Ehrenfest's Theorem

Suppose

$$H = \frac{\vec{p}^{\,2}}{2m} + V(\vec{x}\,).$$

The H-picture $\vec{x}(t)$ is

$$\vec{x}(t) = e^{iHt/\hbar}\,\vec{x}(0)\,e^{-iHt/\hbar}$$

and so

$$\dot{x}_i = e^{iHt/\hbar}\,\frac{i}{\hbar}[H, x_i(0)]\,e^{-iHt/\hbar},$$

Note that

$$H(t) = e^{iHt/\hbar}\,H(0)\,e^{-iHt/\hbar} = H(0).$$

So

$$\dot{x}_i = e^{iHt/\hbar}\,\frac{i}{\hbar}\left[\frac{\vec{p}^{\,2}}{2m}, x_i\right]e^{-iHt/\hbar}$$

$$[\vec{p}^{\,2}, x_i] = [p_i^2, x_i] = p_i\,[p_i, x_i] + [p_i, x_i]\,p_i$$

$$= -i\hbar\, 2p_i = -2i\hbar p_i \qquad \text{so}$$

$$\dot{x}_i = e^{iHt/\hbar}\,\frac{p_i(0)}{m}\,e^{-iHt/\hbar} = \frac{p_i(t)}{m}.$$

Thus

$$\ddot{x}_i = e^{iHt/\hbar}\left[H(0), \frac{p_i(0)}{m}\right]e^{-iHt/\hbar}$$

So

$$m\ddot{x}_i(t) = e^{\frac{iHt}{\hbar}} \frac{i}{\hbar}[V(\vec{x}), p_i(0)]\, e^{-iHt/\hbar}$$

Problem 1.29 a, done in class, shows that

$$[V(\vec{x}), p_i] = i\hbar \frac{\partial V}{\partial x_i}.$$

So we have

$$m\ddot{x}_i(t) = e^{\frac{iHt}{\hbar}} \frac{i}{\hbar} i\hbar \frac{\partial V}{\partial x_i} e^{-iHt}$$

$$= -\, e^{iHt/\hbar} \frac{\partial V}{\partial x_i} e^{-iHt/\hbar}$$

$$= -\frac{\partial V}{\partial x_i}(x(t)).$$

So

$$m\ddot{\vec{x}}(t) = \dot{\vec{p}}(t) = -\vec{\nabla} V(\vec{x}(t)).$$

P. Ehrenfest 1927.

Clearly, if $|\psi\rangle$ is any H-pic state

$$\langle\psi| m\ddot{\vec{x}} |\psi\rangle = -\langle\psi| \vec{\nabla} V(\vec{x}(t)) |\psi\rangle.$$