Particle in an E & M Field

In terms of the 4-vector $(\vec{A}, \phi/c)$, the non-relativistic Lagrangian is

$$L = \frac{1}{2} m \vec{v}^{\,2} + e \frac{\vec{v}}{c} \cdot A - e\phi$$

where $\vec{v} = \dot{\vec{x}} = d\vec{x}/dt$. The action S is

$$S = \int dt \left[\frac{1}{2} m \dot{\vec{x}}^{\,2} + \frac{e}{c} \dot{x} \cdot A - e\phi \right]$$

$$= \int dt \left(\frac{1}{2} m \dot{\vec{x}}^{\,2} - e\phi \right) + \frac{e}{c} \int d\vec{x} \cdot \vec{A} .$$

The canonical momentum $\vec{p}$ is

$$\vec{p} = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + \frac{e}{c} \vec{A}$$

so $\vec{p}$ differs from the mechanical momentum $\vec{\pi} = m\dot{\vec{x}} = \vec{p} - \frac{e}{c}\vec{A}$.

The Hamiltonian H is

$$H = p\dot{x} - L = \frac{p\left(p - \frac{e}{c} A \right)}{m} - \frac{\left(p - \frac{e}{c} A \right)^2}{2m}$$

$$- \frac{e}{c} \frac{\overrightarrow{\left(p - \frac{e}{c} A \right)} \cdot \vec{A}}{m} + e\phi$$

So

$$H = \frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{m} - \frac{\left(p - \frac{e}{c}A\right)^2}{2m} + e\phi$$

or

$$H = \frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m} + e\phi .$$

So H is $e\phi$ plus the mechanical momentum squared over 2m.