

HW1: 123 56

HW2: 8910 12, 13, 14

Time

$$|\alpha, t_0; t\rangle$$

$$t > t_0$$

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$$\lim_{t \rightarrow t_0} |\alpha, t_0; t\rangle = |\alpha\rangle = |\alpha, t_0; t_0\rangle = |\alpha, t_0\rangle$$

Time evolution  $|\alpha, t_0\rangle = |\alpha\rangle \rightarrow |\alpha, t_0; t\rangle$ 

Probability is conserved, so

$$|\alpha, t_0; t\rangle = U(t, t_0) |\alpha, t_0\rangle$$

We expect that  $U(t) |\alpha\rangle$   
still satisfy

$$S_{ij} = \langle \alpha_i | \alpha_j \rangle = \langle \alpha_i | U^\dagger U | \alpha_j \rangle$$

$$|\alpha, t_0\rangle = \sum_{a'} C_{a'}(t_0) |\alpha'\rangle$$

$$|\alpha, t_0; t\rangle = \sum_{a'} C_{a'}(t) |\alpha'\rangle$$

Expect

$$\sum_{a'} P(a', t) = \sum_{a'} |C_{a'}(t)|^2 = \sum_{a'} |C_{a'}(t_0)|^2 = \sum_{a'} P(a', t_0)$$

unitarity

$$\langle \alpha, t_0; t | \alpha, t_0; t \rangle = \langle \alpha, t_0 | \alpha, t_0 \rangle = 1$$

$$U^\dagger(t, t_0) U(t, t_0) = 1$$

$$U(t_2, t_0) = U(t_2, t_1) U(t_1, t_0) \quad t_2 > t_1 > t_0$$

$$|\alpha, t_0; t_0 + dt\rangle = U(t_0 + dt, t_0) |\alpha, t_0\rangle$$

$$\lim_{dt \rightarrow 0} U(t_0 + dt, t_0) = 1$$

 $dt \rightarrow 0$ 

$$\text{Let } U(t_0 + dt, t_0) = 1 - i \Omega dt, \quad \Omega^\dagger = \Omega$$

Then if  $\Omega$  is independent of time

$$U(t_0 + dt_1 + dt_2, t_0) = (1 - i\Omega(dt_1 + dt_2))$$

$$= U(t_0 + dt_1 + dt_2, t_0 + dt_1) U(t_0 + dt_1, t_0)$$

$$= (1 - i\Omega dt_2) (1 - i\Omega dt_1)$$

$$U^\dagger(t_0 + dt, t_0) U(t_0 + dt, t_0) = (1 + i\Omega^\dagger dt) (1 - i\Omega dt)$$

$$= 1 - i(\Omega - \Omega^\dagger) dt = 1 \quad \forall \quad \Omega = \Omega^\dagger$$

The generator of time translation is the energy,

$$\Omega = \frac{H}{\hbar}$$

$$U(t_0 + dt, t_0) = 1 - i\frac{H dt}{\hbar}, \text{ like } \mathcal{T}(dx) = 1 - i\frac{\vec{p} \cdot d\vec{x}}{\hbar}$$

Same to as  $\vec{K} = \frac{\vec{P}}{\hbar}$ , do get  $\langle \dot{x} \rangle = \langle \frac{\vec{P}}{m} \rangle$ .

Sch. Eq.

$$U(t + dt, t_0) = U(t + dt, t) U(t, t_0)$$

$$= \left(1 - i\frac{H dt}{\hbar}\right) U(t, t_0) \quad \text{So}$$

$$U(t + dt, t_0) - U(t, t_0) = -i\frac{H dt}{\hbar} U(t, t_0)$$

$$\text{or } i\hbar \frac{\partial}{\partial t} u(x, t_0) = H u(x, t_0).$$

$$i\hbar \frac{\partial}{\partial t} u(x, t_0) |x, t_0\rangle = H u(x, t_0) |x, t_0\rangle$$

$$i\hbar \frac{\partial}{\partial t} |x, t_0; t\rangle = H |x, t_0; t\rangle$$

3 Cases,

$$1) \frac{\partial H}{\partial t} = 0 \quad \text{then} \quad i\hbar \frac{\partial}{\partial t} u(x, t_0) = H u(x, t_0)$$

is solved by

$$u(x, t_0) = e^{-iH(t-t_0)/\hbar}$$

Proof:

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} [u(x, t_0) - u(x, t_0)] &= i\hbar (e^{-iH(t-t_0)/\hbar} - 1) u(x, t_0) / dt \\ &= i\hbar \left(1 - i \frac{H(t-t_0)}{\hbar}\right) u(x, t_0) = H u(x, t_0). \end{aligned}$$

$$2) \frac{\partial H}{\partial t} \neq 0 \quad \text{but} \quad [H(t), H(t')] = 0, \quad \text{Then by same}$$

argument

$$u(x, t_0) = e^{-\frac{i}{\hbar} \int_{t_0}^t dt' H(t')}$$

(Case 3) Now

$$U(t, t_0) = T \left\{ e^{-\frac{i}{\hbar} \int_{t_0}^t dt' H(t')} \right\}$$

$$= e^{-\frac{i}{\hbar} \int_{t_0}^t H(t) dt} \dots e^{-\frac{i}{\hbar} H(t_0) dt}$$

So again

$$\frac{it\hbar}{dt} [U(t+dt, t_0) - U(t, t_0)] = \frac{it\hbar}{dt} \left( e^{-\frac{i}{\hbar} H(t) dt} - 1 \right) U(t, t_0)$$

$$= \frac{it\hbar}{dt} \left( 1 - \frac{i}{\hbar} H(t) dt - 1 \right) U(t, t_0)$$

$$= H(t) U(t, t_0).$$

This  $U(t, t_0)$  is given by the Dyson series

$$U(t, t_0) = 1 + \sum_{n=1}^{\infty} T \left\{ \left( -\frac{i}{\hbar} \int_{t_0}^t dt' H(t') \right)^n \right\} \frac{1}{n!}$$

$$= 1 + \sum_{n=1}^{\infty} \left( -\frac{i}{\hbar} \right)^n \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n H(t_1) \dots H(t_n).$$

Suppose  $[A, H] = 0$  then if  $A$  or  $H$  is nondeg.  
or if we choose it so that

$$H|a'\rangle = E_{a'}|a'\rangle.$$

Then

$$e^{-\frac{iHt}{\hbar}} = \sum_{a'} e^{-\frac{iHt}{\hbar}} |a' \rangle \langle a'| = \sum_{a'} e^{-\frac{iE_{a'}t}{\hbar}} |a' \rangle \langle a'|.$$

So if  $|\alpha, t_0=0\rangle = \sum |a' \rangle \langle a'| \alpha\rangle = \sum c_{a'} |a'\rangle$ , then

$$|\alpha, t_0=0; t\rangle = e^{-\frac{iHt}{\hbar}} |\alpha, t_0=0\rangle = \sum e^{-\frac{iE_{a'}t}{\hbar}} |a' \rangle \langle a'| \alpha\rangle.$$

$$\text{So } c_{a'}(t) = e^{-\frac{iE_{a'}t}{\hbar}} c_{a'}(0) = e^{-\frac{iE_{a'}t}{\hbar}} c_{a'}.$$

If  $|\alpha, t_0=0\rangle = |a'\rangle$ , then  $A$  stationary state.

$$|\alpha, t_0=0; t\rangle = |a'\rangle e^{-\frac{iE_{a'}t}{\hbar}}$$

remains an e-bet of  $A$  and  $H$  for all  $t$ ,

Observables that commute with  $H$  are constant. If  $|\alpha, t\rangle = e^{-\frac{iHt}{\hbar}} |\alpha\rangle$ , then  $\langle \alpha, t | A | \alpha, t \rangle = \langle \alpha | e^{\frac{iHt}{\hbar}} A e^{-\frac{iHt}{\hbar}} |\alpha\rangle = \langle \alpha | A | \alpha \rangle$ . And  $\langle \alpha, t | A | \alpha, t \rangle = \langle \alpha | A | \alpha \rangle$ . If  $[A, B] = [B, C] = 0$  etc.  $[A, H] = [B, H] = 0$  etc.

then

$$e^{-\frac{iHt}{\hbar}} = \sum_{k'} |k' \rangle \langle k'| e^{-\frac{iE_{k'}t}{\hbar}}$$

$$\langle k | k' \rangle = \delta_{k, k'}$$

where  $K = \{A, B, C\}$   $k' = \{a', b', c', \dots\}$ .

$$[A, H] = 0 \quad [B, H] \neq 0.$$

$$|a', t=0; t\rangle = U(t, t_0) |a'\rangle$$

$$\langle B \rangle_t = \langle a' | U^\dagger(t, t_0) B U(t, t_0) |a'\rangle$$

$$= e^{iE_{a'}t/\hbar} \langle a' | B |a'\rangle e^{-iE_{a'}t/\hbar} = \langle a' | B |a'\rangle_{t_0}.$$

The state  $|a'\rangle$  is stationary in that

$\langle a' | B |a'\rangle$  is time independent given

$$\downarrow [B, H] \neq 0.$$

Say  $|\alpha, t_0=0\rangle = \sum_{a'} c_{a'} |a'\rangle$ . Then

$$\begin{aligned} \langle B \rangle_t &= \sum_{\substack{a' \\ a''}} c_{a'}^* \langle a' | e^{iE_{a'}t/\hbar} B e^{-iE_{a''}t/\hbar} |a''\rangle c_{a''} \\ &= \sum_{a'a''} e^{-i(E_{a''}-E_{a'})t/\hbar} c_{a'}^* c_{a''} \langle a' | B |a''\rangle \end{aligned}$$

in which  $\omega_{a''a'} = \frac{E_{a''}-E_{a'}}{\hbar}$  are Bohr

frequencies.

