21

Renormalization group

21.1 Renormalization Group

First-order perturbation theory gives a good answers when the interaction is weak which is when the coupling constant is small. Suppose, however, that one could compute all the relevant Feynman diagrams for a given process. Then one could approximate the exact result for the process as a first-order Feynman diagram with a coupling constant chosen to represent the Feynman diagrams of all orders. The value of the coupling constant chosen to represent the Feynman diagrams of all orders turns out, as we'll see in this chapter, to vary with the energy of the process. For most nonabelian internal-symmetry groups, the value of the chosen coupling constant gets smaller slowly (as an inverse logarithm) as the energy of the process increases. This property is referred to as the renormalization group.

21.2 Renormalization and Interpolation

Probably because they describe point particles, quantum field theories are divergent. Unknown physics at very short distance scales, removes these infinities. Since these infinities really are absent, we can cancel them consistently in **renormalizable** theories by a procedure called **renormalization**. One starts with an action that contains infinite, unknown charges and masses, such as $e_0 = e - \delta e$ and $m_0 = m - \delta m$, in which $-e$ is the charge of the electron and m is its mass. One then uses δe and δm to cancel unwanted infinite terms as they appear in perturbation theory.

Because the underlying theory is finite, the value of a divergent scattering amplitude may change by a finite amount when we compute it at two different sets of initial and final momenta. This happens, for example, in the theory of a scalar field ϕ with action density

$$\mathcal{L} = -\frac{1}{2}\partial_i\phi\,\partial^i\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4. \tag{21.1}$$

The amplitude for the elastic scattering of two bosons of initial four-momenta p_1 and p_2 into two of final momenta p_1' and p_2' is

$$\begin{aligned} A = g - \frac{g^2}{16\pi^2}\int_0^\infty k^3 dk \int_0^1 dx \Big\{ & [k^2 + m^2 - sx(1-x)]^{-2} \\ & + [k^2 + m^2 - tx(1-x)]^{-2} + [k^2 + m^2 - ux(1-x)]^{-2} \Big\} \end{aligned} \tag{21.2}$$

to one-loop order (Weinberg, 1995, section 12.2). In this formula, s, t, and u are the Mandelstam variables $s = -(p_1 + p_2)^2$, $t = -(p_1 - p_1')^2$, and $u = -(p_1 - p_2')^2$, and $k^2 = k_0^2 + k_1^2 + k_2^2 + k_3^2$ after a Wick rotation.

The amplitude $A(s, t, u)$ diverges logarithmically, but the difference between it and its value $A_0 = A(s_0, t_0, u_0)$ at some point (s_0, t_0, u_0) is finite

$$\begin{aligned} A(s,t,u) - A_0 = -\frac{g^2}{16\pi^2}\int_0^\infty k^3 dk \int_0^1 dx \Bigg\{ & \\ & \frac{x(1-x)(s-s_0)[2k^2 + 2m^2 - (s+s_0)x(1-x)]}{[k^2+m^2-sx(1-x)]^2[k^2+m^2-s_0x(1-x)]^2} \\ & + (s, s_0 \to t, t_0) + (s, s_0 \to u, u_0)\Bigg\}. \end{aligned} \tag{21.3}$$

The second and third terms within the big curly brackets are the same as the first term but with s and s_0 replaced by t and t_0 in the second term and by u and u_0 in the third. The k integral is finite

$$\begin{aligned} A(s,t,u) - A_0 = -\frac{g^2}{32\pi^2}\int_0^1 dx \Bigg\{ & \log\left[\frac{m^2 - s_0x(1-x)}{m^2 - sx(1-x)}\right] \\ & + \log\left[\frac{m^2 - t_0x(1-x)}{m^2 - tx(1-x)}\right] + \log\left[\frac{m^2 - u_0x(1-x)}{m^2 - ux(1-x)}\right]\Bigg\}. \end{aligned} \tag{21.4}$$

If we choose as the renormalization point $s_0 = t_0 = u_0 = -4\mu^2/3$, then we get the usual result (Weinberg, 1995, 1996, sections 12.2, 18.1–2).

