Dirac's Equation

Dirac wanted a linear wave equation that gave

$$E^2 = m^2c^4 + p^2c^2. \qquad 1$$

He found

$$(i p_a \gamma^a + mc)\psi = 0 \qquad 2$$

in which the index a is summed from 0 to 3 and in which the γ's are four 4x4 matrices that obey

$$\{\gamma^a, \gamma^b\} = \gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab} \qquad 3$$

where $\eta^{ab} = (1,1,1,-1)$ or $(-1,1,1,1)$ depending upon where you put the 00 entry.
Here

$$\vec{p} = \frac{\hbar}{i}\vec{\nabla} \text{ and } p^0 = -\frac{\hbar}{i}\frac{\partial}{\partial x^0} = -\frac{\hbar}{ic}\frac{\partial}{\partial t} \qquad 4$$

so

$$(\hbar \partial_a \gamma^a + mc)\psi = 0 \qquad 5$$

is the Dirac equation for a free electron. Now (3) gives

$$p_a\gamma^a p_b\gamma^b = \frac{1}{2}\left(p_a p_b \gamma^a\gamma^b + p_b p_a \gamma^b\gamma^a\right)$$

$$= \frac{1}{2} p_a p_b \left(\gamma^a\gamma^b + \gamma^b\gamma^a\right)$$

$$= \frac{1}{2} p_a p_b \{\gamma^a, \gamma^b\} = p_a p_b \eta^{ab} = p_a p^a. \qquad (6)$$

Thus Dirac's equation (2) implies that

$$(ip_a\gamma^a - mc)(ip_b\gamma^b + mc)\psi = 0$$

$$= (-p_a p^a - m^2c^2)\psi = 0$$

$$= \left(\frac{E^2}{c^2} - \vec{p}^{\,2} - m^2c^2\right)\psi = 0$$

or

$$0 = (E^2 - c^2\vec{p}^{\,2} - m^2c^4)\psi \qquad 7$$

which is what Dirac wanted.

So for a free electron

$$(ip_a\gamma^a + mc)\psi = (i\vec{p}\cdot\vec{\gamma} - ip^0\gamma^0 + mc)\psi = 0$$

and so

$$ip^0\gamma^0\psi = (ip\cdot\gamma + mc)\psi$$

$$+i\frac{E^0}{c}\gamma^0\psi = (ip\cdot\gamma + mc)\psi \qquad 8$$

Since $\gamma^{0\,2} = \eta^{00} = -1$

$$+i\frac{H}{c}\psi = \gamma^0(ip\cdot\gamma + mc)\psi$$

$$H\psi = +ic\gamma^0(ip\cdot\gamma + mc)\psi, \qquad (9)$$

$$= (-c\gamma^0 p\cdot\gamma + imc^2\gamma^0)\psi$$

So for a free electron the hamiltonian H is

$$H = +ic\gamma^0 (i p \cdot \gamma + mc)$$

$$= -c\gamma^0 \vec{p}\cdot\vec{\gamma} + imc^2 \gamma^0. \qquad (10)$$

It is convenient to use an explicit set of Dirac matrices:

$$\vec{\gamma} = -i \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \qquad \gamma^0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (11)$$

To find the hamiltonian for an electron in an electromagnetic field A_μ, Dirac replaced ∂_μ by the covariant derivative

$$\partial_\mu + \frac{ie}{\hbar c} A_\mu. \qquad 12$$

So

$$\vec{p} = \frac{\hbar}{i}\nabla \longrightarrow \frac{\hbar}{i}\left(\vec{\nabla} + \frac{ie}{\hbar c}\vec{A}\right) \qquad 13$$

or

$$\vec{p} \longrightarrow \vec{p} + \frac{e}{c}\vec{A}, \quad \text{and}$$

$$p^0 = -\frac{\hbar}{i}\frac{\partial}{\partial x^0} = -\frac{\hbar}{i}\partial_0 \longrightarrow -\frac{\hbar}{i}\left(\partial_0 + \frac{ie}{\hbar c}A_0\right) \qquad (14)$$

So

$$p^0 \to p^0 - \frac{e}{c} A_0 = p^0 + \frac{e}{c} A^0 . \qquad 15$$

Thus instead of (9) we have

$$H + eA^0 = c p^0 + e A^0 = - c\gamma^0 (\vec{p} + \frac{e}{c}\vec{A}) \cdot \vec{\gamma} + i mc^2 \gamma^0 \qquad 16$$

which means that Dirac's hamiltonian for an electron in an electromagnetic field A_μ is

$$H = c p^0 = - e A^0 - c\gamma^0 (\vec{p} + \frac{e}{c}\vec{A}) \cdot \vec{\gamma} + i mc^2 \gamma^0 . \qquad 17$$

It is this equation that leads to energy levels of the hydrogen atom that are exact apart from radiative corrections.

To get a feel for (17), we will look at

$$\left(\frac{H + eA^0}{c} \right)^2 = \left[- \gamma^0 (\vec{p} + \frac{e}{c}\vec{A}) \cdot \vec{\gamma} + i mc^2 \gamma^0 \right]^2 \qquad 18$$

for a slow electron. The cross terms vanish because by (3)

$$\gamma^0 \gamma^i \gamma^0 + \gamma^0 \gamma^0 \gamma^i = \gamma^0 \{ \gamma^0 \gamma^i \} = 0 . \qquad 19$$

