hbar}} J^2 = D(R)J^2$$

So

$$J^2 D(R)|jm\rangle = D(R)J^2|jm\rangle = j(j+1)\hbar^2 D(R)|jm\rangle$$

So $D(R)$ does not mix j's.

$$\langle j'm'|D(R)|jm\rangle = \begin{pmatrix} [\;] & & \\ & [\;\;] & \\ & & [\;\;\;] \end{pmatrix}$$

is reducible; it breaks up into 1×1, 2×2, 3×3, etc. square matrices.

$D^{(j)}_{m'm}(R)$ is the $(2j+1)\times(2j+1)$ block

called irreducible rep of rot group

$$D^{(j)}(1) = 1 \qquad \left[D^{(j)}(R)\right]^{-1} = D^{(j)}[R^{-1}]$$
$$(\phi \to -\phi).$$

$$\sum_{m'} D^{j}_{m''m'}(R_1)\, D^{j}_{m'm}(R_2)$$

$$= \sum_{m'} \langle j m'' | D(R_1) | j m' \rangle \langle j m' | D(R_2) | j m \rangle$$

$$= \langle j m'' | D(R_1) D(R_2) | j m \rangle$$

$$= \langle j m'' | D(R_1 R_2) | j m \rangle$$

$$= D^{(j)}_{m''m}(R_1 R_2).$$

$$D(R)^{-1} = \left[e^{-i\frac{J\cdot\hat{n}\phi}{\hbar}}\right]^{-1} = e^{i\frac{J\cdot\hat{n}\phi}{\hbar}}$$

$$= \left[e^{-i\frac{J\cdot\hat{n}\phi}{\hbar}}\right]^{\dagger} = D(R)^{\dagger}$$

So $D(R)^{\dagger} = D(R)^{-1}$; $D(R)$ is unitary. Hence

$$D^{(j)}_{m'm}(R^{-1}) = D^{(j)}_{mm'}(R)^{*}$$

$$\parallel$$

$$\langle j m' | D(R^{-1}) | j m \rangle = \langle j m' | D^{\dagger}(R) | j m \rangle = \langle j m | D(R) | j m' \rangle^{*}$$

Euler again

$$D^{(j)}_{m'm}(\alpha\beta\gamma) = \langle j m' | e^{-i\frac{J_3 \alpha}{\hbar}} e^{-i\frac{J_2 \beta}{\hbar}} e^{-i\frac{J_3 \gamma}{\hbar}} | j m \rangle$$

$$= e^{-i(m'\alpha + m\gamma)} \langle j m' | e^{-i\frac{J_2 \beta}{\hbar}} | j m \rangle$$

$$d^{j}_{m'm}(\beta) \equiv \langle j m' | e^{-i\frac{J_2 \beta}{\hbar}} | j m \rangle .$$

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0 | 1

For $j = \frac{1}{2}$

$$e^{-i\frac{J_2 \beta}{\hbar}} = \cos\frac{\beta}{2} - i\sigma_2 \sin\frac{\beta}{2}$$

$$d^{(\frac{1}{2})} = \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\frac{\beta}{2} & \cos\beta/2 \end{pmatrix}$$

$j = 1$

$$e^{-i\frac{\beta J_2}{\hbar}} = 1 - i\frac{J_2}{\hbar}\sin\beta - (1-\cos\beta)\frac{J_2^2}{\hbar^2}$$

which in the $|jm\rangle$ basis gives Eq. (3.5.57).

Orbital $\vec{L}$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$[L_i, L_j] = [\epsilon_{irs} r_r p_s, \epsilon_{jkl} r_k p_l]$$

$$[r_r p_s, r_k p_l] = r_r [p_s, r_k p_l] + [r_r, r_k p_l] p_s$$

$$= r_r(-i\hbar \delta_{sk}) p_l + i\hbar \delta_{rl} r_k p_s$$

$$[L_i, L_j] = \epsilon_{irs} \epsilon_{jkl} i\hbar (\delta_{rl} r_k p_s - \delta_{sk} r_r p_l)$$