## Spin Precession

$$H = - \frac{e}{m_e c} \vec{S} \cdot \vec{B} \quad (e < 0)$$

$B$  static, uniform,  $\vec{B} = B \hat{z}$ .

$$H = - \left( \frac{eB}{m_e c} \right) S_z$$

$$H | \hat{z}, \pm \rangle = \mp \left( \frac{eB}{m_e c} \right) \frac{\hbar}{2} | \hat{z}, \pm \rangle$$

$$E_{\pm} = \mp \frac{e \hbar B}{2 m_e c}$$

Let  $\omega = \frac{|e|B}{m_e c}$ , so that  $H = \omega S_z$ ,

$$U(t, 0) = e^{-i \omega \frac{S_z t}{\hbar}}$$

$$| \alpha \rangle = c_+ | + \rangle + c_- | - \rangle$$

$$| \alpha, t=0; t \rangle = U(t, 0) | \alpha \rangle = e^{-i \omega \frac{S_z t}{\hbar}} (c_+ | + \rangle + c_- | - \rangle)$$

$$= c_+ e^{-i \frac{\omega t}{2}} | + \rangle + c_- e^{i \frac{\omega t}{2}} | - \rangle$$

since  $H | \pm \rangle = \pm \frac{\hbar \omega}{2} | \pm \rangle$ ,

If  $|\alpha\rangle = |+\rangle$ ,  $C_+ = 1$ ,  $C_- = 0$ , then

$$u(t, 0)|+\rangle = e^{-i\omega t/2} |+\rangle \quad \text{for all } t$$

The state is stationary. All mean values of operators (without intrinsic, explicit time dependence) are constant in time.

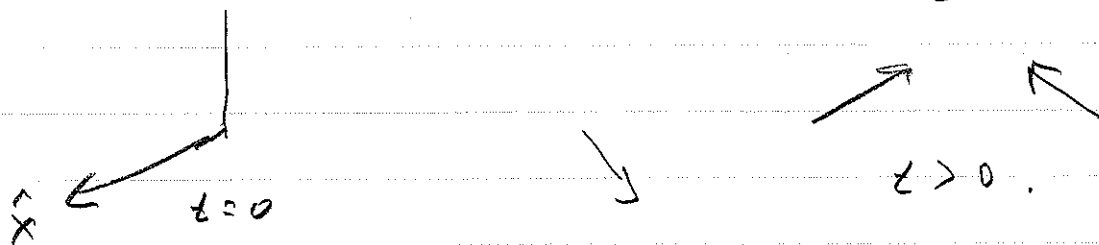
But if  $|\alpha\rangle = |\hat{X}\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ , then

$$|\alpha, t\rangle = e^{-i\omega t/2} \frac{|+\rangle}{\sqrt{2}} + e^{i\omega t/2} \frac{|-\rangle}{\sqrt{2}}, \quad P(\pm, t) = \frac{1}{2}$$

$$P(\hat{X}\pm, t) = |\langle \hat{X}\pm | \alpha, t \rangle|^2$$

$$= \left| \left( \frac{1}{\sqrt{2}} \langle - | + \frac{1}{\sqrt{2}} \langle + | \right) \left( e^{-i\omega t/2} \frac{|+\rangle}{\sqrt{2}} + e^{i\omega t/2} \frac{|-\rangle}{\sqrt{2}} \right) \right|^2$$

$$= \left| \frac{e^{-i\omega t/2}}{2} \pm \frac{e^{i\omega t/2}}{2} \right|^2 = \begin{cases} \cos^2 \frac{\omega t}{2} & + \\ \sin^2 \frac{\omega t}{2} & - \end{cases}$$



$$\langle S_x \rangle = \langle \alpha, t | \frac{\hbar}{2} (1 + X - 1 + 1 - X + 1) | \alpha, t \rangle$$

$$= \frac{\hbar}{2} \left( \frac{e^{i\omega t}}{2} + \frac{e^{-i\omega t}}{2} \right) = \frac{\hbar}{2} \cos \omega t$$

where  $\omega = \omega_+ - \omega_- = \frac{E_+ - E_-}{\hbar}$  a Bohr frequency.



$$\langle S_y \rangle = \langle \alpha, t | \frac{\hbar}{2} (-i1 + X - 1 + i1 - X + 1) | \alpha, t \rangle$$

$$= \frac{\hbar}{2} \left( -i \frac{e^{i\omega t}}{2} + i \frac{e^{-i\omega t}}{2} \right) = \frac{\hbar}{2} \sin \omega t$$

$$\langle S_z \rangle = \langle \alpha, t | \frac{\hbar}{2} (1 + X + 1 - 1 - X - 1) | \alpha, t \rangle$$

$$= \frac{\hbar}{2} \left( \frac{1}{2} - \frac{1}{2} \right) = 0.$$

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Correlation amplitude

$$C(t) = \langle \alpha | \alpha(t, 0) | \alpha \rangle = \langle \alpha | \alpha, t_0 = 0, t \rangle.$$

If  $H|\alpha\rangle = E_\alpha|\alpha\rangle$ , then  $C(t) = e^{-\frac{iE_\alpha t}{\hbar}}$ , and

$|C(t)| = 1$  for all  $t$  in a stationary state.

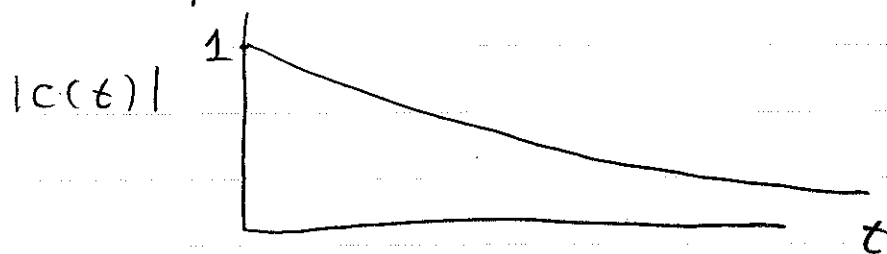
If  $H|a'\rangle = E_{a'}|a'\rangle$ , and  $|\alpha\rangle = \sum_{a'} |a'\rangle \langle a'|\alpha\rangle$ ,

$$\text{then } C(t) = \sum_{a', a''} \langle \alpha | a' \rangle \langle a' | \alpha \rangle e^{-\frac{iE_{a'} t}{\hbar}} \langle a'' | \alpha \rangle \langle \alpha | a'' \rangle$$

$$= \sum_{a'} \langle \alpha | a' \rangle \langle a' | \alpha \rangle e^{-\frac{iE_{a'} t}{\hbar}}$$

$$= \sum_{a'} e^{-\frac{iE_{a'} t}{\hbar}} |\langle a' | \alpha \rangle|^2.$$

So we expect that  $|C(t)|$  decreases from unity as  $t$  grows



Suppose  $\sum_{\alpha'} \rightarrow \int dE \rho(E)$  and  $c_{\alpha'} = \langle \alpha' | \alpha \rangle \rightarrow g(E)$ .

then

$$C(t) = \int dE |g(E)|^2 \rho(E) e^{-i \frac{Et}{\hbar}} \quad \text{with}$$

$$1 = C(0) = \int dE |g(E)|^2 \rho(E) = \langle \alpha | U(0,0) | \alpha \rangle.$$

So if  $|g|^2 \rho$  is peaked at  $E_0$ , then

$$C(t) = e^{-i \frac{E_0 t}{\hbar}} \int dE |g(E)|^2 \rho(E) e^{-i (E-E_0)t/\hbar}$$

and if  $|g(E)|^2 \rho(E) = m_0 e^{-\frac{(E-E_0)^2}{2\Delta E^2}}$ , then

$$C(t) = e^{-i \frac{E_0 t}{\hbar}} m_0 \int dE e^{-\frac{(E-E_0)^2}{2\Delta E^2} - i (E-E_0)t/\hbar}$$

$$= m_0 e^{-i \frac{E_0 t}{\hbar}} \int dE e^{-\left(\frac{E-E_0}{\sqrt{2}\Delta E} + \frac{i t \Delta E}{\sqrt{2}\hbar}\right)^2 - \left(\frac{t \Delta E}{\sqrt{2}\hbar}\right)^2}$$

shift the contour now and get

$$C(t) = m_0 e^{-iE_0 t / \hbar} e^{-\left(\frac{t \Delta E}{\sqrt{2} \hbar}\right)^2} \int dE e^{-\frac{(E-E_0)^2}{2 \Delta E^2}}$$

$$= m_0 e^{-iE_0 t / \hbar} e^{-\frac{\Delta E^2 t^2}{2 \hbar^2}} \int dE e^{-\frac{E^2}{2 \Delta E^2}} \quad x = \frac{E}{\sqrt{2} \Delta E}$$

$$\int dE e^{-\frac{E^2}{2 \Delta E^2}} = \sqrt{2} \Delta E \int dx e^{-x^2} = \sqrt{2\pi} \Delta E$$

$$\left(\int dx e^{-x^2}\right)^2 = \int dx dy e^{-x^2 - y^2} = \int_0^\infty \int_0^\infty 2\pi r dr e^{-r^2}$$

$$= \pi \int_0^\infty dx e^{-x} = \pi \quad S_0$$

$$C(t) = \sqrt{2\pi} \Delta E m_0 e^{-iE_0 t / \hbar} e^{-\frac{\Delta E^2 t^2}{2 \hbar^2}}$$

So when  $\Delta t^2 \Delta E^2 = 2 \hbar^2$ , the correlation is down to  $1/e$ .  $S_0$

$$\Delta t \approx \sqrt{2} \frac{\hbar}{\Delta E}$$

In the spin case, with  $|\alpha\rangle = |\uparrow\rangle$ ,

$$|C(t)|^2 = |\langle \uparrow + | U(t, 0) | \uparrow \rangle|^2 = \cos^2 \frac{\omega t}{2}$$

which is small when

$$\frac{\omega t}{2} = \frac{\pi}{2} \quad \text{or} \quad \pi = t \omega = t \frac{\Delta E}{\hbar} \quad \text{ie}$$

$$\Delta t \approx \frac{\pi \hbar}{\Delta E} \quad \text{But } \Delta E \Delta t \approx \hbar \text{ is not equivalent to } \Delta x \Delta p \approx \frac{\hbar}{2}$$

## Unitary operators

1) change of basis  $|b_i\rangle = U|a_i\rangle$ ,  $U = \sum_i |b_i\rangle\langle a_i|$

2) change state  $|\alpha\rangle \rightarrow U|\alpha\rangle$ ,  $U = \mathcal{T}(\int dx')$

or  $U(t, t_0)$ . (In fact, these may be regarded as examples of (1).)

