Invariances of Dirac's Equation

Dirac's equation

$$\left[\left(\hbar\partial_\mu + i\frac{e}{c}A_\mu(x)\right)\gamma^\mu + mc\right]\psi(x) = 0 \qquad (1)$$

is an equation that describes how the fields $\psi(x)$ and $A_\mu(x)$ that represent electrons, positrons, and photons interact. This equation is invariant both under Lorentz transformations

$$U(\Lambda)\psi_\ell(x)U^{-1}(\Lambda) = \sum_{\ell'} D_{\ell\ell'}(\Lambda^{-1})\psi_{\ell'}(\Lambda x) \qquad 2$$

$$U(\Lambda)A_\mu(x)U^{-1}(\Lambda) = \sum_{\nu=0}^{3} \Lambda^{-1}{}_\mu{}^\nu A_\nu(\Lambda x) \qquad 3$$

and under gauge transformations

$$U(\lambda)\psi_\ell(x)U^{-1}(\lambda) = e^{-ie\lambda(x)/\hbar c}\psi_\ell(x) \qquad 4$$

$$U(\lambda)A_\mu(x)U^{-1}(\lambda) = A_\mu(x) + \partial_\mu\lambda(x), \qquad 5$$

(which actually occur together).

Under a gauge transformation

$$U(\lambda)\left[\left(\hbar\partial_\mu + i\frac{e}{c}A_\mu\right)\gamma^\mu + mc\right]\psi(x)U(\lambda) = 0 \qquad 6$$

Dirac's equation (1) becomes

$$\left[\left(\hbar \partial_\mu + \frac{ie}{c} \left(A_\mu + \partial_\mu \lambda \right) \right) \gamma^\mu + mc \right]_{\ell\ell'} e^{-ie\lambda(x)/\hbar c} \psi_{\ell'}(x) = 0 \qquad 7$$

or

$$e^{-ie\lambda} \left[\left(\hbar \partial_\mu - \frac{ie\hbar}{c} \partial_\mu \lambda + \frac{ie}{c} \left(A_\mu + \partial_\mu \lambda \right) \right) \gamma^\mu + mc \right] \psi = 0 \qquad 8$$

which implies

$$\left[\left(\hbar \partial_\mu + \frac{ie}{c} A_\mu \right) \gamma^\mu + mc \right] \psi = 0 \qquad 9$$

which is (1) again.

To see that Dirac's equation (1) is invariant under Lorentz transformations, we need to recall that γ-matrices transform as vectors under Lorentz transformations

$$D(\Lambda) \, \gamma^\rho \, D^{-1}(\Lambda) = \Lambda_\sigma{}^\rho \, \gamma^\sigma \qquad 10$$

which is (58) of my notes on the Lorentz group and Dirac matrices. We also need to recall that derivatives transform as

$$\frac{\partial}{\partial x'^\mu} = \Lambda^{-1\,\nu}{}_\mu \frac{\partial}{\partial x^\nu} \qquad 11$$

which is (25) of those same notes.

Thus under a Lorentz transformation, Dirac's equation becomes

$$U(\Lambda)\left[(\hbar\partial_\mu + \tfrac{ie}{c}A_\mu)\gamma^\mu + mc\right]U^{-1}(\Lambda)\,U(\Lambda)\psi(x)U^{-1}(\Lambda) = 0$$

$$= \left[(\hbar\partial_\mu + i\tfrac{e}{c}UA_\mu U^{-1})\gamma^\mu + mc\right]U\psi U^{-1} = 0 \qquad 12$$

So

$$0 = \left[(\hbar\partial_\mu + i\tfrac{e}{c}\,\Lambda_\mu{}^\nu A_\nu(\Lambda x))\gamma^\mu + mc\right]D(\Lambda^{-1})\psi(\Lambda x). \qquad 13$$

So

$$D^{-1}(\Lambda)\left[(\hbar\partial_\mu + i\tfrac{e}{c}\Lambda_\mu{}^\nu A_\nu(\Lambda x))D(\Lambda)\gamma^\mu D^{-1}(\Lambda) + mc\right]\psi(\Lambda x) = 0. \quad (14)$$

By (10), we have

$$\left[(\hbar\partial_\mu + i\tfrac{e}{c}\Lambda_\mu{}^\nu A_\nu(\Lambda x))\Lambda_\sigma{}^\mu\gamma^\sigma + mc\right]\psi(\Lambda x) = 0. \qquad 15$$

Now

$$x^\mu = \Lambda^{-1\,\mu}{}_\nu(\Lambda x)^\nu \Leftrightarrow \Lambda^\nu{}_\mu x^\mu = (\Lambda x)^\nu \qquad 16$$

so

$$\frac{\partial}{\partial x^\mu} = \frac{\partial(\Lambda x)^\nu}{\partial x^\mu}\frac{\partial}{\partial(\Lambda x)^\nu} = \Lambda^\nu{}_\mu\frac{\partial}{\partial\Lambda x^\nu} \qquad 17$$

whence

$$\left[(\hbar\Lambda^\nu{}_\mu\frac{\partial}{\partial\Lambda x^\nu} + \tfrac{ie}{c}\Lambda_\mu{}^\nu A_\nu(\Lambda x))\Lambda_\sigma{}^\mu\gamma^\sigma + mc\right]\psi(\Lambda x) = 0. \qquad 18$$

Let $\quad x'^{\mu} = (\Lambda x)^{\mu}.$ 19

Then (18) is

$$\left[\left(\hbar \Lambda^{\nu}{}_{\mu} \frac{\partial}{\partial x'^{\nu}} + \frac{ie}{c} \Lambda^{-1}{}_{\mu}{}^{\nu} A_{\nu}(x') \right) \Lambda_{\sigma}{}^{\mu} \gamma^{\sigma} + mc \right] \psi(x') = 0. \quad 20$$

Now the defining condition for a Lorentz matrix Λ is

$$\eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma} \quad 21$$

so

$$\eta^{\alpha\rho} \eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = \eta^{\alpha\rho} \eta_{\rho\sigma} = \delta^{\alpha}_{\sigma} \quad 22$$

so

$$\eta_{\mu\nu} \Lambda^{\mu\alpha} \Lambda^{\nu}{}_{\sigma} = \delta^{\alpha}_{\sigma} \quad 23$$

or

$$\Lambda_{\nu}{}^{\alpha} \Lambda^{\nu}{}_{\sigma} = \delta^{\alpha}_{\sigma}. \quad 24$$

That is,

$$(\Lambda^{-1})^{\alpha}{}_{\nu} = \Lambda_{\nu}{}^{\alpha}. \quad 25$$

Thus

$$\Lambda^{\nu}{}_{\mu} \Lambda_{\sigma}{}^{\mu} = \Lambda^{\nu}{}_{\mu} \Lambda^{-1\,\mu}{}_{\sigma} = \delta^{\nu}_{\sigma}. \quad 26$$

And clearly

$$\Lambda_{\sigma}{}^{\mu} \Lambda^{-1}{}_{\mu}{}^{\nu} = \delta^{\nu}_{\sigma}. \quad 27$$

Thus (20) is really Eq. (1) with $x \to x'$:

$$\left[\left(\hbar \frac{\partial}{\partial x'^{\sigma}} + \frac{ie}{c} A_{\sigma}(x') \right) \gamma^{\sigma} + mc \right] \psi(x') = 0. \quad (28)$$