21.3 Renormalization group in quantum field theory

We can use the scattering amplitude (21.4) to define a **running coupling constant** g_μ at energy scale μ as the experimentally measured, finite, physical amplitude A at $s_0 = t_0 = u_0 = -\mu^2$. Then with $g_\mu \equiv A(s_0, t_0, u_0)$, the scattering amplitude (21.4) is finite to order g^2

$$A(s,t,u) = g_\mu - \frac{g^2}{32\pi^2}\int_0^1 dx \left\{ \log\left[\frac{1+(\mu/m)^2x(1-x)}{1-s/m^2x(1-x)}\right] + \log\left[\frac{1+(\mu/m)^2x(1-x)}{1-t/m^2x(1-x)}\right] + \log\left[\frac{1+(\mu/m)^2x(1-x)}{1-u/m^2x(1-x)}\right]\right\}. \tag{21.5}$$

Callan (Callan, 1970) and Symanzik (Symanzik, 1970) noticed that this scattering amplitude, like any physical quantity, is **independent of the sliding scale** μ. Thus its derivative with respect to μ vanishes

$$\begin{aligned} 0 = \frac{\partial A(s,t,u)}{\partial\mu} &= \frac{\partial g_\mu}{\partial\mu} - \frac{3g^2}{32\pi^2}\frac{\partial}{\partial\mu}\int_0^1 dx\, \log\left[1+(\mu/m)^2x(1-x)\right] \\ &= \frac{\partial g_\mu}{\partial\mu} - \frac{3g^2}{32\pi^2}\int_0^1 dx\, \frac{(2\mu/m^2)x(1-x)}{1+(\mu/m)^2x(1-x)}. \end{aligned} \tag{21.6}$$

For $\mu \gg m$, the integral is $2/\mu$. So at high energies, the running coupling constant obeys the differential equation

$$\mu\frac{\partial g_\mu}{\partial\mu} \equiv \beta(g_\mu) = \frac{3g^2}{16\pi^2} = \frac{3g_\mu^2}{16\pi^2} \tag{21.7}$$

in which the last equality holds to second order in g_μ. Integrating the beta function $\beta(g_\mu)$, we get

$$\log\frac{E}{M} = \int_M^E \frac{d\mu}{\mu} = \int_{g_M}^{g_E}\frac{dg_\mu}{\beta(g_\mu)} = \frac{16\pi^2}{3}\int_{g_M}^{g_E}\frac{dg_\mu}{g_\mu^2} = \frac{16\pi^2}{3}\left(\frac{1}{g_M} - \frac{1}{g_E}\right). \tag{21.8}$$

So the running coupling constant g_μ at energy $\mu = E$ is

$$g_E = \frac{g_M}{1 - 3\,g_M\,\log(E/M)/16\pi^2}. \tag{21.9}$$

As the energy $E = \sqrt{s}$ rises above M, while staying below the singular value $E = M\exp(16\pi^2/3g_M)$, the running coupling constant g_E slowly increases, as does the scattering amplitude, $A \approx g_E$.

Example 21.1 (Quantum electrodynamics) Vacuum polarization makes the one-loop amplitude for the scattering of two electrons proportional to $A(q^2) = e^2\left[1+\pi(q^2)\right]$ rather than to e^2 (Gell-Mann and Low, 1954), (Weinberg, 1995, section 11.2). Here e is the renormalized charge, $q = p_1' - p_1$ is the four-momentum transferred to the first electron, and

$$\pi(q^2) = \frac{e^2}{2\pi^2}\int_0^1 x(1-x)\log\left[1+\frac{q^2x(1-x)}{m^2}\right]dx \tag{21.10}$$

represents the polarization of the vacuum. One defines the square of the running coupling constant e_μ^2 to be the amplitude $A(q^2)$ at $q^2 = \mu^2$

$$e_\mu^2 = A(\mu^2) = e^2 \left[1 + \pi(\mu^2)\right]. \tag{21.11}$$

For $q^2 = \mu^2 \gg m^2$, the vacuum-polarization term $\pi(\mu^2)$ is (exercise 21.1)

$$\pi(\mu^2) \approx \frac{e^2}{6\pi^2}\left[\log\frac{\mu}{m} - \frac{5}{6}\right]. \tag{21.12}$$

