Feynman Path Integrals

Let's start by inserting the identity operator

$$I = \int_{-\infty}^{\infty} dx' \, |x'\rangle\langle x'|$$

into the time-evolution operator $e^{-iHt/\hbar}$ N times in such a way as to divide the time interval t into N tiny bits

$$\epsilon = \frac{t}{N}.$$

Imagine $N \to \infty$. So

$$\langle x_f | e^{-i\epsilon H/\hbar} | x_i \rangle = \langle x_f | \int_{-\infty}^{\infty} dx_N |x_N\rangle\langle x_N| e^{-iH\epsilon/\hbar} \cdots$$

$$\cdots e^{-i\frac{\epsilon H}{\hbar}} \int_{-\infty}^{\infty} dx_j \, |x_j\rangle\langle x_j| \, e^{-i\epsilon H/\hbar} \cdots e^{-iH\epsilon/\hbar} \int_{-\infty}^{\infty} dx_0 |x_0\rangle\langle x_0|x_i\rangle$$

This is an integral over the N x_j's of products of the form

$$\langle x_{j+1} | e^{-i\epsilon H/\hbar} | x_j \rangle.$$

Suppose

$$H = \frac{p^2}{2m} + V(x),$$

so we're talking about a particle of mass m moving in one dimension under the influence of the potential $V(x)$.

Since ϵ is very small, we can write

$$e^{-i\epsilon H/\hbar} = e^{-i\epsilon \frac{V}{2\hbar}} e^{-i\epsilon \frac{P^2}{2m\hbar}} e^{-i\epsilon \frac{V}{2\hbar}},$$

Now we insert the identity operator

$$I = \int_{-\infty}^{\infty} dp_j \; |p_j\rangle\langle p_j| .$$

and we get

$$\langle x_{j+1}| e^{-i\epsilon H/\hbar} |x_j\rangle = \int_{-\infty}^{\infty} dp_j \, \langle x_{j+1}| e^{-i\epsilon \frac{V}{2\hbar}} |p_j\rangle\langle p_j| e^{-i\epsilon \frac{P^2}{2m\hbar}}$$
$$\times e^{-i\epsilon \frac{V}{2\hbar}} |x_j\rangle$$

$$= e^{-i\epsilon \frac{\bar{V}_j}{\hbar}} \int_{-\infty}^{\infty} dp_j \, e^{-i\epsilon \frac{p_j^2}{2m\hbar}} \langle x_{j+1}|p_j\rangle\langle p_j|x_j\rangle$$

where $\bar{V}_j = \frac{1}{2}\left(V(x_{j+1}) + V(x_j)\right)$. So

$$\langle x_{j+1}| e^{-i\epsilon H/\hbar} |x_j\rangle = e^{-i\epsilon \frac{\bar{V}_j}{\hbar}} \int_{-\infty}^{\infty} \frac{dp_j}{2\pi\hbar} \, e^{-\frac{i\epsilon p_j^2}{2m\hbar}} e^{i p_j (x_{j+1} - x_j)/\hbar}$$

We complete the square in the exponential.

$$-\frac{i\epsilon p_j^2}{2m\hbar} + i p_j \frac{(x_{j+1}-x_j)}{\hbar} = -i\epsilon\frac{(p_j-\alpha)^2}{2m\hbar} + \beta$$

So

$$i p_j \frac{(x_{j+1}-x_j)}{\hbar} = \frac{i\epsilon}{m\hbar} p_j \alpha$$

and

$$\frac{i\epsilon\alpha^2}{2m\hbar} = \beta.$$

Well,

$$\alpha = m\frac{(x_{j+1}-x_j)}{\epsilon}$$

and

$$\beta = \frac{i\epsilon\alpha^2}{2m\hbar} = \frac{i\epsilon}{2m\hbar} m^2 \left[\frac{(x_{j+1}-x_j)}{\epsilon}\right]^2$$

$$= \frac{i\epsilon}{\hbar}\,\frac{1}{2}m\left(\frac{x_{j+1}-x_j}{\epsilon}\right)^2.$$

Now $\dfrac{x_{j+1}-x_j}{\epsilon}$ is an effective velocity.

So we set $v_j = \dfrac{x_{j+1}-x_j}{\epsilon}$.

So

$$\langle x_{j+1} | e^{-\frac{i\epsilon H}{\hbar}} | x_j \rangle = e^{-\frac{i\epsilon \bar{V}_j}{\hbar}} \, e^{\frac{i\epsilon \frac{1}{2} m v_j^2}{\hbar}}$$

$$\times \int_{-\infty}^{\infty} \frac{dp_j'}{2\pi\hbar} \, e^{-\frac{i\epsilon (p_j - \alpha)^2}{2m\hbar}}$$

$$= e^{\frac{i\epsilon}{\hbar}\left[\frac{m}{2} v_j^2 - \bar{V}_j\right]} \int_{-\infty}^{\infty} \frac{dp_j'}{2\pi\hbar} \, e^{-\frac{i\epsilon p_j'^2}{2m\hbar}} .$$

The integral converges to (2.5.46)

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} \, e^{-\frac{i\epsilon p^2}{2m\hbar}} = \sqrt{\frac{m}{2\pi i \hbar \epsilon}} = C(\epsilon)$$

but the value of $C(\epsilon)$ paradoxically is not important since in practice one computes ratios of path integrals; in such ratios, the products $C(\epsilon)^N$ cancel.

We have

$$\langle x_{j+1} | e^{-\frac{i\epsilon H}{\hbar}} | x_j \rangle = e^{\frac{i\epsilon}{\hbar}\left[\frac{m}{2} v_j^2 - \bar{V}_j\right]} \sqrt{\frac{m}{2\pi i \hbar \epsilon}} .$$

The quantity inside [ ] is the lagrangian

$$L = \frac{m}{2}\dot{x}^2 - V$$ of the hamiltonian H.