$$= i\hbar \left[\epsilon_{irs} \epsilon_{jkr} r_k p_s - \epsilon_{irs} \epsilon_{jsl} r_r p_l\right]$$

$$= i\hbar \left[\epsilon_{irs} \epsilon_{jls} r_r p_l - \epsilon_{ris} \epsilon_{rjk} r_k p_s\right]$$

$$= i\hbar \left[(\delta_{ij} \delta_{rl} - \delta_{il} \delta_{rj}) r_r p_l - (\delta_{ij} \delta_{sk} - \delta_{ik} \delta_{sj}) r_k p_s\right]$$

$$= i\hbar \left[\delta_{ij} \vec{r} \cdot \vec{p} - r_j p_i - \delta_{ij} r \cdot p + r_i p_j\right]$$

$$= i\hbar (r_i p_j - r_j p_i)$$

$$i\hbar \epsilon_{ijk} L_k = i\hbar \epsilon_{ijk} \epsilon_{klm} r_l p_m$$

$$= i\hbar (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) r_l p_m$$

$$= i\hbar (r_i p_j - r_j p_i)$$

Spherical coordinates

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times \frac{\hbar}{i} \vec{\nabla}$$

$$\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin\theta} \frac{\partial}{\partial \phi}$$

$$\vec{L} = r\hat{r} \times \frac{\hbar}{i} \left[\hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin\theta} \frac{\partial}{\partial \phi} \right]$$

$$= \frac{\hbar}{i} \left(\hat{r} \times \hat{\theta} \frac{\partial}{\partial \theta} + \frac{\hat{r} \times \hat{\phi}}{\sin\theta} \frac{\partial}{\partial \phi} \right) = \frac{\hbar}{i} \left(\hat{\phi} \frac{\partial}{\partial \theta} - \frac{\hat{\theta}}{\sin\theta} \frac{\partial}{\partial \phi} \right)$$

$$\hat{r} = \frac{\vec{r}}{r} \qquad \hat{\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

$$\hat{\phi} = \hat{r} \times \hat{\theta} = (-\sin\phi, \cos\theta, 0) \qquad \text{so}$$

$$\vec{L} = \frac{\hbar}{i} \left[(-\sin\phi, \cos\phi, 0) \frac{\partial}{\partial \theta} - (\cot\theta\cos\phi, \cot\theta\sin\phi, -1) \frac{\partial}{\partial \phi} \right]$$

$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \qquad L_x = -\frac{\hbar}{i} \left(\sin\phi \frac{\partial}{\partial \theta} + \cot\theta\cos\phi \frac{\partial}{\partial \phi} \right)$$

$$L_y = \frac{\hbar}{i} \left(\cos\phi \frac{\partial}{\partial \theta} - \cot\theta\sin\phi \frac{\partial}{\partial \phi} \right)$$

$$L_+ = L_x + iL_y = \frac{\hbar}{i} \left[(i\cos\phi + i^2\sin\phi) \frac{\partial}{\partial \theta} - (\cos\phi + i\sin\phi)\cot\theta \frac{\partial}{\partial \phi} \right]$$

$$L_+ = \frac{\hbar}{i} e^{i\phi} \left(i \frac{\partial}{\partial \theta} - \cot\theta \frac{\partial}{\partial \phi} \right)$$

$$L_- = -\frac{\hbar}{i} e^{-i\phi}\left(i\frac{\partial}{\partial\theta} + \cot\theta \frac{\partial}{\partial\phi}\right)$$

$$L_\pm = \pm\frac{\hbar}{i} e^{\pm i\phi}\left(i\frac{\partial}{\partial\theta} \mp \cot\theta \frac{\partial}{\partial\phi}\right)$$