The amplitude then is

$$A(q^2) = e_\mu^2 \frac{1 + \pi(q^2)}{1 + \pi(\mu^2)}, \tag{21.13}$$

and since it must be independent of μ, we have

$$0 = \frac{d}{d\mu}\frac{A(q^2)}{1 + \pi(q^2)} = \frac{d}{d\mu}\frac{e_\mu^2}{1 + \pi(\mu^2)} \approx \frac{d}{d\mu}\left\{e_\mu^2\left[1 - \pi(\mu^2)\right]\right\}. \tag{21.14}$$

So by differentiating e_μ and the vacuum-polarization term (21.12), we find

$$0 = 2e_\mu\left(\frac{de_\mu}{d\mu}\right)\left[1 - \pi(\mu^2)\right] - e_\mu^2\frac{d\pi(\mu^2)}{d\mu} = 2e_\mu\left(\frac{de_\mu}{d\mu}\right)\left[1 - \pi(\mu^2)\right] - e_\mu^2\frac{e^2}{6\pi^2\mu}. \tag{21.15}$$

But by (21.10) the vacuum-polarization term $\pi(\mu^2)$ is of order e^2, which is the same as e_μ^2 to lowest order in e_μ. Thus we arrive at the Callan-Symanzik equation

$$\mu\frac{de_\mu}{d\mu} \equiv \beta(e_\mu) = \frac{e_\mu^3}{12\pi^2} \tag{21.16}$$

which we can integrate

$$\log\frac{E}{M} = \int_M^E \frac{d\mu}{\mu} = \int_{e_M}^{e_E}\frac{de_\mu}{\beta(e_\mu)} = 12\pi^2\int_{e_M}^{e_E}\frac{de_\mu}{e_\mu^3} = 6\pi^2\left(\frac{1}{e_M^2} - \frac{1}{e_E^2}\right)$$

to

$$e_E^2 = \frac{e_M^2}{1 - e_M^2\,\log(E/M)/6\pi^2}. \tag{21.17}$$

The fine-structure constant $e_\mu^2/4\pi$ slowly rises from $\alpha = 1/137.036$ at m_e to

$$\frac{e^2(45.5\text{GeV})}{4\pi} = \frac{\alpha}{1 - 2\alpha\log(45.5/0.00051)/3\pi} = \frac{1}{134.6} \tag{21.18}$$

at $\sqrt{s} = 91$ GeV. When all light charged particles are included, one finds that the fine-structure constant rises to $\alpha = 1/128.87$ at $E = 91$ GeV. □

Example 21.2 (Quantum chromodynamics) The beta functions of scalar field theories and of quantum electrodynamics are positive, and so interactions in these theories become stronger at higher energy scales. But Yang-Mills theories have beta functions that can be negative because of the cubic interactions of the gauge fields and the ghost fields (20.287). If the gauge group is $SU(3)$, then the beta function is

$$\mu \frac{dg_\mu}{d\mu} \equiv \beta(g_\mu) = -\frac{11g^3}{16\pi^2} = -\frac{11g_\mu^3}{16\pi^2} \tag{21.19}$$

to lowest order in g_μ. Integrating, we find

$$\log \frac{E}{M} = \int_M^E \frac{d\mu}{\mu} = \int_{g_M}^{g_E} \frac{dg_\mu}{\beta(g_\mu)} = -\frac{16\pi^2}{11} \int_{g_M}^{g_E} \frac{dg_\mu}{g_\mu^3} = \frac{8\pi^2}{11} \left(\frac{1}{g_M^2} - \frac{1}{g_E^2} \right) \tag{21.20}$$

and

$$g_E^2 = g_M^2 \left[1 + \frac{11g_M^2}{8\pi^2} \log \frac{E}{M} \right]^{-1} \tag{21.21}$$

which shows that as the energy E of a scattering process increases, the running coupling slowly **decreases**, going to zero at infinite energy, an effect called **asymptotic freedom** (Gross and Wilczek, 1973; Politzer, 1973).

