

23.19 Solutions to the exercises on path integrals

1. From (20.1), derive the multiple gaussian integral for real a_j and b_j

$$\int_{-\infty}^{\infty} \exp \left(\sum_{j=1}^n i a_j x_j^2 + 2 i b_j x_j \right) \prod_{j=1}^n dx_j = \prod_{j=1}^n \sqrt{\frac{i\pi}{a_j}} e^{-ib_j^2/a_j}. \quad (23.471)$$

Solution: The integral (20.1) tells us that for each j

$$\int_{-\infty}^{\infty} \exp (i a_j x_j^2 + 2i b_j x_j) dx_j = \sqrt{\frac{i\pi}{a_j}} \exp (-ib_j^2/a_j). \quad (23.472)$$

So multiplying such integrals together, we get the result (23.471).

2. Use (20.285) (or equivalently (23.471)) to derive the multiple imaginary gaussian integral (20.3). Hint: Any real symmetric matrix s can be diagonalized by an orthogonal transformation $a = oso^\top$. Let $y = ox$.

Solution: Any real symmetric matrix s can be diagonalized by an orthogonal transformation $a = oso^\top$, so we let $y = ox$ and write $s = o^\top ao$, $x = o^\top y$, and $d = oc$. Then the quadratic form appearing in the first exponential of (20.3) takes the form of a sum over i and j from 1 to n

$$\begin{aligned} is_{jk}x_jx_k + 2ic_jx_j &= ix^\top sx + 2ic^\top x = ix^\top o^\top aox + 2ic^\top o^\top y \\ &= iy^\top ay + 2id^\top y = ia_{jj}y_j^2 + 2id_jy_j. \end{aligned} \quad (23.473)$$

The jacobian (section 1.21) of the transformation from x to y is the determinant of an orthogonal matrix the absolute value of which is unity. Thus

$$\begin{aligned} \int_{-\infty}^{\infty} e^{is_{jk}x_jx_k + 2ic_jx_j} dx_1 \dots dx_n &= \int_{-\infty}^{\infty} e^{ia_{jj}y_j^2 + 2id_jy_j} dy_1 \dots dy_n \\ &= \prod_{j=1}^n \sqrt{\frac{i\pi}{a_{jj}}} e^{-id_j^2/a_{jj}}. \end{aligned} \quad (23.474)$$

But $a_{11} \dots a_{nn} = \det a$ and $d^\top a^{-1}d = c^\top o^\top os^{-1}o^\top oc = c^\top s^{-1}c$, so we have

$$\int_{-\infty}^{\infty} e^{is_{jk}x_jx_k + 2ic_jx_j} dx_1 \dots dx_n = \sqrt{\frac{(i\pi)^n}{\det s}} e^{-ic_j(s^{-1})_{jk}c_k}. \quad (23.475)$$

3. Use (20.2) to show that for positive a_j

$$\int_{-\infty}^{\infty} \exp \left(\sum_j -a_j x_j^2 + 2ib_j x_j \right) \prod_{j=1}^n dx_j = \prod_{j=1}^n \sqrt{\frac{\pi}{a_j}} e^{-b_j^2/a_j}. \quad (23.476)$$

Solution: For each j , (20.2) says that

$$\int_{-\infty}^{\infty} e^{-a_j x_j^2 + 2ib_j x_j} dx_j = \sqrt{\frac{\pi}{a_j}} e^{-b_j^2/a_j}. \quad (23.477)$$

So the product of n such integrals is (23.476).

4. Use (20.286) to derive the many variable real gaussian integral (20.4). Same hint as for exercise 20.2.

Solution: The positive, real, symmetric matrix s has the diagonal form $a = oso^\top$, so we let $y = ox$ and write $s = o^\top ao$, $x = o^\top y$, and $d = oc$. Since s is positive, all its eigenvalues a_{jj} are positive. The absolute value of the jacobian (section 1.21) is unity because o is an orthogonal matrix. Thus

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-s_{jk}x_j x_k + 2ic_j x_j} dx_1 \dots dx_n &= \int_{-\infty}^{\infty} e^{-a_{jj}y_j^2 + 2id_j y_j} dy_1 \dots dy_n \\ &= \prod_{j=1}^n \sqrt{\frac{\pi}{a_{jj}}} e^{-d_j^2/a_{jj}}. \end{aligned} \quad (23.478)$$

But $a_{11} \dots a_{nn} = \det a$ and $d^\top a^{-1} d = c^\top o^\top os^{-1} o^\top oc = c^\top s^{-1} c$, so we have

$$\int_{-\infty}^{\infty} e^{-s_{jk}x_j x_k + 2ic_j x_j} dx_1 \dots dx_n = \sqrt{\frac{\pi^n}{\det s}} e^{-c_j(s^{-1})_{jk} c_k}. \quad (23.479)$$

5. Show that the vector \bar{Y} that makes the argument $-iY^\top SY + iD^\top Y$ of the multiple gaussian integral

$$\int_{-\infty}^{\infty} \exp(-iY^\top SY + iD^\top Y) \prod_{i=1}^n dy_i = \sqrt{\frac{\pi^n}{\det(iS)}} \exp\left(\frac{i}{4} D^\top S^{-1} D\right). \quad (23.480)$$

stationary is $\bar{Y} = S^{-1}D/2$, and that the multiple gaussian integral (20.287) is equal to its exponential $\exp(-iY^\top SY + iD^\top Y)$ evaluated at its stationary point $Y = \bar{Y}$ apart from a prefactor involving $\det iS$.