N.B.  $\langle \beta | \alpha \rangle \rightarrow \langle \beta | U^\dagger U | \alpha \rangle = \langle \beta | \alpha \rangle$ ,

$$\langle \beta | Y | \alpha \rangle \rightarrow (\langle \beta | U^\dagger) Y (U | \alpha \rangle) = \langle \beta | (U^\dagger Y U) | \alpha \rangle.$$

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associativity of matrix algebra

Two equivalent ways of thinking:

(1)  $|\alpha\rangle \rightarrow U|\alpha\rangle$        $Y \rightarrow Y$

(2)  $Y \rightarrow U^\dagger Y U$       or       $|\alpha\rangle \rightarrow |\alpha\rangle$ .

The second approach is closer to classical physics where there are no states.

(1)  $|\alpha\rangle \rightarrow (1 - i \frac{p \cdot dx'}{\hbar}) |\alpha\rangle$        $\vec{x} \rightarrow \vec{x}$

(2)  $|\alpha\rangle \rightarrow |\alpha\rangle$  but  $\vec{x} \rightarrow (1 + i \frac{p \cdot dx'}{\hbar}) \vec{x} (1 - i \frac{p \cdot dx'}{\hbar})$

$$x_i \rightarrow x_i + \sum_{j=1}^3 \frac{dx'_j}{\hbar} [p_j, x_i] = x_i + \sum_j \frac{dx'_j}{\hbar} (-i\hbar \delta_{ij})$$

$$x_i \rightarrow x_i + dx'_i \quad \text{whence} \quad \langle \vec{x}' \rangle = \langle \vec{x} + d\vec{x}' \rangle = \langle \vec{x} \rangle + \langle d\vec{x}' \rangle.$$

Sch

Heis

$$\text{Let } U(t, t_0) = U(t, 0) = U(t) = e^{-iHt/\hbar} \quad (\text{Suppose } t_0 = 0)$$

$$A^{(M)}(t) = U^\dagger(t) A^{(S)} U(t) \quad U(0) = 1.$$

$$A^{(M)}(0) = U^\dagger(0) A^{(S)} U(0) = A^{(S)}$$

$$|\alpha, t_0=0; t\rangle_M = |\alpha, t_0=0\rangle \text{ is independent of } t$$

$$|\alpha, t_0=0; t\rangle_S = U(t) |\alpha, t_0=0\rangle.$$

$$\langle \alpha, t_0=0; t | A^{(S)} | \alpha, t_0=0; t \rangle_S = \langle \alpha, t_0=0 | U^\dagger(t) A^{(S)} U(t) | \alpha, t_0=0 \rangle$$

$$= {}_M \langle \alpha, t_0=0; t | A^{(M)}(t) | \alpha, t_0=0; t \rangle_M$$

Sch

Heis.

$$|\alpha, t\rangle_S = e^{-iHt/\hbar} |\alpha\rangle = U(t) |\alpha\rangle$$

$$|\alpha, t\rangle = |\alpha\rangle$$

$$A^{(S)}(t) = A$$

$$A^{(M)}(t) = e^{iHt/\hbar} A e^{-iHt/\hbar}$$

$$= U^\dagger(t) A U(t)$$

$$A^{(M)}(t) = U^\dagger(t) A U(t)$$

Assume  $\frac{dA}{dt} = 0$ .

$$\frac{d}{dt} A^{(M)}(t) = \frac{d}{dt} U^\dagger(t) A U(t) + U^\dagger(t) A \frac{d}{dt} U(t).$$

$$u^\dagger(t) = e^{iHt/\hbar} \quad \frac{du^\dagger}{dt} = e^{iHt/\hbar} \frac{iH}{\hbar}$$

$$u = e^{-iHt/\hbar} \quad \frac{du}{dt} = -\frac{iH}{\hbar} e^{-iHt/\hbar}$$

So

$$\frac{d}{dt} A^H(t) = u^\dagger(t) \frac{iH}{\hbar} A u(t) - u^\dagger(t) A \frac{iH}{\hbar} u(t)$$

$$= \frac{i}{\hbar} u^\dagger(t) [H, A] u(t)$$

$$= \frac{i}{\hbar} [H, u^\dagger(t) A u(t)] = \frac{i}{\hbar} [H, A^H(t)]$$

$$= \frac{1}{i\hbar} [A^H(t), H] \quad H \text{ is eq. of mot.}$$

do problem 2.1

It's like the classical eq. ( $\forall \partial A / \partial t = 0$ )

$$\begin{aligned} \frac{dA}{dt} &= \sum_i \frac{\partial A}{\partial q_i} \dot{q}_i + \frac{\partial A}{\partial p_i} \dot{p}_i \\ &= \sum_i \frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q_i} = \{A, H\}_{P.B.} \end{aligned}$$

using  $[, ] = i\hbar \{, \}_{P.B}$  we get

$$\frac{dA}{dt} = \frac{1}{i\hbar} [A, H]$$

We may derive classical mechanics from QM by replacing  $[, ]$ 's by  $i\hbar \{, \}$ 's.

$$[x_i, F(\vec{p})] = i\hbar \{x_i, F(\vec{p})\} = i\hbar \frac{\partial F(p)}{\partial p_i}$$

$$[G(\vec{x}), p_i] = i\hbar \{G(\vec{x}), p_i\} = i\hbar \frac{\partial G(x)}{\partial x_i}$$

E.g. if  $G = \vec{x}^2$

$$[G(\vec{x}), p_i] = \sum_{j=1}^3 [x_j^2, p_i] = \sum_{j=1}^3 x_j [x_j, p_i] + [x_j, p_i] x_j$$

$$= \sum_j 2i\hbar \delta_{ij} x_j = 2i\hbar x_i = i\hbar \frac{\partial G}{\partial x_i}$$

Free particle:

Let  $H = \frac{\vec{p}^2}{2m} = \sum_{i=1}^3 \frac{p_i^2}{2m}$ , Then

$$\frac{dp_i}{dt} = -\frac{i}{\hbar} [H, p_i] = 0$$

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{i}{\hbar} [H, x_i] = \frac{i}{\hbar} \left[ \frac{\vec{p}^2}{2m}, x_i \right] = \frac{i}{2\hbar m} (-i\hbar) \frac{\partial p^2}{\partial p_i} \\ &= \frac{1}{2m} \frac{\partial p^2}{\partial p_i} = \frac{2p_i}{2m} = \frac{p_i}{m} = \frac{p_i(0)}{m} \end{aligned}$$

$$\text{So } x_i(t) = x_i(0) + \frac{p_i(0)}{m} t.$$

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$$\text{NB } [x_i(0), x_j(0)] = 0 \quad \text{But}$$

$$[x_i(t), x_j(0)] = [x_i(0) + \frac{p_i(0)}{m} t, x_j(0)]$$

$$= \frac{t}{m} (-i\hbar) \delta_{ij} = -i\hbar \frac{t}{m} \delta_{ij}.$$

So by (1.4.53)

$$\langle (\Delta x_i(t))^2 \rangle \langle (\Delta x_i(0))^2 \rangle \geq \frac{1}{4} | \langle [x_i(t), x_i(0)] \rangle |^2$$

$$\geq \frac{\hbar^2 t^2}{4 m^2}$$

$$\langle (\Delta x_i)^2 \rangle_t \langle (\Delta x_i)^2 \rangle_0 \geq \frac{\hbar^2 t^2}{4 m^2}. \quad \text{Interesting}$$

$$H = \frac{\vec{p}^2}{2m} + V(\vec{x})$$

$$\frac{dp_i}{dt} = \frac{i}{\hbar} [H, p_i] = \frac{i}{\hbar} [V(x), p_i] = - \frac{\partial V(\vec{x})}{\partial x_i}$$

while still

$$\frac{dx_i}{dt} = \frac{1}{\hbar} [H, x_i] = \frac{p_i}{m}$$



$$\begin{aligned} \text{So } \frac{d^2 x_i}{dt^2} &= \frac{i}{\hbar} [H, \frac{dx_i}{dt}] = \frac{i}{\hbar} [H, \frac{p_i}{m}] = \frac{1}{m} \frac{dp_i}{dt} \\ &= \frac{i}{\hbar m} \frac{\partial V}{\partial x_i} = -\frac{1}{m} \frac{\partial V}{\partial x_i} \end{aligned}$$

So

$$m \frac{d^2 x_i}{dt^2} = \frac{dp_i}{dt} = -\frac{\partial V}{\partial x_i} \quad \text{or}$$

$$m \ddot{\vec{x}} = \dot{\vec{p}} = -\vec{\nabla} V(\vec{x}) \quad \text{and so}$$

$$\begin{aligned} \text{in } \frac{d^2}{dt^2} \langle \vec{x} \rangle &= \frac{d}{dt} \langle \vec{p} \rangle = -\langle \vec{\nabla} V(\vec{x}) \rangle \\ &= -\vec{\nabla} \langle V(\vec{x}) \rangle. \end{aligned}$$

No  $\hbar$ 's here. Centers of wave packets obey classical mechanics.

Sub:  $A|a'\rangle = a'|a'\rangle$  so base kets, e-kets of  $A$  don't move.

Heis:  $A^H(t) = U^\dagger A(0)U$  so  $A|a'\rangle = a'|a'\rangle$  implies

$$U^\dagger A U U^\dagger |a'\rangle = a' U^\dagger |a'\rangle, \quad \text{or } A^H(t) U^\dagger(t) |a'\rangle = a' U^\dagger(t) |a'\rangle$$

So  $U^\dagger(t) |a'\rangle$  is the base ket for the Heis rep.

$|a', t\rangle_M = U^\dagger(t) |a'\rangle$  is the  $H$ -picture basis ket.

$$i\hbar \frac{d}{dt} |a', t\rangle_M = -H |a', t\rangle_M$$

$$A^H(t) |a', t\rangle_M = a' |a', t\rangle_M$$

$|a', t\rangle_M$  is the state that has e-val  $a'$  at time  $t$ .