So

$$\left(\frac{H + eA^0}{c} \right)^2 = \left[(\vec{p} + \frac{e}{c}\vec{A}) \cdot \gamma^0 \vec{\gamma} \right]^2 + m^2 c^4 \qquad 20$$

since by (3)

$$(i mc^2 \gamma^0)^2 = m^2 c^4 . \qquad 21$$

Now by (3)

$$\gamma^0 \gamma^i \gamma^0 \gamma^j = -\gamma^0 \gamma^0 \gamma^i \gamma^j = \gamma^i \gamma^j, \qquad 22$$

so (20) is

$$\left(\frac{H + eA^0}{c}\right)^2 = \left(p_i + \frac{e}{c}A_i\right)\left(p_j + \frac{e}{c}A_j\right)\gamma^i \gamma^j + m^2 c^2. \qquad 23$$

Note that

$$Q_i Q_j \gamma^i \gamma^j = \frac{1}{4}\left(Q_i Q_j + Q_j Q_i\right)\left(\gamma^i \gamma^j + \gamma^j \gamma^i\right) + \frac{1}{4}\left(Q_i Q_j - Q_j Q_i\right)\left(\gamma^i \gamma^j - \gamma^j \gamma^i\right) \qquad (24)$$

$$= \frac{1}{4}\left(Q_i Q_j + Q_j Q_i\right) 2\eta^{ij} + \frac{1}{4}[Q_i, Q_j][\gamma^i, \gamma^j]$$

$$= \vec{Q} \cdot \vec{Q} + \frac{1}{4}[Q_i, Q_j][\gamma^i, \gamma^j]. \qquad (25)$$

What is $[\gamma^i, \gamma^j]$? By (10), we find

$$[\gamma^i, \gamma^j] = (-i)^2 \left[\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}\right]$$

$$= -\begin{pmatrix} -[\sigma_i, \sigma_j] & 0 \\ 0 & -[\sigma_i, \sigma_j] \end{pmatrix} = \begin{pmatrix} [\sigma_i, \sigma_j] & 0 \\ 0 & [\sigma_i, \sigma_j] \end{pmatrix}. \qquad 26$$

Now $$[\sigma_i, \sigma_j] = 2i\,\epsilon_{ijk}\,\sigma_k$$

where a sum over k from 1 to 3 is understood, So

$$[\gamma^i, \gamma^j] = 2i\,\epsilon_{ijk}\begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}. \qquad 27$$

If we use $\bar{\sigma}_{4k}$ for this matrix

$$\bar{\sigma}_{4k} = \begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}, \qquad 28$$

then we have

$$[\gamma^i, \gamma^j] = 2i\,\epsilon_{ijk}\,\bar{\sigma}_{4k}. \qquad 29$$

So (20), (25), and (28) now give

$$\left(\frac{H + eA^0}{c}\right)^2 = \left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 + m^2c^4 + \frac{1}{4}\left[p_i + \frac{e}{c}A_i,\ p_j + \frac{e}{c}A_j\right]2i\,\epsilon_{ijk}\,\bar{\sigma}_{4k}, \qquad (30)$$

Now

$$\left[p_i + \frac{e}{c}A_i,\ p_j + \frac{e}{c}A_j\right] = \frac{e}{c}[p_i, A_j] + \frac{e}{c}[A_i, p_j]$$

$$= \frac{\hbar}{i}\frac{e}{c}\left(\partial_i A_j - \partial_j A_i\right). \qquad 31$$

Thus

$$\left(\frac{H+eA^0}{c}\right)^2 = \left(\vec{p}+\frac{e}{c}\vec{A}\right)^2 + m^2c^2$$
$$+ \frac{i}{2}\epsilon_{ijk}\,\bar{\sigma}_{4k}\,\frac{\hbar}{i}\frac{e}{c}\left(\partial_i A_j - \partial_j A_i\right)$$

$$= \left(\overrightarrow{p+\frac{e}{c}A}\right)^2 + m^2c^2 + \frac{e\hbar}{c}\vec{\sigma}_4\cdot(\nabla\times A) \qquad 32$$

or with

$$B = \nabla \times A \qquad 33$$

for the magnetic field

$$\left(\frac{H+eA^0}{c}\right)^2 = \left(\vec{p}+\frac{e}{c}\vec{A}\right)^2 + m^2c^2 + \frac{e\hbar}{c}\vec{\sigma}_4\cdot\vec{B}, \qquad 34$$

So far, this is exact. Now we consider a slow electron for which

$$H = mc^2 + H_1 \qquad (35)$$

in which H_1 is small compared to mc^2 so that

$$\left(\frac{H+eA^0}{c}\right)^2 = \left(mc + \frac{H_1+eA^0}{c}\right)^2 \approx m^2c^2 + 2m(H_1+eA^0). \qquad (36)$$

So

$$2m(H_1+eA^0) = \left(\overrightarrow{p+\frac{e}{c}A}\right)^2 + \frac{e\hbar}{c}\vec{\sigma}_4\cdot\vec{B} \qquad 37$$

or

$$H_1 = -eA^0 + \frac{\left(\overrightarrow{p + \frac{e}{c}A}\right)^2}{2m} + \frac{e\hbar}{2mc}\vec{\sigma}_4 \cdot \vec{B} . \quad (38)$$

Thus if we identify the spin part of the electron's angular momentum as

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}_4 = \frac{\hbar}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \quad (39)$$

then

$$H_1 = -eA^0 + \frac{(p + \frac{e}{c}A)^2}{2m} + \frac{e}{mc}\vec{S} \cdot \vec{B} \quad (40)$$

which is experimentally correct as shown by both the Stern-Gerlach experiment and by Zeeman's effect.

This result is surprising because for orbital angular momentum $\vec{L}$, the analogous term is

$$\frac{e}{2mc}\vec{L} \cdot \vec{B} . \quad (41)$$

Note that (5) and (13) give

$$\left[\left(\hbar\, \partial_\mu + \frac{ie}{c}A_\mu\right)\gamma^\mu + mc\right]\psi = 0 \quad 42$$

as Dirac's equation for an electron in an EdM field A_μ.