Now

$$\frac{\partial\hat{\phi}}{\partial\phi} = \frac{\partial}{\partial\phi}(-\sin\phi, \cos\phi, 0) = -(\cos\phi, \sin\phi, 0)$$

$$\frac{\partial\hat{\theta}}{\partial\theta} = \frac{\partial}{\partial\theta}(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

$$= -(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = -\hat{r}$$

$$\frac{\partial\hat{\phi}}{\partial\theta} = 0 \qquad \hat{\theta}\cdot\frac{\partial\hat{\phi}}{\partial\phi} = -\left[\cos\theta\cos^2\phi + \cos\theta\sin^2\phi\right] = -\cos\theta$$

$$\frac{\partial\hat{\theta}}{\partial\phi} = \frac{\partial}{\partial\phi}(-\cos\theta\sin\phi, \cos\theta\cos\phi, 0) \text{ so } \quad \hat{\theta}\cdot\frac{\partial\hat{\theta}}{\partial\phi} = 0$$

whence

$$\vec{L}\cdot\vec{L} = -\hbar^2\left(\hat{\phi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(\hat{\phi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$= -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} - \frac{1}{\sin\theta}\left(\hat{\phi}\cdot\frac{\partial\hat{\theta}}{\partial\theta}\right)\frac{\partial}{\partial\phi} - \frac{1}{\sin\theta}\left(\hat{\theta}\cdot\frac{\partial\hat{\phi}}{\partial\phi}\right)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

So $[L_i, L_j] = i\hbar\,\epsilon_{ijk} L_k = i\hbar\,(r_i p_j - r_j p_i)$.

Now consider

$$1 - i\frac{\delta\phi}{\hbar} L_z = 1 - i\frac{\delta\phi}{\hbar}(x p_y - y p_x)$$

$$\left(1 - i\frac{\delta\phi L_z}{\hbar}\right)|x\,y\,z\rangle = |x - y\,\delta\phi,\; y + x\,\delta\phi,\; z\rangle$$

since

$$\left(1 - i\frac{a\cdot p}{\hbar}\right)|\vec{x}\rangle = |\vec{x} + \vec{a}\rangle$$

by (1.6.12) and (1.6.32). So $\left(1 - i\frac{\delta\phi L_z}{\hbar}\right)$ just rotates $|\vec{x}\rangle$ by $\delta\phi$ about $\hat{z}$.
counter-clockwise, right-handed rotation,

$$\langle x\,y\,z|\left(1 - i\frac{\delta\phi}{\hbar}L_z\right)|\alpha\rangle = \langle x + y\,\delta\phi,\; y - x\,\delta\phi,\; z|\alpha\rangle$$

ie

$$\langle r\,\theta\,\phi|\left(1 - i\frac{\delta\phi L_z}{\hbar}\right)|\alpha\rangle = \langle r, \theta, \phi - \delta\phi|\alpha\rangle$$

$$\doteq \langle r\,\theta\,\phi|\alpha\rangle - \delta\phi\,\frac{\partial}{\partial\phi}\langle r\,\theta\,\phi|\alpha\rangle$$

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Thus

$$+\frac{i}{\hbar}L_z = \frac{\partial}{\partial\phi} \quad \text{or} \quad L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}.$$

Now $\hat{x}$:

$$\left(1 - i\frac{\delta\phi_x}{\hbar}L_x\right)|x\,y\,z\rangle = \left(1 - i\frac{\delta\phi_x}{\hbar}(y p_z - z p_y)\right)|x\,y\,z\rangle$$

$$= |x,\; y - \delta\phi_x z,\; z + \delta\phi_x y\rangle$$

So

$$\langle xyz| \left(1 - i\frac{\delta\phi_x}{\hbar}L_x\right)|\alpha\rangle = \langle x, y + \delta\phi_x z, z - \delta\phi_x y|\alpha\rangle$$

Also

$$\left(1 - i\frac{\delta\phi_y}{\hbar}L_y\right)|xyz\rangle = \left(1 - i\frac{\delta\phi_y}{\hbar}(zp_x - xp_z)\right)|xyz\rangle$$

$$= |x + z\delta\phi_y, y, z - x\delta\phi_y\rangle \qquad \text{so}$$

$$\langle xyz|\left(1 - i\frac{\delta\phi_y}{\hbar}L_y\right)|\alpha\rangle = \langle x - z\delta\phi_y, y, z + x\delta\phi_y|\alpha\rangle \text{ So}$$

$$-i\frac{\delta\phi_x}{\hbar}\langle xyz|L_x|\alpha\rangle = \langle x, y + \delta\phi_x z, z - \delta\phi_x y|\alpha\rangle - \langle xyz|\alpha\rangle$$

$$= (0, z\delta\phi_x, -y\delta\phi_x)\cdot\vec{\nabla}\langle xyz|\alpha\rangle$$