So the e-values do not change with time.

Operators that are unitarily equivalent have the same e-values.

$$A^H(t) = \sum a' |a', t\rangle \langle a', t|$$

$$= \sum a' U^\dagger(t) |a'\rangle \langle a'| U(t)$$

$$= U^\dagger A^{(S)} U = e^{\frac{iMt}{\hbar}} A e^{-\frac{iMt}{\hbar}}$$

Coefficients.  $\langle a', t=0 | a', t \rangle = \sum |a' \langle a', t=0 | U(t) | a', t=0 \rangle|$

$$c_{a'}(t) = \langle a' | U(t) | a', t_0=0 \rangle$$

$$= \langle a' | U(t) | a', t_0=0 \rangle$$

$$= \langle a', t=0 | a', t_0=0 \rangle = \langle a' | U(t) | a', t_0=0 \rangle.$$

$$\langle x' | a', t \rangle = \langle x' | U(t) | a \rangle = \langle x' | U(t) | a \rangle.$$

Sch Heis

$$A|a'\rangle = a'|a'\rangle$$

$$B|b'\rangle = b'|b'\rangle$$

at  $t=0$   $|\alpha\rangle = |a'\rangle$ . Then amplitude is

$$\langle b'|U(t)|a'\rangle = \langle b'|U(t)|a'\rangle$$

$$P_{b|a} = |\langle b'|U(t)|a'\rangle|^2.$$

One may say that  $|a',t\rangle_M$

$$|a',t\rangle_M = U^\dagger(t)|a'\rangle = e^{i\frac{Ht}{\hbar}}|a'\rangle$$

is the state that at time  $t$  is an  
e-hel of  $A$  with e-val  $a'$ .

$$A e^{-i\frac{Ht}{\hbar}}|a',t\rangle_M = A U(t)|a',t\rangle_M$$

$$= A U(t) U^\dagger(t)|a'\rangle = A|a'\rangle = a'|a'\rangle$$

$$= a' U(t) U^\dagger(t)|a'\rangle = a' U(t)|a',t\rangle_M$$

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left( x + \frac{i p}{m\omega} \right) = \frac{1}{\sqrt{2\hbar}} \left( \sqrt{m\omega} x + \frac{i p}{\sqrt{m\omega}} \right)$$

$$a^\dagger = \frac{1}{\sqrt{2\hbar}} \left( \sqrt{m\omega} x - i \frac{p}{\sqrt{m\omega}} \right) = \sqrt{\frac{m\omega}{2\hbar}} \left( x - \frac{i p}{m\omega} \right)$$

$$[a, a^\dagger] = \frac{1}{2\hbar} \left( i [p, x] - i [x, p] \right) = 1$$

$$N = a^\dagger a$$

$$a^\dagger a = \frac{m\omega}{2\hbar} \left( x - \frac{i p}{m\omega} \right) \left( x + \frac{i p}{m\omega} \right)$$

$$= \frac{m\omega}{2\hbar} \left( x^2 + \frac{p^2}{m^2\omega^2} + i \frac{[x, p]}{m\omega} \right)$$

$$\hbar \omega a^\dagger a = \frac{1}{2} m \omega^2 x^2 + \frac{p^2}{2m} - \frac{\hbar \omega}{2}$$

That is

$$H = \hbar \omega \left( a^\dagger a + \frac{1}{2} \right) = \hbar \omega \left( N + \frac{1}{2} \right).$$

$$N |n\rangle = n |n\rangle$$

$$H |n\rangle = \hbar \omega \left( n + \frac{1}{2} \right) |n\rangle.$$

$$E_n = \hbar \omega \left( n + \frac{1}{2} \right)$$

$$[N, a] = [a^\dagger a, a] = [a^\dagger, a] a = -a$$

$$[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger [a, a^\dagger] = a^\dagger$$

$$\begin{aligned} N a^\dagger |n\rangle &= ([N, a^\dagger] + a^\dagger N) |n\rangle \\ &= (a^\dagger + a^\dagger n) |n\rangle = (n+1) a^\dagger |n\rangle \end{aligned}$$

$$\begin{aligned} N a |n\rangle &= ([N, a] + a N) |n\rangle = (-a + n a) |n\rangle \\ &= (n-1) a |n\rangle. \end{aligned}$$

annihilation / creation

lowering / raising

$$a |n\rangle = c |n-1\rangle$$

$$\langle n | a^\dagger a |n\rangle = |c|^2 = n \quad \text{if state } \langle m | m \rangle = \langle m+1 | m+1 \rangle = 1$$

Take  $c > 0$  then

$$a |n\rangle = \sqrt{n} |n-1\rangle$$

So

$$\begin{aligned} \langle n | a^\dagger |m\rangle &= \langle m | a |n\rangle^* = \sqrt{n} \langle m | n-1 \rangle^* = \sqrt{n} \delta_{m, n-1} \\ &= \sqrt{n+1} \delta_{m+1, n} \end{aligned}$$

$$\text{Then } a^\dagger |m\rangle = \sqrt{m+1} |m+1\rangle$$

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle.$$

$$a^2 |n\rangle = \sqrt{n(n-1)} |n-2\rangle \text{ etc.}$$

$$\text{But } n = \langle n | a^\dagger a | n \rangle = \langle n | a \rangle^\dagger (a | n \rangle) \geq 0.$$

So  $n$  must be a non-negative integer.

$$E_0 = \frac{1}{2} \hbar \omega \quad \text{zero-point energy}$$

$$a^\dagger |0\rangle = \sqrt{0+1} |1\rangle = |1\rangle$$

$$a^\dagger |1\rangle = \sqrt{1+1} |2\rangle = \sqrt{2} |2\rangle$$

$$|2\rangle = \frac{1}{\sqrt{2}} a^\dagger |1\rangle = \frac{1}{\sqrt{2}} (a^\dagger)^2 |0\rangle.$$

$$a^\dagger |2\rangle = \sqrt{2+1} |3\rangle = \sqrt{3} |3\rangle \quad \text{so}$$

$$|3\rangle = \frac{1}{\sqrt{3}} a^\dagger |2\rangle = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{2}} (a^\dagger)^2 |1\rangle = \frac{1}{\sqrt{3 \cdot 2}} (a^\dagger)^3 |0\rangle$$

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle.$$

$$E_n = (n + \frac{1}{2}) \hbar \omega.$$

$$\langle n' | a | n \rangle = \sqrt{n} \delta_{n', n-1}$$

$$\langle n' | a^\dagger | n \rangle = \sqrt{n+1} \delta_{n', n+1}$$

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) \quad p = i \sqrt{\frac{m\hbar\omega}{2}} (a^\dagger - a)$$

So

$$\langle n' | x | n \rangle = \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n} \delta_{n', n-1} + \sqrt{n+1} \delta_{n', n+1})$$

$$\langle n' | p | n \rangle = i \sqrt{\frac{m\hbar\omega}{2}} (-\sqrt{n} \delta_{n', n-1} + \sqrt{n+1} \delta_{n', n+1})$$

Neither is diagonal in  $N$ -rep.

But neither  $x, p$  nor  $a, a^\dagger$  commute with  $N$ .

$$a|0\rangle = 0$$

$$0 = \langle x' | a | 0 \rangle = \sqrt{\frac{m\omega}{2\hbar}} \langle x' | x + \frac{i p}{m\omega} | 0 \rangle$$

Thus

$$\frac{i}{m\omega} \langle x' | p | 0 \rangle = -x' \langle x' | 0 \rangle$$

||

$$\frac{\hbar}{m\omega} \frac{d}{dx'} \langle x' | 0 \rangle = -x' \langle x' | 0 \rangle - \frac{m\omega}{2\hbar} x'^2 \langle x' | 0 \rangle$$

$$\langle x' | 0 \rangle = \text{const. } e^{-\frac{m\omega}{2\hbar} x'^2}$$

Let  $x_0 = \sqrt{\frac{\hbar}{m\omega}}$ . Then

$$\langle x' | 0 \rangle = \frac{1}{\sqrt{x_0 \sqrt{\pi}}} e^{-\frac{1}{2} \left( \frac{x'}{x_0} \right)^2}$$

$$\langle x' | 1 \rangle = \langle x' | a^\dagger | 0 \rangle \quad a^\dagger = \frac{1}{\sqrt{2} x_0} \left( x - \frac{i p x_0^2}{\hbar} \right)$$

$$= \frac{1}{\sqrt{2} x_0} \langle x' | x - \frac{i}{\hbar} x_0^2 p | 0 \rangle$$

$$= \frac{1}{\sqrt{2} x_0} \left( x' - x_0^2 \frac{d}{dx'} \right) \langle x' | 0 \rangle$$

$$\langle x' | n \rangle = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \frac{1}{x_0^{n+\frac{1}{2}}} \left( x' - x_0^2 \frac{d}{dx'} \right)^n e^{-\frac{1}{2} \left( \frac{x'}{x_0} \right)^2}$$

As indicated:  $x^2 = \frac{\hbar}{2m\omega} (a+a^\dagger)^2$  and

$$\langle 0 | x^2 | 0 \rangle = \frac{x_0^2}{2} = \frac{\hbar}{2m\omega}$$

$$\langle 0 | p^2 | 0 \rangle = -\frac{m\hbar\omega}{2} \langle 0 | (a^\dagger - a)^2 | 0 \rangle = \frac{\hbar m\omega}{2}$$