If the gauge group is $SU(N)$, and the theory has n_f flavors of quarks with masses below μ, then the beta function is

$$\beta(g_\mu) = -\frac{g_\mu^3}{4\pi^2} \left(\frac{11N}{12} - \frac{n_f}{6} \right) \tag{21.22}$$

which is negative as long as $n_f < 11N/2$. Using this beta function with $N = 3$ and again integrating, we get instead of (21.21)

$$g_E^2 = g_M^2 \left[1 + \frac{(11 - 2n_f/3)g_M^2}{16\pi^2} \log \frac{E^2}{M^2} \right]^{-1}. \tag{21.23}$$

So with

$$M^2 \equiv \Lambda^2 \exp \left(\frac{16\pi^2}{(11 - 2n_f/3)g_M^2} \right) \tag{21.24}$$

we find (exercise 21.2)

$$\alpha_s(E) \equiv \frac{g^2(E)}{4\pi} = \frac{12\pi}{(33 - 2n_f) \log(E^2/\Lambda^2)} \tag{21.25}$$

which expresses the dimensionless QCD coupling constant $\alpha_s(E)$ appropriate to energy E in terms of a parameter Λ that has the dimension of energy. Sidney Coleman called this **dimensional transmutation**. For $\Lambda = 230$

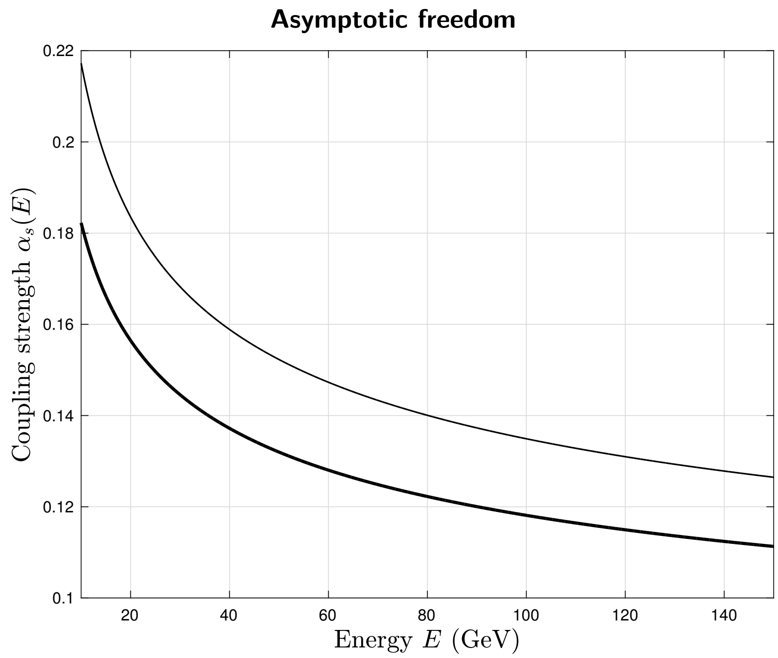

Figure 21.1 The strong-structure constant $\alpha_s(E)$ as given by the one-loop formula (21.25) (thin curve) and by a three-loop formula (thick curve) with $\Lambda = 230$ MeV and $n_f = 5$ is plotted for $m_b \ll E \ll m_t$. This chapter's Matlab scripts are in Renormalization_group at `github.com/kevinecahill`.

MeV and $n_f = 5$, Fig. 21.1 displays $\alpha_s(E)$ in the range $4.19 = m_b \ll E \ll m_t = 172$ GeV. The thin curve is the one-loop formula (21.25), and the thick curve is a three-loop formula (Weinberg, 1996, p. 156). □

21.4 Renormalization group in lattice field theory

Let us consider a quantum field theory on a lattice (Gattringer and Lang, 2010, chap. 3) in which the strength of the nonlinear interactions depends upon a single dimensionless coupling constant g. The spacing a of the lattice regulates the infinities, which return as $a \to 0$. The value of an observable P computed on this lattice will depend upon the lattice spacing a and on the coupling constant g, and so will be a function $P(a, g)$ of these two parameters. The *right* value of the coupling constant is the value that makes the result of the computation be as close as possible to the physical value P. So the correct coupling constant is not a constant at all, but rather a function $g(a)$ that varies with the lattice spacing or cutoff a. Thus as we vary the lattice spacing and go to the continuum limit $a \to 0$, we must adjust the coupling function $g(a)$ so that what we compute, $P(a, g(a))$, is equal to the physical value P. That is, $g(a)$ must vary with a so as to keep $P(a, g(a))$ constant at $P(a, g(a)) = P$

$$\frac{dP(a, g(a))}{da} = 0. \tag{21.26}$$

Writing this condition as a dimensionless derivative

$$a \frac{dP(a, g(a))}{da} = \frac{da}{d\log a} \frac{dP(a, g(a))}{da} = \frac{dP(a, g(a))}{d\log a} = 0 \tag{21.27}$$

we arrive at the **Callan-Symanzik equation**

$$0 = \frac{dP(a, g(a))}{d\log a} = \left(\frac{\partial}{\partial \log a} + \frac{dg}{d\log a} \frac{\partial}{\partial g} \right) P(a, g(a)). \tag{21.28}$$