So

$$L_x \doteq \frac{\hbar}{i}(0, -z, y)\cdot\vec{\nabla}$$

$$= \frac{\hbar}{i}\; r(0, -\cos\theta, +\sin\theta\sin\phi)\cdot\left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right]$$

So since $\hat{y} = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$ and $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$ we have

$$L_x = \frac{\hbar}{i}\, r(0, -\cos\theta, \sin\theta\sin\phi)\cdot\left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right]$$

$$r(0, -\cos\theta, \sin\theta\sin\phi) = r\left[-\cos\theta\,\hat{y} + \sin\theta\sin\phi\,\hat{z}\right]$$

$$= -r\cos\theta\left[\hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi\right]$$
$$+ r\sin\theta\sin\phi\left[\hat{r}\cos\theta - \hat{\theta}\sin\theta\right]$$

So

$$r(0,-\cos\theta, \sin\theta\sin\phi) = \hat{\theta}\left(-r\cos^2\theta\sin\phi - r\sin^2\theta\sin\phi\right)$$
$$- r\cos\theta\cos\phi\,\hat{\phi}$$
$$= -r\hat{\theta}\sin\phi \quad - r\cos\theta\cos\phi\,\hat{\phi}$$

So

$$L_x \doteq \frac{\hbar}{i} r\left[-\hat{\theta}\sin\phi - \hat{\phi}\cos\theta\cos\phi\right]\cdot\left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right]$$
$$= \frac{\hbar}{i}\left[-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right]$$

Similarly using $\hat{x} = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$

$$-\frac{i}{\hbar}\delta\phi_y \langle xyz|L_y|\alpha\rangle = -\langle xyz|\alpha\rangle + \langle x - z\,\delta\phi_y, y, z + x\,\delta\phi_y|\alpha\rangle$$
$$= (-z\,\delta\phi_y, 0, +x\,\delta\phi_y)\cdot\vec{\nabla}\langle xyz|\alpha\rangle$$

So

$$L_y \doteq \frac{\hbar}{i}(z, 0, -x)\cdot\vec{\nabla}$$
$$= \frac{\hbar}{i}\left[r\cos\theta\,\hat{x} - r\sin\theta\cos\phi\,\hat{z}\right]\cdot\vec{\nabla}$$
$$= \frac{\hbar}{i}\left[r\cos\theta\left(\hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi\right)\right.$$
$$\left. - r\sin\theta\cos\phi\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)\right]\cdot\left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right]$$

So

$$L_y \doteq \frac{\hbar}{i}\left[\hat{\theta}\, r\cos\phi - \hat{\phi}\, r\cos\theta\sin\phi\right]\cdot\left(\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$= \frac{\hbar}{i}\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

So

$$L_\pm = L_x \pm iL_y \doteq \frac{\hbar}{i}\left[\left(-\sin\phi \pm i\cos\phi\right)\frac{\partial}{\partial\theta} + \cot\theta\left(-\cos\phi \mp i\sin\phi\right)\frac{\partial}{\partial\phi}\right]$$

$$= \frac{\hbar}{i}\left[\pm i\left(\cos\phi \pm i\sin\phi\right)\frac{\partial}{\partial\theta} - \cot\theta\left(\cos\phi \pm i\sin\phi\right)\frac{\partial}{\partial\phi}\right]$$

$$L_\pm \doteq \frac{\hbar}{i}e^{\pm i\phi}\left(\pm i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)$$