So  $\langle 0 | \frac{p^2}{2m} | 0 \rangle = \frac{\hbar\omega}{4} = \frac{\langle 0 | H | 0 \rangle}{2}$  and

$$\langle 0 | \frac{m\omega^2 x^2}{2} | 0 \rangle = \frac{\hbar\omega}{4} = \frac{\langle 0 | H | 0 \rangle}{2}$$

$$V(x) = \frac{1}{2} k x^2 = \frac{1}{2} m \omega^2 x^2$$

$$H = \frac{p^2}{2m} + V(x)$$

as in virial theorem:  $\langle V \rangle = \frac{2E}{k+2}$        $\langle T \rangle = \frac{kE}{k+2}$



f0/02  
0/1  
↓

$$\langle 0 | x | 0 \rangle = \langle 0 | p | 0 \rangle = 0$$

$$S_0 \quad (\Delta A \equiv A - \langle A \rangle)$$

$$\langle 0 | (\Delta x)^2 | 0 \rangle = \langle 0 | x^2 | 0 \rangle = \frac{\hbar}{2m\omega}$$

$$\langle 0 | (\Delta p)^2 | 0 \rangle = \langle 0 | p^2 | 0 \rangle = \hbar \frac{m\omega}{2}$$

$$\langle 0 | (\Delta x)^2 | 0 \rangle \langle 0 | (\Delta p)^2 | 0 \rangle = \frac{\hbar^2}{4}$$

$$\langle n | x^2 | n \rangle = \langle n | \frac{\hbar}{2m\omega} (a+a^\dagger)^2 | n \rangle = \frac{\hbar}{2m\omega} \langle n | a a^\dagger + a^\dagger a | n \rangle = \frac{(2n+1)\hbar}{2m\omega} = (n+\frac{1}{2}) \frac{\hbar}{m\omega}$$

$$\langle n | p^2 | n \rangle = -\frac{m\hbar\omega}{2} \langle n | (a^\dagger - a)^2 | n \rangle = \frac{m\hbar\omega}{2} \langle n | a^\dagger a + a a^\dagger | n \rangle = (n+\frac{1}{2}) m \hbar \omega$$

$$\langle n | (\Delta x)^2 | n \rangle \langle n | (\Delta p)^2 | n \rangle = (n+\frac{1}{2})^2 \hbar^2$$

Time evolution H's eq.  $\Rightarrow$  (2.2, 32, 33)  $\Rightarrow$

$$\frac{dp}{dt} = -\frac{\partial V}{\partial x} = -m\omega^2 x \quad \text{and} \quad \frac{dx}{dt} = \frac{p}{m}$$

or more simply

$$\frac{da}{dt} = \frac{i}{\hbar} [H, a] = \frac{i}{\hbar} [\hbar\omega(a^\dagger a + \frac{1}{2}), a]$$

$$= i\omega a [a^\dagger, a] = -i\omega a$$

$$\text{so} \quad a(t) = e^{-i\omega t} a(0)$$

$$a^\dagger(t) = e^{i\omega t} a^\dagger(0)$$

Clearly  $H(\epsilon) = \hbar\omega (a^\dagger(\epsilon)a(\epsilon) + \frac{1}{2})$   
 $= \hbar\omega (e^{i\omega t} a^\dagger a e^{-i\omega t} + \frac{1}{2}) = H(0).$

Now  $x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger)$  so

$$x(t) = \sqrt{\frac{\hbar}{2m\omega}} (a e^{-i\omega t} + a^\dagger e^{i\omega t})$$

$$p(t) = i \sqrt{\frac{\hbar m\omega}{2}} (-a e^{-i\omega t} + a^\dagger e^{i\omega t})$$

and since

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left( x + \frac{i p}{m\omega} \right) \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left( x - \frac{i p}{m\omega} \right)$$

we get

$$x(t) = \frac{1}{2} \left[ \left( x + \frac{i p}{m\omega} \right) e^{-i\omega t} + \left( x - \frac{i p}{m\omega} \right) e^{i\omega t} \right]$$

$$= x \cos \omega t + \frac{p}{m\omega} \sin \omega t$$

and

$$p(t) = \frac{i m \omega}{2} \left[ - \left( x + \frac{i p}{m\omega} \right) e^{-i\omega t} + \left( x - \frac{i p}{m\omega} \right) e^{i\omega t} \right]$$

$$p(t) = p \cos \omega t - m\omega x \sin \omega t$$

as one would expect classically.

One may also use the BH lemma

$$e^{G/A} e^{-G} = A + [G, A] + \frac{1}{2!} [G, [G, A]] + \dots + \frac{1}{n!} [G, [G, [G, \dots [G, A]]]]$$

$$e^{\frac{iHt}{\hbar}} x e^{-\frac{iHt}{\hbar}} = x + \frac{it}{\hbar} [H, x] + \frac{(it)^2}{\hbar^2} \frac{1}{2!} [H, [H, x]]$$

$$[H, x] = -\frac{i\hbar p}{m} \quad [H, p] = i\hbar m\omega^2 x$$

$$e^{\frac{iHt}{\hbar}} x e^{-\frac{iHt}{\hbar}} = x + \frac{p}{m} t - \frac{t^2 \omega^2}{2!} x - \frac{1}{3!} t^3 \omega^2 \frac{p}{m} + \dots$$

$$= (\cos \omega t) x + \frac{\sin \omega t}{m\omega} p.$$

Let  $D(\alpha) = e^{a\alpha^\dagger - \alpha a}$  By BM

$$D(-\alpha) a D(\alpha) = D^{-1}(\alpha) a D(\alpha) = D^\dagger(\alpha) a D(\alpha) \\ = a + [\alpha^\dagger a - a a^\dagger, a] + \frac{1}{2!} [\alpha^\dagger a - a a^\dagger, [\alpha^\dagger a - a a^\dagger, a]] + \dots$$

$$= a + [\alpha^\dagger a - a a^\dagger, a] = a - \alpha [a^\dagger, a] = a + \alpha$$

Now  $a|0\rangle = 0$  So  $D^{-1}(\alpha) a D(\alpha)|0\rangle = (a+\alpha)|0\rangle = \alpha|0\rangle$

or

$a D(\alpha)|0\rangle = \alpha D(\alpha)|0\rangle$ . So  $|\alpha\rangle = D(\alpha)|0\rangle$  is an e-vec of  $a$  with e-val  $\alpha \in \mathbb{C}$ .

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Thus  $\langle n|\alpha\rangle|^2 = \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2}$  is a Poisson distribution.

$$D(\lambda\alpha) = e^{-\lambda(\alpha a^\dagger - \alpha^* a)}, \quad \lambda \text{ real}$$

Let

$$a(\lambda, \alpha) = D^\dagger(\lambda\alpha) a D(\lambda\alpha)$$

$$\frac{da(\lambda, \alpha)}{d\lambda} = \frac{d}{d\lambda} \begin{bmatrix} e^{-\lambda(\alpha a^\dagger - \alpha^* a)} & \lambda(\alpha a^\dagger - \alpha^* a) \\ e & a e \end{bmatrix}$$

$$= D^\dagger(\alpha) \left[ -(\alpha a^\dagger - \alpha^* a) a + a (\alpha a^\dagger - \alpha^* a) \right] D(\alpha)$$

$$= D^\dagger(\alpha) \alpha [a, a^\dagger] D(\alpha) = \alpha D^\dagger(\alpha) D(\alpha) = \alpha$$

So

$$a(\lambda, \alpha) = a + \lambda\alpha \quad \text{and}$$

$$a(1, \alpha) = D^\dagger(\alpha) a D(\alpha) = a + \alpha.$$

$$\langle \alpha | a(t) | \alpha \rangle = \langle \alpha | a e^{-i\omega t} | \alpha \rangle = e^{-i\omega t} \alpha$$

$$\langle \alpha | a^\dagger(t) | \alpha \rangle = \langle \alpha | a^\dagger e^{i\omega t} | \alpha \rangle = \alpha^* e^{i\omega t}$$

Note

$$\langle \beta | \alpha \rangle = e^{-\frac{1}{2} |\alpha|^2 - \frac{1}{2} |\beta|^2 + \beta^* \alpha}$$

$$1 = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|$$

A spin  $\frac{1}{2}$  particle on a line in SUSY QM:

$$[H, Q_i] = 0 \quad i = 1, 2$$

$$[Q_i, Q_j]_+ = \delta_{ij} H$$

$$Q_1 \doteq \frac{1}{2} \left( \frac{\sigma_1}{\sqrt{m}} p + \sigma_2 \sqrt{m} W(x) \right)$$

$$Q_2 \doteq \frac{1}{2} \left( \frac{\sigma_2}{\sqrt{m}} p - \sigma_1 \sqrt{m} W(x) \right)$$

$$Q_i^\dagger = Q_i$$

The superpotential  $|W(x)| \rightarrow \infty$  as  $|x| \rightarrow \infty$

$$\langle x | \psi \rangle = \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$$

$$\langle x | p | \psi \rangle = \frac{\hbar}{i} \frac{d\psi}{dx}$$

$$[Q_1, Q_1]_+ = 2Q_1^2 = H = \frac{p^2}{2m} + m \frac{W(x)^2}{2} + \frac{\hbar}{2} \sigma_3 \frac{dW}{dx}$$

$$= [Q_2, Q_2]$$

$$[Q_1, Q_2]_+ = 0.$$

Susy is exact if  $Q_1|\psi\rangle = 0.$

Then  $H|\psi\rangle = 2Q_1^2|\psi\rangle = 0.$  So since  $H = 2Q_2^2$

we have  $0 = \langle \psi | Q_2^2 | \psi \rangle = (\langle \psi | Q_2)^\dagger (Q_2 | \psi \rangle) = 0$  so  $Q_2 | \psi \rangle = 0$  too.

$$e^{-i(\theta_1 Q_1 + \theta_2 Q_2)} |\psi\rangle = |\psi\rangle.$$

$H = 2Q_1^\dagger Q_1$  so all  $E'_n \geq 0.$

$$Q_1 |\psi\rangle = 0$$

$$\frac{1}{2} \left( \frac{\sigma_1 p}{\sqrt{m}} + \sigma_2 \sqrt{m} W(x) \right) |\psi\rangle = 0$$

$$\frac{\sigma_1}{\sqrt{m}} \frac{\hbar}{i} \frac{d\psi}{dx} + \sigma_2 W \psi = 0$$

$$\frac{\hbar}{i} \frac{d\psi}{dx} = -m \sigma_1 \sigma_2 W \psi = -i m \sigma_3 W(x) \psi(x)$$

$$\frac{\hbar}{i} \frac{d\psi}{dx} = m \sigma_3 W \psi$$

$$\psi(x) = e^{\left( m \int_0^x dy W(y) \sigma_3 \right)} \psi(0).$$

$\frac{10}{q}$   

$$\psi(\vec{x}', t) = \langle \vec{x}' | \alpha, t_0; t \rangle.$$

$$H = \frac{p^2}{2m} + V(\vec{x})$$

$$\langle \vec{x}'' | V(\vec{x}) | \vec{x}' \rangle = V(\vec{x}') \delta^3(\vec{x}'' - \vec{x}')$$

$$i\hbar \frac{\partial}{\partial t} \langle \vec{x}' | \alpha, t_0; t \rangle = \langle \vec{x}' | H | \alpha, t_0; t \rangle$$

$$= \langle \vec{x}' | \frac{p^2}{2m} | \alpha, t_0; t \rangle + \langle \vec{x}' | V(\vec{x}) | \alpha, t_0; t \rangle$$

$$= -\frac{\hbar^2}{2m} \nabla'^2 \langle \vec{x}' | \alpha, t_0; t \rangle + V(\vec{x}') \langle \vec{x}' | \alpha, t_0; t \rangle$$

or

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{x}', t) = -\frac{\hbar^2}{2m} \nabla'^2 \psi(\vec{x}', t) + V(\vec{x}') \psi(\vec{x}', t)$$

