The Lorentz Group and the Dirac Matrices

A Lorentz transformation turns

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \qquad 1$$

into

$$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu \qquad 2$$

(where the repeated index ν is summed from 0 to 3) while preserving the Minkowski dot product

$$x' \cdot y' = x' \cdot y - x'^0 y'^0 = \vec{x} \cdot \vec{y} - x^0 y^0 \qquad 3$$

in which y' is related to y by

$$y'^\mu = \Lambda^\mu{}_\nu y^\nu . \qquad 4$$

Now $x' \cdot y'$ is

$$x' \cdot y' = x'^\mu y'^\nu \eta_{\mu\nu} \qquad 5$$

where

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \qquad 6$$

So to satisfy (3) the Lorentz matrix $\Lambda^\mu{}_\nu$ must obey the constraint

$$x^\mu y^\nu \eta_{\mu\nu} = x'^a y'^b \eta_{ab}$$
$$= \Lambda^a{}_\mu x^\mu \Lambda^b{}_\nu y^\nu \eta_{ab} \qquad (7)$$

which holds as long as

$$\eta_{\mu\nu} = \Lambda^a{}_\mu \Lambda^b{}_\nu \eta_{ab} . \qquad (8)$$

These matrices Λ form a group because if Λ_1 and Λ_2 preserve all 4-dot products, then so will their product

$$\Lambda_{12}{}^a{}_b = \Lambda_1{}^a{}_c \Lambda_2{}^c{}_b \qquad (9)$$

preserve all 4dot products

$$x'^\mu = \Lambda_{12}{}^\mu{}_\nu x^\nu$$

$$y'^\mu = \Lambda_{12}{}^\mu{}_a y^a$$

$$x'^\mu x'^\nu \eta_{\mu\nu} = x^\mu y^\nu \eta_{\mu\nu}$$

$$= \vec{x}' \cdot \vec{y}' - x'^0 y'^0 = \vec{x} \cdot \vec{y} - x^0 y^0 . \qquad 10$$

These matrices form a group with identity element

$$\Lambda_I{}^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} \qquad 11$$

or

$$\Lambda_I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad 12$$

and inverse elements being those that reverse others. That is, if

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \qquad 13$$

then Λ^{-1} is the Lorentz transformation that takes x' back to x

$$x^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu}\, x'^{\nu} . \qquad 14$$

This group of Lorentz transformations is the homogeneous Lorentz group. If we add in the translations so that

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} + a^{\mu} \qquad (15)$$

then these transformations form the inhomogeneous Lorentz group (aka, the Poincaré group).

A representation of the homogeneous Lorentz group is a set of matrices satisfying the multiplication law

$$D(\Lambda_2)\, D(\Lambda_1) = D(\Lambda_2 \Lambda_1) \qquad (16)$$

of the homogeneous Lorentz group.

A tiny Lorentz transformation is of the form

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu , \qquad (17)$$

where the $\omega^\mu{}_\nu$ are tiny. Neglecting terms like ω^2, we see that this Λ will satisfy the key condition (5) if

$$\begin{aligned} \eta_{\mu\nu} &= \left(\delta^a{}_\mu + \omega^a{}_\mu\right)\left(\delta^b{}_\nu + \omega^b{}_\nu\right) \eta_{ab} \\ &= \delta^a{}_\mu \delta^b{}_\nu \eta_{ab} + \omega^a{}_\mu \delta^b{}_\nu \eta_{ab} \\ &\quad + \delta^a{}_\mu \omega^b{}_\nu \eta_{ab} \\ &= \eta_{\mu\nu} + \omega^a{}_\mu \eta_{a\nu} + \omega^b{}_\nu \eta_{\mu b} . \end{aligned} \qquad (18)$$