Now

$$\langle\vec{x}|L^2|\alpha\rangle = \langle\vec{x}|\tfrac{1}{2}L_+L_- + \tfrac{1}{2}L_-L_+ + L_z^2|\alpha\rangle$$

$$= \frac{1}{2}\left(-\hbar^2\right)e^{i\phi}\left(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)e^{-i\phi}\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)\langle x|\alpha\rangle$$

$$-\frac{\hbar^2}{2}e^{-i\phi}\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)e^{i\phi}\left(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)\langle x|\alpha\rangle$$

$$-\hbar^2\frac{\partial^2}{\partial\phi^2}\langle\vec{x}|\alpha\rangle.$$

$$\langle x' | L^2 | \alpha \rangle = -\hbar^2 \left\{ \frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \cot^2\theta \frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\varphi^2} \right\} \langle x | \alpha \rangle$$

$$= -\hbar^2 \left\{ \left(\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} \right) \frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} \right] \langle x | \alpha \rangle$$

$$= -\hbar^2 \left\{ \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} \right] \langle x | \alpha \rangle.$$

Now

$$L^2 = (X \wedge P)^2 = \epsilon_{ijk} \epsilon_{irs} \, x_j p_k x_r p_s$$

$$= \left(\delta_{jr} \delta_{ks} - \delta_{js} \delta_{kr} \right) x_j p_k x_r p_s$$

$$= x_j p_k x_j p_k - x_j p_k x_k p_j$$

$$= x_j \left(x_j p_k + [p_k, x_j] \right) p_k - x_j \left(x_k p_k + [p_k, x_k] \right) p_j$$

$$= x^2 p^2 - i\hbar \, x \cdot p - x_j \, x \cdot p \, p_j + 3 i\hbar \, x \cdot p$$

$$= x^2 p^2 + 2 i\hbar \, x \cdot p - x_j \left(p_j \, x \cdot p + [x \cdot p, p_j] \right)$$

$$= x^2 p^2 - (x \cdot p)^2 + 2 i\hbar \, x \cdot p - i\hbar \, x \cdot p$$

$$= x^2 p^2 - (x \cdot p)^2 + i\hbar \, x \cdot p$$

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$$\langle x' | x \cdot p | \alpha \rangle = x' \frac{\hbar}{i} \nabla' \langle x' | \alpha \rangle = \frac{\hbar}{i} \vec{r}' \cdot \left[\hat{r}' \frac{\partial}{\partial r'} + \cdots \right] \langle x' | \alpha \rangle$$

$$\langle \vec{r} \,|\, x\cdot p \,|\, \alpha \rangle = \frac{\hbar}{i}\, r \frac{\partial}{\partial r} \langle \vec{r} \,|\, \alpha \rangle$$

And

$$\langle \vec{r} \,|\, (x\cdot p)^2 \,|\, \alpha \rangle = \frac{\hbar}{i}\, r \frac{\partial}{\partial r} \frac{\hbar}{i} r \frac{\partial}{\partial r} \langle \vec{r} \,|\, \alpha \rangle$$

$$= -\hbar^2 \left(r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} \right) \langle \vec{r} \,|\, \alpha \rangle$$

So

$$\langle \vec{r} \,|\, \vec{L}^2 \,|\, \alpha \rangle = \langle \vec{r} \,|\, \vec{x}^2 \vec{p}^2 - (\vec{x}\cdot\vec{p})^2 + i\hbar\, \vec{x}\cdot\vec{p} \,|\, \alpha \rangle$$

$$= r^2 \langle \vec{r} \,|\, \vec{p}^2 \,|\, \alpha \rangle + \hbar^2 \left(r^2 \frac{\partial^2}{\partial r^2} + 2r \frac{\partial}{\partial r} \right) \langle \vec{r} \,|\, \alpha \rangle$$

whence

$$\langle \vec{r} \,|\, \frac{\vec{p}^2}{2m} \,|\, \alpha \rangle = -\frac{\hbar^2}{2m} \vec{\nabla}^2 \langle \vec{r} \,|\, \alpha \rangle$$

$$= -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) \langle \vec{r} \,|\, \alpha \rangle + \frac{1}{2mr^2} \langle \vec{r} \,|\, \vec{L}^2 \,|\, \alpha \rangle$$

$$= -\frac{\hbar^2}{2m} \left[\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) \langle \vec{r} \,|\, \alpha \rangle - \frac{1}{\hbar^2 r^2} \langle \vec{r} \,|\, \vec{L}^2 \,|\, \alpha \rangle \right]$$