E. S. 1926.

Say  $H|\alpha\rangle = E_\alpha|\alpha\rangle$ . Then

$$|\alpha, 0; t\rangle = e^{-\frac{iHt}{\hbar}} |\alpha, 0\rangle = e^{-\frac{iE_\alpha t}{\hbar}} |\alpha, 0\rangle$$

and so

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \langle \vec{x}' | \alpha, t_0; t \rangle &= i\hbar \left( -\frac{iE_\alpha}{\hbar} \right) \langle \vec{x}' | \alpha, t_0; t \rangle \\ &= E_\alpha \langle \vec{x}' | \alpha, t_0; t \rangle. \end{aligned}$$

So

$$E_\alpha \langle \vec{x}' | \alpha, t_0; t \rangle = -\frac{\hbar^2}{2m} \nabla'^2 \langle \vec{x}' | \alpha, t_0; t \rangle + V(\vec{x}') \langle \vec{x}' | \alpha, t_0; t \rangle$$

$$\text{Letting } \langle \vec{x}' | \alpha, t_0; t \rangle = e^{-\frac{iE_\alpha(t-t_0)}{\hbar}} \phi_{E_\alpha}(\vec{x}'),$$

we have

$$E \phi_E(\vec{x}') = -\frac{\hbar^2}{2m} \nabla'^2 \phi_E(\vec{x}') + V(\vec{x}') \phi_E(\vec{x}')$$

$$\text{If } E < \lim_{|\vec{x}'| \rightarrow \infty} V(\vec{x}')$$

then we expect

$$\phi_E(\vec{x}') \rightarrow 0 \quad \text{as } |\vec{x}'| \rightarrow \infty.$$

The particle is confined and we expect that the time-independent S. eq. will have solutions that are normalizable only for discrete values of  $E$ .

If  $V(\vec{x}') = V(r)$  is spherically symmetric, then as in A.5, we use spherical coordinates

$$E \Psi_E(r, \theta, \phi) = -\frac{\hbar^2}{2m} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \Psi_E(r, \theta, \phi)}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial \Psi_E(r, \theta, \phi)}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Psi_E(r, \theta, \phi)}{\partial \phi^2} \right] + V(r) \Psi_E(r, \theta, \phi)$$

$$\text{We assume } \lim_{r \rightarrow \infty} r^2 V(r) = 0.$$

Let  $\Psi_E(r, \theta, \phi) = R(r) Y_e^m(\theta, \phi)$  in which

$$-\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y_e^m(\theta, \phi) = e(e+1) Y_e^m(\theta, \phi)$$



for  $l = 0, 1, 2, \dots$  and  $m = -l, -(l-1), \dots, (l-1), l$

and

$$-i \frac{\partial}{\partial \phi} Y_l^m(\theta, \phi) = m Y_l^m(\theta, \phi).$$

and

$$\int d\Omega Y_l^{m'}(\theta, \phi) Y_l^m(\theta, \phi) = \delta_{ll'} \delta_{mm'}$$

then

$$\begin{aligned} E R(l) Y_l^m(\theta, \phi) &= -\frac{\hbar^2}{2m} \left[ Y_l^m(\theta, \phi) \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} R(l) \right. \\ &\quad \left. - \frac{l(l+1)}{r^2} R(l) Y_l^m(\theta, \phi) \right] \\ &\quad + V(r) R(l) Y_l^m(\theta, \phi) \end{aligned}$$

or

$$E R(l) = -\frac{\hbar^2}{2m} \left[ \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} R(l) - \frac{l(l+1)}{r^2} R(l) \right] + V(r) R(l)$$

Let  $u_E(r) = r R(l)$ . Then

$$E u_E(r) = -\frac{\hbar^2}{2m} \frac{d^2 u_E(r)}{dr^2} + \left[ V(r) + \frac{l(l+1)\hbar^2}{2m r^2} \right] u_E(r)$$

with  $u_E(0) = 0$ .

If  $V(r) = 0$ ,  $R(r) = c_2 j_l(kr)$  ( $R$  not  $u$  in)

where  $k = \sqrt{\frac{2mE}{\hbar^2}}$  (free-particle case)

is a spherical Bessel function. These are not too bad

$$j_0(p) = \frac{\sin p}{p}, \quad j_1(p) = \frac{\sin p}{p^2} - \frac{\cos p}{p}, \text{ etc}$$

$$j_l(p) \sim \frac{p^l}{(2l+1)!!} \quad (p \gg 0)$$

If  $V(r) = \begin{cases} 0 & r < R \\ \infty & r > R, \end{cases}$

then

$$R_{nl}(r) = c_2 j_l(\alpha_n r) \quad \text{with}$$

$$\alpha_n = \sqrt{\frac{2mE_n}{\hbar^2}} \quad \text{in which } \alpha_n R = x_{nl}$$

is a root of the equation  $0 = j_l'(x_{nl})$ .

H atom (Bohr)

$$V(r) = -\frac{Ze^2}{r} \quad \rho = \left(\frac{8me|E|}{\hbar^2}\right)^{\frac{1}{2}} r$$

then

$$\Psi_{nlm}(r, \theta, \phi) = R_{nl}(r) Y_l^m(\theta, \phi)$$

$$E_n = -\frac{1}{2} mc^2 \frac{Z^2 \alpha^2}{n^2} = -\frac{1}{2} mc^2 Z^2 \left(\frac{e^2}{\hbar c}\right)^2$$

where  $\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$  in these units.

If

$$V(r) = \begin{cases} -V_0 & r < R \\ 0 & r > R, \end{cases} \quad \text{then}$$

for  $r < R$

$$R(r) = A_0(r) = N j_e(\alpha r), \quad \alpha, E < 0,$$

$$\alpha = \left[ \frac{2m(V_0 - |E|)}{\hbar^2} \right]^{1/2}, \quad N \text{ a constant.}$$

But for  $r > R$

$$R(r) = A_0(r) = N_e h_e^{(1)}(i k r) \quad \text{where}$$

$$h_e^{(1)}(p) = j_e(p) + i n_e(p) \quad \text{and}$$

$$k = \left( \frac{2m|E|}{\hbar^2} \right)^{1/2}.$$

The Hankel function  $h_e^{(1)}(p)$  as  $p \rightarrow \infty$  is

$$h_e^{(1)}(p) \sim \frac{1}{p} e^{i[p - (l+1)\pi/2]}$$

These are nice functions,

$$h_0^{(1)}(i k r) = -\frac{1}{k r} e^{-k r}$$

$$h_1^{(1)}(i k r) = i \left( \frac{1}{k r} + \frac{1}{k^2 r^2} \right) e^{-k r}$$

$$Y_l^m(\theta, \varphi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi} \quad m \geq 0$$

$$P_l^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_l(x) \quad m \geq 0$$

$$P_l(x) = \frac{(-1)^l}{2^l l!} \frac{d^l (1-x^2)^l}{dx^l}$$

for  $m < 0$

$$Y_l^m(\theta, \varphi) = (-1)^{|m|} Y_l^{|m|}(\theta, \varphi)$$

Always  $Y_l^m(\theta, \varphi) \sim e^{im\varphi}$ .

$$R_{nl}(r) = N_{nl} e^{-\rho/2} \rho^l L_{n-l}^{2l+1}(\rho) \quad \rho = \frac{2Zn}{na_0}$$

$$N_{nl} = - \left\{ \left( \frac{2Z}{na_0} \right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3} \right\}^{\frac{1}{2}} \quad \text{the minus sign, is cosmetic}$$

$$a_0 = \frac{\hbar^2}{me^2} = \text{Bohr radius} \quad n \geq l+1$$

$$E_n = \frac{-Z^2 e^2}{2n^2 a_0} \quad L_p^q(\rho) = \frac{d^q}{d\rho^q} L_p(\rho)$$

$$L_p(\rho) = e^{\rho} \frac{d^p}{d\rho^p} (\rho^p e^{-\rho}) \quad L_1(\rho) = e^{\rho} \frac{d}{d\rho} (\rho e^{-\rho}) = e^{\rho} (e^{-\rho} - \rho e^{-\rho}) = -\rho + 1$$

General properties of solutions of radial equation for bound states in spherically symmetric potentials:

$$E u(r) = -\frac{\hbar^2}{2m} u''(r) + \left[ V(r) + \frac{l(l+1)\hbar^2}{2mr^2} \right] u(r)$$

If  $V(r) \rightarrow 0$  as  $r \rightarrow \infty$ , then as  $r \rightarrow \infty$

$$E u(r) \approx -\frac{\hbar^2}{2m} u''(r) \quad r$$

$$u(r) \sim e^{-\sqrt{\frac{2m|E|}{\hbar^2}} r}$$

If  $r^2 V(r) \rightarrow 0$  as  $r \rightarrow 0$ , then as  $r \rightarrow 0$

$$E u(r) \approx -\frac{\hbar^2}{2m} u''(r) + \frac{l(l+1)\hbar^2}{2mr^2} u(r) \quad r$$

$$u''(r) = \frac{l(l+1)}{r^2} u(r) \quad \text{let } u = r^\alpha$$

$$\alpha(\alpha-1) = l(l+1) \quad \alpha^2 - \alpha - l(l+1) = 0$$

$$\alpha = \frac{1 \pm \sqrt{1 + 4l(l+1)}}{2} = \frac{1 \pm \sqrt{4l^2 + 4l + 1}}{2} = \frac{1 \pm \sqrt{(2l+1)^2}}{2}$$

$$= \frac{1 \pm (2l+1)}{2} = \begin{cases} l+1 \\ -l \end{cases} \quad \text{nonsingular choice}$$