So the amplitude

$$\langle x_f | e^{-\frac{iHt}{\hbar}} | x_i \rangle = \prod_{j=0}^{N} \int_{-\infty}^{\infty} dx_j \sqrt{\frac{mN}{2\pi i \hbar t}}\, e^{\frac{it}{\hbar N}\left(\frac{m}{2} v_j^2 - \bar{V}_j\right)} \delta(x_N - x_f) \times \delta(x_0 - x_i)$$

is a product of N integrals over dx_j of exponentials of

$$\frac{\epsilon}{\hbar}\left(\frac{m}{2} v_j^2 - \bar{V}_j\right) = \frac{L\, dt}{\hbar}.$$

The sum over j

$$\sum_{j=1}^{N} \frac{\epsilon}{\hbar}\left(\frac{m}{2} v_j^2 - \bar{V}_j\right) = \frac{1}{\hbar}\int_0^t dt' L = \frac{S}{\hbar}$$

gives the action S.

The delta functions in the top equation just insure that all paths go from x_i at $t=0$ to x_f at t. Equivalently

$$\langle x_f | e^{-\frac{iHt}{\hbar}} | x_i \rangle = \left(\frac{mN}{2\pi i \hbar t}\right)^{\frac{N}{2}} \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} dx_j\, e^{\frac{it}{\hbar N}\left(\frac{m}{2} v_j^2 - \bar{V}_j\right)}.$$

People often write this as

$$\langle x_f | e^{-\frac{iHt}{\hbar}} | x_i \rangle = \int \mathcal{D}[x(t')]\, e^{i\int_0^t dt' \left[\frac{1}{2} m \dot{x}^2 - V(x(t'))\right]/\hbar}$$

in which all paths $x(t')$ go from x_i to x_f.

One may generalize this discussion to virtually any hamiltonian of interest. So a particle of mass m interacting with a potential $V(\vec{x})$ in 3 dimensions, has an amplitude

$$\langle x_f | e^{-iHt/\hbar} | x_i \rangle = \int \mathcal{D}\vec{x}\, e^{iS/\hbar}$$

where

$$S = \int_0^t dt' \, \tfrac{1}{2} m \dot{\vec{x}}^{\,2} - V(\vec{x}).$$

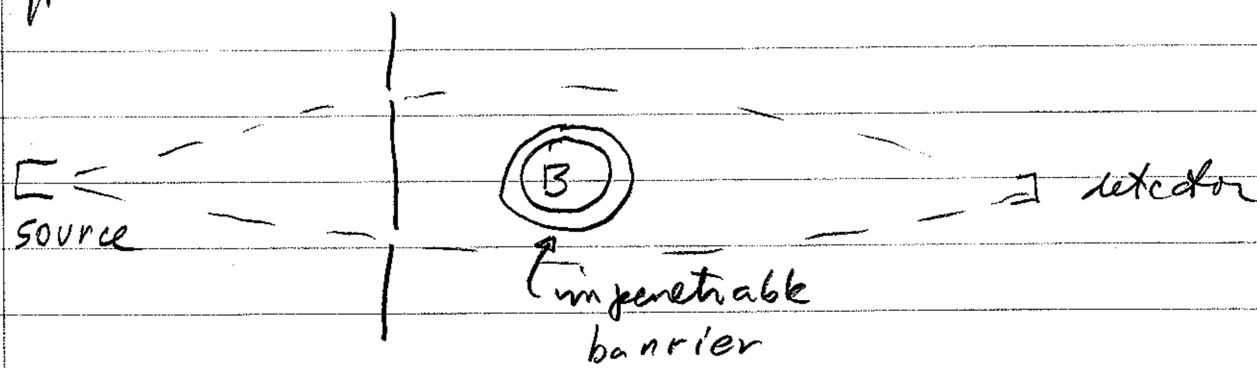
Application. Where will the amplitude be bright? At points $\vec{x}_f$ that can be reached from $\vec{x}_i$ in time t by a classical path, one has

$$0 = \delta S \quad \text{to first order in } \delta x(t).$$

For such points $\vec{x}_f$, an infinity of nearly identical paths will add up with nearly the same phase $S/\hbar$. Note that this classical trajectory is defined to order $\hbar$. So the path integral implies classical physics which may be summarized as

$$\delta S = 0.$$

Application. The Aharonov-Bohm effect:



Suppose a B field points up out of the paper and the source, slits, and detector are symmetrically arranged. Then if $B \neq 0$, we must include the action of the particle interacting with the E&M field. If the particle has charge e then

$$S = \underset{A=0}{S} + \frac{ie}{\hbar c}\int dt\left(\dot{\vec{x}}\cdot\vec{A} - A_0\right).$$

We may ignore A_0 here and set $dx = \dot{x}\,dt$ so that

$$S = \underset{A=0}{S} + \frac{ie}{\hbar c}\int d\vec{x}\cdot A(\vec{x})$$

in which we assumed $\dot{A} = 0$. The difference in phase due to the B-field is

$$\Delta S = \frac{ie}{\hbar c}\oint d\vec{x}\cdot\vec{A} = \frac{ie}{\hbar c}\int \overrightarrow{\nabla\times A}\cdot d\vec{a} = \frac{ie}{\hbar c}\int \vec{B}\cdot d\vec{a}.$$

But the integral is the magnetic flux

$$\int B \cdot da = \Phi_B$$

so

$$\Delta S = \frac{ie}{\hbar c} \Phi_B .$$

Thus as B is varied, we should see the detector go on and off.