The coefficient of the second partial derivative with a minus sign is the lattice beta function

$$\beta_L(g) \equiv -\frac{dg}{d\log a}. \tag{21.29}$$

Since the lattice spacing a and the energy scale μ are inversely related, the lattice beta function differs from the continuum beta function by a minus sign.

In $SU(N)$ gauge theory, the first two terms of the lattice beta function for small g are $\beta_L(g) = -\beta_0\, g^3 - \beta_1\, g^5$ where for n_f flavors of light quarks

$$\begin{aligned} \beta_0 &= \frac{1}{(4\pi)^2} \left(\frac{11}{3} N - \frac{2}{3} n_f \right) \\ \beta_1 &= \frac{1}{(4\pi)^4} \left(\frac{34}{3} N^2 - \frac{10}{3} N n_f - \frac{N^2 - 1}{N} n_f \right) \end{aligned} \tag{21.30}$$

and $N = 3$ in quantum chromodynamics.

Combining the definition (21.29) of the beta function with its expansion $\beta_L(g) = -\beta_0\, g^3 - \beta_1\, g^5$ for small g, one gets the differential equation

$$\frac{dg}{d\log a} = \beta_0\, g^3 + \beta_1\, g^5 \tag{21.31}$$

which one may integrate

$$\int d\log a = \log a - \log c = \int \frac{dg}{\beta_0 g^3 + \beta_1 g^5} = -\frac{1}{2\beta_0 g^2} + \frac{\beta_1}{2\beta_0^2} \log \left(\frac{\beta_0 + \beta_1 g^2}{g^2} \right)$$

to find that the lattice spacing has an essential singularity at $g = 0$

$$a(g) = c \left(\frac{\beta_0 + \beta_1 g^2}{g^2} \right)^{\beta_1/2\beta_0^2} e^{-1/(2\beta_0 g^2)} \tag{21.32}$$

in which c is a constant of integration. The term $\beta_1 g^2$ is of higher order in g, and if one drops it and absorbs a power of β_0 into a new constant of integration Λ, then one finds

$$a(g) = \frac{1}{\Lambda}\left(\beta_0 g^2\right)^{-\beta_1/2\beta_0^2} e^{-1/(2\beta_0 g^2)}. \tag{21.33}$$

As $g \to 0$, the lattice spacing $a(g)$ goes to zero *very fast* (as long as $n_f < 17$ for $N = 3$). The inverse of this relation (21.33)

$$g(a) \approx \left[\beta_0 \log(a^{-2}\Lambda^{-2}) + (\beta_1/\beta_0)\log\left(\log(a^{-2}\Lambda^{-2})\right)\right]^{-1/2} \tag{21.34}$$

shows that the coupling constant slowly goes to zero with a, which is a lattice version of **asymptotic freedom**. □

21.5 Renormalization group in condensed-matter physics

In classical statistical mechanics, the partition function $Z(\beta)$ is a sum over all configurations c of $\exp(-\beta H(c))$ in which $H(c)$ is the energy of the configuration c. A configuration might be a d-dimensional array of spins or a field in d-dimensional space (not spacetime)

$$\phi(\boldsymbol{x}) = \int_\Lambda e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,\phi(k)\,d^d\boldsymbol{k} \tag{21.35}$$

in which the integral is over Fourier modes $\boldsymbol{k}$ with a cutoff $k < \Lambda$ at an inverse lattice spacing. The hamiltonian might be

$$H[\phi] = \int \left[\tfrac{1}{2}(\nabla\phi)^2 + g_2\phi^2 + g_4\phi^4 + g_6\phi^6 + \cdots\right] d^d\boldsymbol{x}. \tag{21.36}$$