So as  $r \rightarrow 0$   $u(r) \approx r^{\ell+1}$   
 and as  $r \rightarrow \infty$   $u(r) \approx e^{-\sqrt{\frac{2m|E|}{\hbar^2}} r}$

So

$$R(r) = \frac{u(r)}{r} \sim r^{\ell} \quad \text{as } r \rightarrow 0$$

and

$$R(r) = \frac{u(r)}{r} \sim \frac{e^{-\sqrt{\frac{2m|E|}{\hbar^2}} r}}{r} \quad \text{as } r \rightarrow \infty.$$

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Let  $\rho = r/a_0$        $a_0 = \frac{\hbar^2}{m e^2}$        $m = \frac{m_e m_p}{m_e + m_p}$

$\lambda_{k,l} = \sqrt{-E_{k,l}/E_I}$        $E_I = \frac{m e^4}{2 \hbar^2}$

then

$$\left[ \frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2} + \frac{2}{\rho} - \lambda_{k,l}^2 \right] u_{k,l}(\rho) = 0$$

Let

$u_{k,l}(\rho) = e^{-\rho} \lambda_{k,l} y_{k,l}(\rho)$       then

$$\left\{ \frac{d^2}{d\rho^2} - 2\lambda_{k,l} \frac{d}{d\rho} + \left[ \frac{2}{\rho} - \frac{l(l+1)}{\rho^2} \right] \right\} y_{k,l}(\rho) = 0$$

$y_{k,l}(0) = 0$        $y_{k,l}(\rho) = \rho^s \sum_{q=0}^{\infty} c_q \rho^q$

find as before

$s = l+1$

and get

$$q(q+2l+1)c_q = 2 \sum_{r=0}^{q-1} (q+r) \lambda_{k,l}^{-1} c_{q-r}$$

termination requires

$\lambda_{k,l} = \frac{1}{q+l}$  or  $\lambda_{k,l} = \frac{1}{k+l}$

$E_{k,l} = -\frac{E_I}{(k+l)^2}$

$k+l = n$

Probability  $p(x,t) = |\psi(x,t)|^2$

$$\frac{\partial p}{\partial t} = \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}$$

$$= \psi^* \left( -\frac{iH}{\hbar} \right) \psi + \psi \frac{\partial}{\partial t} \langle \psi | e^{\frac{iHt}{\hbar}} | x \rangle$$

$$= -\frac{i}{\hbar} \psi^* H \psi + \frac{i}{\hbar} \psi H \psi^*$$

$$= -\frac{i}{\hbar} \psi^*(x,t) \left( -\frac{\hbar^2}{2m} \nabla^2 + V(x) \right) \psi(x,t)$$

$$+ \frac{i}{\hbar} \psi(x,t) \left( -\frac{\hbar^2}{2m} \nabla^2 + V(x) \right) \psi^*(x,t)$$

$$= \frac{i\hbar}{2m} \left( \psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* \right)$$

$$= \frac{i\hbar}{2m} \vec{\nabla} \cdot \left( \psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right) \equiv -\vec{\nabla} \cdot \vec{J}$$

$$\vec{J} = -\frac{i\hbar}{2m} \left( \psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right)$$

$$= \frac{\hbar}{m} \operatorname{Im} (\psi^* \nabla \psi)$$

$$0 = \frac{\partial p}{\partial t} + \vec{\nabla} \cdot \vec{J}$$



$$\int d^3x j(x,t) = \int d^3x \frac{-i\hbar}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*)$$

$$= \int d^3x \frac{-i\hbar}{m} \psi^* \nabla \psi = \int d^3x \psi^* \frac{\vec{p}}{m} \psi$$

$$= \frac{\langle \vec{p} \rangle}{m}$$

Let  $\psi(x,t) = \sqrt{\rho(x,t)} e^{iS(x,t)/\hbar}$  real  
real & positive This is always possible.

Then

$$\psi^* \nabla \psi = e^{-iS/\hbar} \sqrt{\rho} \vec{\nabla} \sqrt{\rho} e^{iS/\hbar}$$

$$= e^{-iS/\hbar} \sqrt{\rho} \left[ e^{iS/\hbar} \nabla \sqrt{\rho} + \sqrt{\rho} \frac{i}{\hbar} \nabla S e^{iS/\hbar} \right]$$

$$= \underbrace{\sqrt{\rho} \vec{\nabla} \sqrt{\rho}}_{\text{real}} + \frac{i}{\hbar} \rho \vec{\nabla} S$$

So

$$\vec{j} = \frac{\hbar}{m} \text{Im} \psi^* \nabla \psi = \frac{\hbar}{m} \vec{\nabla} S$$

$F_{cl}$  - plane wave

$$\psi(x,t) = c e^{i \frac{p \cdot x}{\hbar} - i \frac{Et}{\hbar}}$$

$$E = \frac{p^2}{2m}$$

$$S = p \cdot x - Et \quad \vec{\nabla} S = \vec{p}$$

$$\vec{j} = \frac{p}{m} \psi^* \psi = \frac{p \psi^* \psi}{m}$$

Call  $\vec{v} \equiv \frac{\nabla S}{m}$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0 + 0 = 0.$$

Classical limit

$$\psi(x,t) = \sqrt{\rho(x,t)} e^{i S(x,t)/\hbar}$$

$\swarrow$  real  
 $\nwarrow$  pos

$H\psi = i\hbar \partial\psi/\partial t$  gives

$$-\frac{\hbar^2}{2m} \left[ \nabla^2 \sqrt{\rho} + \frac{2i}{\hbar} \vec{\nabla} \sqrt{\rho} \cdot \vec{\nabla} S - \frac{1}{\hbar^2} \sqrt{\rho} \nabla^2 S^2 + \frac{1}{\hbar} \sqrt{\rho} \nabla^2 S \right]$$

$$+ \sqrt{\rho} V = i\hbar \left[ \frac{\partial \sqrt{\rho}}{\partial t} + \frac{1}{\hbar} \sqrt{\rho} \frac{\partial S}{\partial t} \right].$$

Dropping terms  $O(\hbar)$  and  $O(\hbar^2)$ , assuming  $\hbar |\nabla^2 S| \ll |\nabla S|^2$

$$\frac{1}{2m} \sqrt{\rho} \nabla^2 S^2 + \sqrt{\rho} V = -\sqrt{\rho} \frac{\partial S}{\partial t} \sim$$

$$\frac{1}{2m} (\nabla S)^2 + V(x) + \frac{\partial S(x,t)}{\partial t} = 0$$

M-J. 1836

Stationary state case

Classically for a

constant classical  $H$ ,  $S(x,t) = W(x) - Et$

$W(x)$  is  $H$ 's characteristic function

$$\vec{p}_{class} = \vec{\nabla} S = \vec{\nabla} W$$

But back in 1836 there was no  $t$ .

No idea particles waned.

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WKB

For a stationary state of energy  $E$

$$S(x,t) = W(x) - Et \quad \text{and so}$$

$$\frac{1}{2m} (\vec{\nabla} S)^2 + V(x) + \frac{\partial S}{\partial t} = 0 \quad \text{becomes}$$

$$\frac{1}{2m} (\vec{\nabla} W)^2 + V(x) = E$$

In one dimension this is

$$\left(\frac{dW}{dx}\right)^2 = 2m(E - V(x)) \quad \text{whence}$$

$$W(x) = \pm \int^x dx' \sqrt{2m(E - V(x'))} \quad \text{and}$$

$$S(x,t) = \pm \int^x dx' \sqrt{2m(E - V(x'))} - Et$$

For a stationary state, all the  $t$  dependence is in  $e^{-iEt/\hbar}$ .

So

$$\dot{\rho} = 0.$$

Thus the exact formula

$$\vec{j} = \frac{\hbar}{m} \nabla S \xrightarrow{d=1} j = \frac{\hbar}{m} \frac{dS}{dx} = \frac{\hbar}{m} w'$$

But

$$0 = \dot{\rho} + \nabla \cdot \vec{j} \xrightarrow{d=1} 0 = \dot{\rho} + j' = j' \text{ where}$$

$$0 = (\rho w')' = \rho' w' + \rho w''$$

so

$$\rho w' = \text{constant}$$

$$\rho(x) = \frac{\text{constant}}{\pm \sqrt{2m(E-V(x))}}$$

or

$$\sqrt{\rho} = \frac{\text{constant}'}{(\pm \sqrt{2m(E-V(x))})^{1/2}} \propto \frac{1}{\sqrt{v}} \text{ classical.}$$

Particle density  $\rho \sim 1/v$  is least where speed is greatest.

So the JWKB solution is

$$\psi(x,t) \approx \frac{\text{constant}}{[E-V(x)]^{1/4}} e^{\left\{ \pm \frac{i}{\hbar} \int dx' \sqrt{2m[E-V(x')]} - \frac{iEt}{\hbar} \right\}}$$

This is good when the terms we dropped are small, i.e.,

$$\hbar |\nabla^2 S| \ll (\nabla S)^2, \text{ etc.}$$

This is a short-wavelength limit.

for  $d=1$ , it is  $\hbar|W''| \ll (W')^2$

Now

$$\lambda = \frac{\hbar}{p} = \frac{\hbar}{\sqrt{2m(E-V(x))}}$$

and

$$W = \pm \int^x dx' \sqrt{2m(E-V(x'))} \quad \text{SO}$$

$$W' = \pm \sqrt{2m(E-V(x))} \quad \text{SO}$$

$$W'' = \pm \frac{1}{2} \frac{(-2m V'(x))}{\sqrt{2m(E-V(x))}} = \mp \frac{m V'}{\sqrt{2m(E-V)}}$$

So the condition is

$$\frac{\hbar |V'|}{\sqrt{2m(E-V)}} \ll 2(E-V) \quad \text{or}$$

$$\hbar |V'| \approx |dV| \ll 2(E-V).$$

So in  $\lambda$ ,  $dV$  should be small compared to  $2(E-V)$ .  $\hbar |V'|/2mE \ll 1$ .