As we saw in A.5, for spherically symmetric potentials energy efunctions are of the form

$\langle \vec{r} \,|\, n l m \rangle = R_{nl}(r) \, Y_{lm}(\theta, \phi)$

radial quantum number or energy of free particle

This is because $-\frac{\hbar^2}{2m} \vec{\nabla}^2 + V(r)$ is separable

in spherical coordinates. Equivalently

$$H = \frac{p^2}{2m} + V(r) \quad \text{so} \quad [H, \vec{L}] = 0 \quad \text{so}$$

we can have simultaneous ekets of $H, \vec{L}^2, L_z$.

Isolate $\langle \hat{n} \,|\, l m \rangle = \langle \theta, \phi \,|\, l m \rangle = Y_{lm}(\theta, \phi) \equiv Y_l^m(\theta, \phi)$

$|\hat{n}\rangle = |\theta, \phi\rangle$ is a direction eket.

$Y_l^m(\theta, \phi)$ is amplitude for $|l m\rangle$ to have θ, ϕ.

$$L_z |l, m\rangle = m\hbar |l m\rangle$$

$$\langle \hat{n} | L_z | l m \rangle = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \langle \hat{n} | l m \rangle = \frac{\hbar}{i} \frac{\partial}{\partial \phi} Y_l^m(\theta, \phi)$$

So $\quad = m\hbar \langle \hat{n} | l m \rangle = m\hbar \, Y_l^m(\theta, \phi)$

So $\quad Y_l^m(\theta, \phi) \sim e^{im\phi} f(\theta).$

$$\vec{L}^2 |lm\rangle = l(l+1)\hbar^2 |lm\rangle$$

$$\langle \hat{n}|\vec{L}^2|lm\rangle = l(l+1)\hbar^2\, Y_l^m(\theta,\phi)$$

$$= -\hbar^2 \left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right] Y_l^m(\theta\phi)$$

or

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right] Y_l^m(\theta,\phi) = 0$$

or $\quad [(\sin\theta)^{-1}\partial_\theta(\sin\theta\,\partial_\theta) - m^2(\sin\theta)^{-2} + l(l+1)]\, Y_l^m(\theta,\phi) = 0$

$$\langle l'm'|lm\rangle = \delta_{ll'}\delta_{mm'}$$

$$= \int d\Omega\, \langle l'm'|\hat{n}\rangle\langle\hat{n}|lm\rangle$$

$$= \int d\Omega\, Y_{l'}^{m'*}(\theta\phi)\, Y_l^m(\theta,\phi)$$

$$= \int_{-1}^{1} d\cos\theta \int_0^{\pi} d\phi\, Y_{l'}^{m'*}(\theta\phi)\, Y_l^m(\theta\phi).$$

$$1 = \int d\Omega\, |\hat{n}\rangle\langle\hat{n}| = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} |lm\rangle\langle lm|.$$

To ~~get~~ Y_l^m, try $m = l$

$$\langle\hat{n}|L_+|ll\rangle = 0$$

$$0 = \frac{\hbar}{i} e^{i\phi}\left(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right) Y_l^l(\theta,\phi)$$

since $Y_l^l(\theta,\phi) \sim e^{il\phi}$

$$\frac{\partial}{\partial\theta} Y_l^l(\theta,\phi) = \frac{1}{i}\cot\theta \frac{\partial}{\partial\phi} Y_l^l = l\cot\theta\, Y_l^l(\theta\phi)$$

$$\frac{\partial}{\partial\theta} Y_l^l(\theta,\phi) = l\cot\theta\, Y_l^l(\theta,\phi)$$

$$\frac{dy}{d\theta} = l\cot\theta\, y \qquad\qquad \frac{dy}{y} = l\cot\theta\, d\theta$$

$$\ln y = l\int\cot\theta\, d\theta = l\int\frac{\cos\theta\, d\theta}{\sin\theta} = l\int\frac{d\sin\theta}{\sin\theta}$$

$$= l\ln\sin\theta = \ln\sin^l\theta$$

So

$$y \sim \sin^l\theta$$

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$$Y_l^l(\theta,\phi) = c_l\sin^l\theta\, e^{il\phi} = \langle\theta\phi|ll\rangle.$$