Expressions like $\omega^a{}_\mu \eta_{a\nu}$ occur so often that they have a special notation

$$\omega^a{}_\mu \eta_{a\nu} = \omega_{\nu\mu} . \qquad (19)$$

Since $\eta_{\mu\nu}$ is by (6) the same as the identity matrix except for a -1 instead of a $+1$ in the 00 spot

$$\omega_{\nu\mu} = \omega^{\nu}{}_{\mu} \quad \text{if } \nu = 1, 2, \text{ or } 3 \qquad (20)$$

while

$$\omega_{\nu\mu} = -\omega^{\nu}{}_{\mu} \quad \text{if } \nu = 0 . \qquad (21)$$

So $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$ is a tiny Lorentz transformation if

$$\eta_{\mu\nu} = \eta_{\mu\nu} + \omega_{\nu\mu} + \omega_{\mu\nu} \qquad (22)$$

that is, if ω is anti-symmetric

$$\omega_{\mu\nu} = -\omega_{\nu\mu} , \qquad (23)$$

How do derivatives transform? If

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \quad \text{then} \quad x^{\nu} = \Lambda^{-1\,\nu}{}_{\mu} x'^{\mu} \qquad (24)$$

and so by the chain rule

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}} = \Lambda^{-1\,\nu}{}_{\mu} \frac{\partial}{\partial x^{\nu}} . \qquad (25)$$

We may greatly simplify some of what follows by setting both $\hbar$ and c equal to unity. We then can write the unitary operator $U(1+\omega)$ that represents the tiny Lorentz transformation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \qquad (26)$$

as

$$U(1+\omega) = e^{\frac{1}{2} i \omega_{\mu\nu} J^{\mu\nu}} \approx 1 + \frac{i}{2} \omega_{\mu\nu} J^{\mu\nu}. \qquad (27)$$

Since ω is anti-symmetric (21), the Lorentz generators $J^{\mu\nu}$ must also be anti-symmetric

$$J^{\mu\nu} = - J^{\nu\mu} \qquad (28)$$

for any symmetric piece of J would cancel in (27). By considering the product

$$U(\Lambda)\, U(1+\omega)\, U^{-1}(\Lambda) = U\left(\Lambda (1+\omega) \Lambda^{-1}\right) \qquad (29)$$

one can show that

$$U(\Lambda)\, J^{\rho\sigma}\, U^{-1}(\Lambda) = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma J^{\mu\nu}. \qquad (30)$$

By now letting Λ itself be tiny, one further may show that

$$i\left[\tfrac{1}{2}\omega_{\mu\nu}J^{\mu\nu},\, J^{\rho\sigma}\right] = \omega_\mu{}^\rho J^{\mu\sigma} + \omega_\nu{}^\sigma J^{\rho\nu} \tag{31}$$

and so that the generators $J^{\mu\nu}$ of Lorentz transformations satisfy

$$i\left[J^{\mu\nu}, J^{\rho\sigma}\right] = \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu}. \tag{32}$$

Similar work on the inhomogeneous Lorentz group with generators $J^{\mu\nu}$ and P^ρ shows that

$$i\left[P^\mu, J^{\rho\sigma}\right] = \eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho \tag{33}$$

and

$$\left[P^\mu, P^\nu\right] = 0. \tag{34}$$

In all these equations, the summation convention

$$X_a Y^a = \sum_{a=0}^{3} X_a Y^a \tag{35}$$

is used.

So we may find a representation of the homogeneous Lorentz group if we can find matrices $\mathcal{J}^{\mu\nu}$ that are anti-symmetric

$$\mathcal{J}^{\mu\nu} = -\mathcal{J}^{\nu\mu} \tag{36}$$

and that satisfy (32)

$$i\,[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = \eta^{\nu\rho}\mathcal{J}^{\mu\sigma} - \eta^{\mu\rho}\mathcal{J}^{\nu\sigma} - \eta^{\sigma\mu}\mathcal{J}^{\rho\nu} + \eta^{\sigma\nu}\mathcal{J}^{\rho\mu}. \tag{37}$$

We need only set

$$D(\Lambda) = e^{\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}} \tag{38}$$

or

$$D(1+\omega) = 1 + \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu} \tag{39}$$

if ω is tiny.