When  $V(x) > E$ , we get

$$\psi(x, E) \approx \frac{\text{constant}}{(V(x)-E)^{1/4}} C \left( \pm \frac{1}{\hbar} \int^x dx' \sqrt{2m[V(x')-E]} - i \frac{E}{\hbar} \right)$$

where  $V(x) = E$ , neither solution is valid.

The fix is:

1) approximate  $V(x)$  near  $x_0$  by  $V(x_0) + (x-x_0)V'(x_0)$ .

2) Solve

$$-\frac{\hbar^2}{2m} u''(x) + V_0 + (x-x_0)V'(x_0) u(x) = E u(x)$$

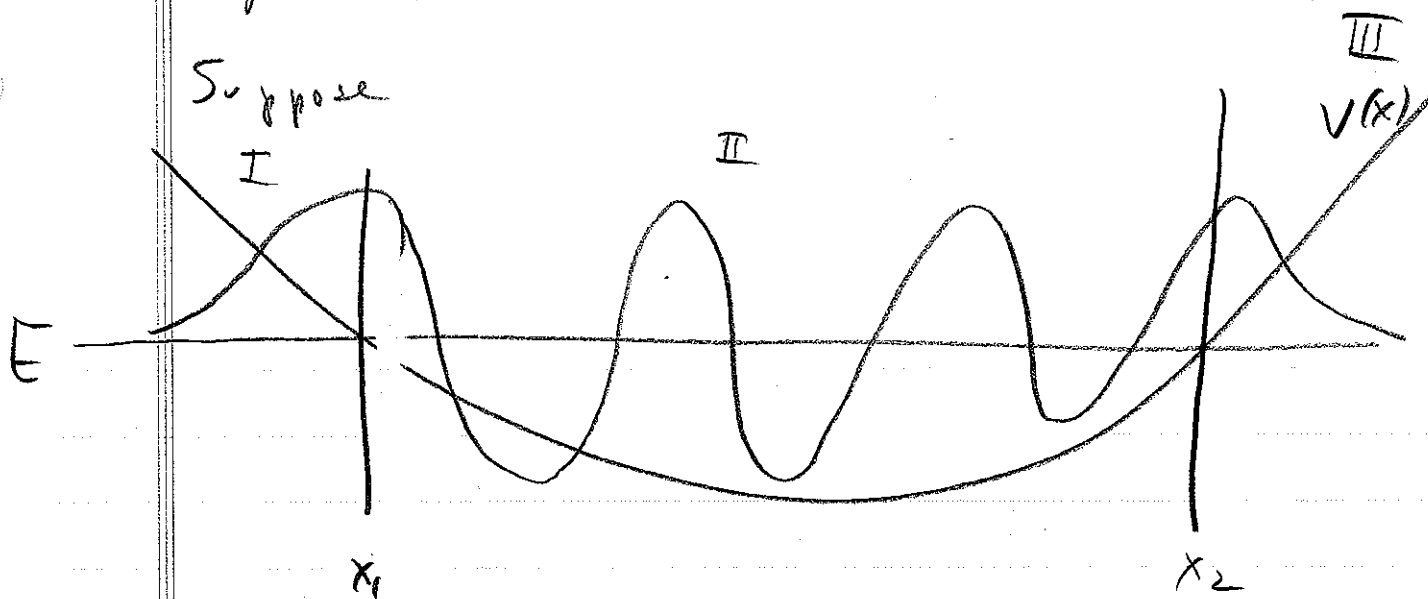
or

$$u''(x) = \frac{2m}{\hbar^2} V'(x_0) (x-x_0) u(x)$$

in terms of a Bessel function of order  $\pm \frac{1}{3}$  near  $x_0$ .

3) Match this solution to the other two by adjusting the constants of integration.

Suppose



In region I

$$\frac{1}{(V(x)-E)^{\frac{1}{4}}} \exp \left[ -\frac{1}{\hbar} \int_x^{x_1} dx' \sqrt{2m(V(x')-E)} \right] \text{ matches}$$

$$\frac{2}{(E-V(x))^{\frac{1}{4}}} \cos \left[ \frac{1}{\hbar} \int_{x_1}^x dx' \sqrt{2m(E-V(x'))} - \frac{\pi}{4} \right]$$

in II

And in III

$$\text{III} = \frac{1}{[V(x)-E]^{\frac{1}{4}}} \exp \left[ -\frac{i}{\hbar} \int_{x_2}^x dx' \sqrt{2m[V(x')-E]} \right] \text{ matches}$$

$$\text{II}_3 = \frac{2}{[E-V(x)]^{\frac{1}{4}}} \cos \left[ -\frac{i}{\hbar} \int_x^{x_2} dx' \sqrt{2m[E-V(x')]} + \frac{\pi}{4} \right] \text{ in II}$$

The functions in II agree if

$$\frac{i}{\hbar} \int_{x_1}^x ( ) - \frac{\pi}{4} = -\frac{i}{\hbar} \int_x^{x_2} ( ) + \frac{\pi}{4} \quad \text{or}$$

$$\int_{x_1}^{x_2} dx' \sqrt{2m[E-V(x')]} = \frac{\pi}{2} \hbar + n\pi \hbar \quad \text{or}$$

The  $n\pi$  is because one could apply (-1) to  $\text{II}_3$  and  $\text{III}$ .

$$\oint dx' \sqrt{2m[E-V(x')]} = (2n+1)\pi \hbar = (n+\frac{1}{2})h$$

which is to be compared with the <sup>A.</sup> Sommerfeld  
W. Wilson condition

$$\oint p dx = nh$$

1915.

$$V(x) = \begin{cases} mgx & x > 0 \\ \infty & x < 0 \end{cases}$$

has solutions that are the same as the odd-parity solutions of

$$V(x) = mg|x|$$

which has  $x_1 = -\frac{E}{mg}$   $x_2 = \frac{E}{mg}$ . Then

$$\int_{-E/mg}^{E/mg} dx \sqrt{2m(E - mg|x|)} = (n_{\text{odd}} + \frac{1}{2}) \pi \hbar \quad \text{or}$$

$$\int_0^{E/mg} dx \sqrt{2m(E - mgx)} = \frac{1}{2} (2n' + 1 + \frac{1}{2}) \pi \hbar \quad n' = 0, 1, 2, 3$$

$$= (n' + \frac{3}{4}) \pi \hbar = (n - \frac{1}{4}) \pi \hbar$$

$$\frac{2}{3(2m^2g)} \left[ 2m(E - mgx) \right]^{\frac{3}{2}} \Big|_{\frac{E}{mg}}^0 = (n - \frac{1}{4}) \pi \hbar \quad n = 1, 2, 3, \dots$$

$$(2mE)^{3/2} = 3(n - \frac{1}{4}) \pi \hbar m^2 g$$

$$E_n = \frac{(3(n - \frac{1}{4}) \pi \hbar m^2 g)^{2/3}}{2m}$$

$$E_n = \frac{1}{2} [3(n - \frac{1}{4}) \pi]^2 (mg^2 \hbar^2)^{1/3} \quad n = 1, 2, 3, \dots$$

exact is  $E_n = \frac{\lambda_n}{2^{1/3}} (mg^2 \hbar^2)^{1/3}$  where  $A_1(-\lambda_n) = 0$ .  
Better agreement at higher  $n$ . But good at  $n = 1, 2, 3$ .



$q\bar{q}$  potential may be

$$V(r) = \sigma r$$

with  $\sigma = \frac{1 \text{ GeV}}{1 \text{ fm}} \approx 1.6 \times 10^5 \text{ N} \approx 16 \text{ tons}$ .

Gravity gives  $0.98 \text{ N}$  on ball of  $0.1 \text{ kg}$ .

Propagators

$$K(x', t'; x, t) = \langle x' | e^{-i\frac{H(t'-t)}{\hbar}} | x \rangle$$

If  $H|a'\rangle = E_{a'}|a'\rangle$ , then

$$\begin{aligned} K(x', t'; x, t) &= \langle x' | \sum |a'\rangle \langle a'| e^{-i\frac{H(t'-t)}{\hbar}} | x \rangle \\ &= \sum \langle x' | a' \rangle e^{-i\frac{E_{a'}(t'-t)}{\hbar}} \langle a' | x \rangle. \end{aligned}$$

Note that  $K(x', t; x, t) = \delta^3(\vec{x}' - \vec{x})$ .

If we add the boundary condition, reducing physics

$$K(x', t'; x, t) = \langle x' | e^{-iH(t'-t)/\hbar} | x \rangle \theta(t'-t),$$

then

$$\begin{aligned} \left[ -\frac{\hbar^2}{2m} \Delta' + V(x') \right] K(x', t'; x, t) &= \sum_{a'} \langle x' | H | a' \rangle e^{-iE_{a'}(t'-t)/\hbar} \langle a' | x \rangle \theta(t'-t) \\ &= \sum_{a'} \langle x' | a' \rangle E_{a'} e^{-iE_{a'}(t'-t)/\hbar} \langle a' | x \rangle \theta(t'-t) \end{aligned}$$

while

$$\begin{aligned} -i\hbar \frac{\partial}{\partial t'} K &= \sum_{a'} \langle x' | a' \rangle (-E_{a'}) e^{-iE_{a'}(t'-t)/\hbar} \langle a' | x \rangle \\ &= -i\hbar \sum_{a'} \langle x' | a' \rangle e^{-i\frac{E_{a'}(t'-t)}{\hbar}} \langle a' | x \rangle \delta(t'-t) \end{aligned}$$

So

$$\left[ -\frac{\hbar^2}{2m} \Delta' + V(x') - i\hbar \frac{\partial}{\partial t'} \right] K(x', t'; x, t) = -i\hbar \delta^3(x' - x) \delta(t' - t).$$