In fact

$$c_l = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)(2l)!}{4\pi}}$$

Next

$$\langle\hat{n}|l,m-1\rangle = \frac{\langle\hat{n}|L_-|l,m\rangle}{\hbar\sqrt{(l+m)(l-m+1)}}$$

$$\langle \hat{n} | l, m-1 \rangle = \frac{e^{-i\phi}}{\sqrt{(l+m)(l-m+1)}} \left(-\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \phi} \right) \langle \hat{n} | l, m \rangle$$

and so we get them all: For $m \geq 0$

$$Y_l^m(\theta,\phi) = \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l+1)}{4\pi} \frac{(l+m)!}{(l-m)!}} \frac{e^{im\phi}}{\sin^m\theta} \frac{d^{l-m}}{d(\cos\theta)^{l-m}} (\sin\theta)^{2l}$$

and

$$Y_l^{-m}(\theta,\phi) = (-1)^m \left[Y_l^m(\theta,\phi) \right]^*$$

Always, any m $\quad Y_l^m(\theta,\phi) \propto \sin\theta^{|m|} \, \mathrm{Poly}(\cos\theta)$
(order $l-|m|$.

$m = 0$:

$$Y_l^0(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}} \, P_l(\cos\theta)$$

For spin $\frac{1}{2}$ $\qquad$ m half integer

$$e^{i2\pi \frac{J_z}{\hbar}} |\alpha\rangle = e^{i2\pi m} |\alpha\rangle = -|\alpha\rangle$$

but if

$$J = r \times p \quad \text{then}$$

$$\langle \vec{r} | e^{-iJ_z \frac{2\pi}{\hbar}} |\alpha\rangle = \langle \vec{r} | \alpha \rangle,$$

So if $J = r \times p$, then j must be an integer, not a half integer.

Y's & D's

$$|\hat{n}\rangle = D(R)|\hat{z}\rangle = D(R)\sum_{\ell m} |\ell m\rangle\langle \ell m|\hat{z}\rangle$$

$$\langle \ell m'|\hat{n}\rangle = \sum_{m} \langle \ell m'|D(R)|\ell m\rangle\langle \ell m|\hat{z}\rangle.$$

But

$$\langle \ell m|\hat{z}\rangle = Y_\ell^{m*}(\theta=0,\phi)\,\delta_{m0}$$ because

$$\langle \ell m|L_z|\hat{z}\rangle = 0 = m\hbar\langle \ell m|\hat{z}\rangle. \quad \text{So}$$

$$\langle \ell m|\hat{z}\rangle = \delta_{m0} Y_\ell^{0*}(0,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\; P_\ell(1)\,\delta_{m0}$$

Since $P_\ell(1) = 1$, this is

$$\langle \ell m'|\hat{n}\rangle = \langle \ell m'|D(R)|\ell 0\rangle \sqrt{\frac{2\ell+1}{4\pi}}$$

So

$$D^{(\ell)}_{m'0}(R) = \langle \ell m'|\hat{n}\rangle\sqrt{\frac{4\pi}{2\ell+1}} = \sqrt{\frac{4\pi}{2\ell+1}}\; Y_\ell^{m'*}(\theta,\phi)$$

any R that takes $\hat{z}$ to $\hat{n}$. of $\hat{n}$

Euler case, for any γ,

$$D^{(\ell)}_{m'0}(\phi,\theta,\gamma) = \sqrt{\frac{4\pi}{2\ell+1}}\; Y_\ell^{m'*}(\theta,\phi).$$ For $m=0$

(See, A.5)

$$D^{(\ell)}_{00}(\phi,\theta,\gamma) = d^{\ell}_{00}(\theta) = P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}\; Y_\ell^{0*}(\theta,\phi) = P_\ell^0(\cos\theta).$$