The partition function $Z(\beta)$ is an integral over all such configurations

$$Z(\beta) = \int \exp\left[-\beta H[\phi]\right] D\phi. \tag{21.37}$$

We change variables from $\phi(\boldsymbol{x})$ to a stretched field $\phi(\boldsymbol{x}/\ell)$ with $\ell > 1$

$$\phi_\ell(\boldsymbol{x}) = a_\ell\,\phi(\boldsymbol{x}/\ell) = a_\ell \int_\Lambda e^{i\boldsymbol{k}\cdot\boldsymbol{x}/\ell}\,\phi(k)\,d^d\boldsymbol{k} \tag{21.38}$$

in which a_ℓ is a factor that keeps the kinetic part of the hamiltonian invariant (Fisher, 1974, 1998; Kosterlitz et al., 1976; Kadanoff, 2009; Wilson, 1971, 1975). To have $H_k[\phi_\ell] = H_k[\phi]$, we need derivative terms to remain invariant

$$H_k[\phi_\ell] = \int \left(\frac{\partial \phi_\ell(\boldsymbol{x})}{\partial x_i}\right)^2 d^d\boldsymbol{x} = \int a_\ell^2 \left(\frac{\partial \phi(\boldsymbol{x}/\ell)}{\partial x_i}\right)^2 d^d\boldsymbol{x} \tag{21.39}$$

$$= \int \ell^d a_\ell^2 \left(\frac{\partial \phi(x/\ell)}{\ell \partial x_i/\ell}\right)^2 d^d(\boldsymbol{x}/\ell) = \int \ell^{d-2} a_\ell^2 \left(\frac{\partial \phi(x')}{\partial x'_i}\right)^2 d^d(\boldsymbol{x}') = H_k[\phi].$$

That is, we need $a_\ell = \ell^{(2-d)/2}$.

How do the various potential-energy terms change? A term $H_n[\phi]$ with ϕ^n changes to

$$H_n[\phi_\ell] = \int g_n\, \phi_\ell^n(\boldsymbol{x})\, d^d\boldsymbol{x} = \int g_n\, a_\ell^n\, \phi^n(\boldsymbol{x}/\ell)\, d^d\boldsymbol{x} = \int g_n\, a_\ell^n\, \phi^n(\boldsymbol{x}/\ell)\, \ell^d\, d^d(\boldsymbol{x}/\ell)$$

$$= \ell^d\, a_\ell^n \int g_n\, \phi^n(\boldsymbol{x}')\, d^d\boldsymbol{x}' = \ell^{d+n(2-d)/2}\, H_n[\phi]. \tag{21.40}$$

In effect, the coupling constant changes to $g_n(\ell) = \ell^{d+n(2-d)/2} g_n$. Similar reasoning shows that the coupling constant $g_{n,p}$ of a term with n factors of the field ϕ and p spatial derivatives of ϕ should vary as $g_{n,p}(\ell) = \ell^{d+n(2-d)/2-p} g_{n,p}$.

Coupling constants with positive exponents $d + n(d-2)/2 - p > 0$ become more important at greater spatial scales and are said to be **relevant**. Those with negative exponents become less important at greater spatial scales and are said to be **irrelevant**. Those with vanishing exponents are **marginal**. The mass term $g_2(\ell)\phi^2 = \ell^2 g_2 \phi^2$ is always relevant. The quartic term $g_4(\ell)\phi^4 = \ell^{4-d} g_4 \phi^4$ is relevant in fewer than 4 dimensions, marginal in 4 dimensions, and irrelevant in more than 4 dimensions. The term $g_6(\ell)\phi^6 = \ell^{6-2d} g_6 \phi^6$ is relevant in fewer than 3 dimensions, marginal in 3 dimensions, and irrelevant in more than 3.

Further reading

Quantum Field Theory in a Nutshell (Zee, 2010, chapters III & VI), *An Introduction to Quantum Field Theory* (Peskin and Schroeder, 1995, chapter 12), *The Quantum Theory of Fields* (Weinberg, 1995, 1996, sections 12.2 & 18.1–2).

Exercises

21.1 Show that for $\mu^2 \gg m^2$, the vacuum polarization term (21.10) reduces to (21.12). Hint: Use $\log a\, b = \log a + \log b$ when integrating.

21.2 Show that by choosing the energy scale Λ according to (21.24), one can derive (21.25) from (21.23).