To find the matrices $\mathcal{J}^{\mu\nu}$ which represent the generators $J^{\mu\nu}$, we introduce four 4X4 matrices — the Dirac matrices — that satisfy the anti-commutation relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \{\gamma^\mu, \gamma^\nu\} = [\gamma^\mu, \gamma^\nu]_+$$
$$= 2\eta^{\mu\nu}. \tag{40}$$

One may verify that the matrices

$$\gamma^0 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \vec{\gamma} = -i\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \tag{41}$$

in which 1 is the 2×2 identity matrix and the $\vec{\sigma}$ are the 3 Pauli matrices. For instance,

$$\gamma^1 = -i\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \;\&\; \gamma^2 = -i\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}$$

and so γ^1 and γ^2 anti-commute

$$\gamma^1\gamma^2 = -\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}$$

$$= -\begin{pmatrix} -\sigma_1\sigma_2 & 0 \\ 0 & -\sigma_1\sigma_2 \end{pmatrix} \tag{42}$$

Since

$$\sigma_i\sigma_j = \delta_{ij} + \sum_{k=1}^{3} i\,\epsilon_{ijk}\,\sigma_k, \tag{43}$$

$\sigma_1\sigma_2 = i\sigma_3$ and so

$$\gamma^1\gamma^2 = i\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \tag{44}$$

But

$$\gamma^2\gamma^1 = \begin{pmatrix} \sigma_2\sigma_1 & 0 \\ 0 & \sigma_2\sigma_1 \end{pmatrix} = -i\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \tag{45}$$

and so

$$\{\gamma^1, \gamma^2\} = 2\eta^{12} = 0. \tag{46}$$

Also

$$\{\gamma^0, \gamma^0\} = 2\gamma^{0\,2} = -2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= -2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2\eta^{00}. \tag{47}$$

Any set of 4 4×4 matrices γ^a that satisfy

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{48}$$

are called Dirac matrices; any set of Dirac matrices suffices to form a representation of the Lorentz group. Note that if S is any 4×4 non-singular matrix (i.e., matrix with an inverse) then if the γ^μ's obey (40) then the γ_s^μ's defined by

$$\gamma_s^\mu = S \gamma^\mu S^{-1} \qquad 49$$

also obey (40) because if

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \qquad 50$$

then

$$S(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) S^{-1} = 2\eta^{\mu\nu} S S^{-1} = 2\eta^{\mu\nu} \qquad (51)$$

and so

$$S\gamma^\mu S^{-1} S \gamma^\nu S^{-1} + S\gamma^\nu S^{-1} S \gamma^\mu S^{-1} = 2\eta^{\mu\nu} \qquad (52)$$

or

$$\gamma_s^\mu \gamma_s^\nu + \gamma_s^\nu \gamma_s^\mu = 2\eta^{\mu\nu}, \qquad (53)$$

So there are infinitely many sets of Dirac matrices.

Take any four 4x4 Dirac matrices γ^0, γ^1, γ^2, & γ^3 and let

$$\mathcal{J}^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]. \qquad (54)$$

These 4x4 matrices $\mathcal{J}^{\mu\nu}$ are anti-symmetric

$$\mathcal{J}^{\mu\nu} = -\mathcal{J}^{\nu\mu} \qquad 55$$

and they satisfy

$$[\mathcal{J}^{\mu\nu}, \gamma^\rho] = -i\gamma^\mu \eta^{\nu\rho} + i\gamma^\nu \eta^{\mu\rho}. \qquad 56)$$

One may now use this relation (56) to show that the $\mathcal{J}^{\mu\nu}$'s obey the commutation relation (37) and so furnish a 4×4 representation of the Lorentz group

$$D(1+\omega) = 1 + \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}. \qquad 57)$$

Rule (56) implies that

$$D(\Lambda)\gamma^\rho D^{-1}(\Lambda) = \Lambda_\sigma{}^\rho \gamma^\sigma \qquad (58)$$

which means that in the representation $D(\Lambda)$ the Dirac matrices transform as a vector. Obviously

$$D(\Lambda)\, 1\, D^{-1}(\Lambda) = 1 \qquad (59)$$

and so the unit matrix is a scalar.