Solution: To find \bar{Y} , we write

$$0 = \frac{\partial}{\partial y_j} (-iY^T SY + iD^T Y)$$

and find, since $S^T = S$, that

$$2S_{jk}\bar{Y}_k = D_k$$

or $\bar{Y} = S^{-1}D/2$. Thus, since $S^T = S$, we have

$$\begin{aligned} \exp(-i\bar{Y}^T SY + iD^T \bar{Y}) &= \exp\left(-i\frac{D^T S^{-1} S S^{-1} D}{4} + i\frac{D^T S^{-1} D}{2}\right) \\ &= \exp\left(-i\frac{D^T S^{-1} D}{4} + i\frac{D^T S^{-1} D}{2}\right) \\ &= \exp\left(i\frac{D^T S^{-1} D}{4}\right). \end{aligned}$$

6. Show that the vector \bar{Y} that makes the argument $-Y^T SY + D^T Y$ of the multiple gaussian integral

$$\int_{-\infty}^{\infty} \exp(-Y^T SY + D^T Y) \prod_{i=1}^n dy_i = \sqrt{\frac{\pi^n}{\det(S)}} \exp\left(\frac{1}{4} D^T S^{-1} D\right) \quad (23.481)$$

stationary is $\bar{Y} = S^{-1}D/2$, and that the multiple gaussian integral (20.288) is equal to its exponential $\exp(-Y^T SY + D^T Y)$ evaluated at its stationary point $Y = \bar{Y}$ apart from a prefactor involving $\det S$.

Solution: To find \bar{Y} , we write

$$0 = \frac{\partial}{\partial y_j} (-Y^T SY + D^T Y)$$

and find, since $S^T = S$, that

$$2S_{jk}\bar{Y}_k = D_k$$

or $\bar{Y} = S^{-1}D/2$. Thus, since $S^T = S$, we have

$$\begin{aligned} \exp(-\bar{Y}^T SY + D^T \bar{Y}) &= \exp\left(-\frac{D^T S^{-1} S S^{-1} D}{4} + \frac{D^T S^{-1} D}{2}\right) \\ &= \exp\left(-\frac{D^T S^{-1} D}{4} + \frac{D^T S^{-1} D}{2}\right) \\ &= \exp\left(\frac{D^T S^{-1} D}{4}\right). \end{aligned}$$

7. Do the q_2 integral (20.27).

Solution: In the exponential,

$$(q_3 - q_2)^2 + (q_2 - q_a)^2/2 = \frac{3}{2}q_2^2 - q_2(2q_3 + q_a) + q_3^2 + \frac{1}{2}q_a^2,$$

so

$$\int e^{im[(q_3 - q_2)^2 + (q_2 - q_a)^2/2]/(2\hbar\epsilon)} dq_2 = \int e^{im[\frac{3}{2}q_2^2 - q_2(2q_3 + q_a) + q_3^2 + \frac{1}{2}q_a^2]/(2\hbar\epsilon)} dq_2$$

using the integral (20.1) with

$$a = \frac{3m}{4\hbar\epsilon} \quad \text{and} \quad b = -\frac{m(2q_3 + q_a)}{2\hbar\epsilon},$$

we find

$$\int e^{im[(q_3 - q_2)^2 + (q_2 - q_a)^2/2]/(2\hbar\epsilon)} dq_2 = \sqrt{\frac{4\pi i\hbar\epsilon}{3m}} \exp\left(\frac{im}{6\hbar\epsilon}(q_3 - q_a)^2\right)$$

and so

$$\begin{aligned} & \frac{m}{2\sqrt{2}\pi i\hbar\epsilon} \int e^{im[(q_3 - q_2)^2 + (q_2 - q_a)^2/2]/(2\hbar\epsilon)} dq_2 \\ &= \sqrt{\frac{m}{2\pi i\hbar 3\epsilon}} e^{im(q_3 - q_a)^2/(2\hbar 3\epsilon)}. \end{aligned}$$

8. Insert the identity operator in the form of an integral (20.10) of outer products $|p\rangle\langle p|$ of eigenstates of the momentum operator p between the exponential and the state $|q_a\rangle$ in the matrix element (20.25) and so derive for that matrix element $\langle q_b | \exp(-i(t_b - t_a)H/\hbar) | q_a \rangle$ the formula (20.28). Hint: use the inner product $\langle q | p \rangle = \exp(iqp/\hbar)/\sqrt{2\pi\hbar}$, and do the resulting Fourier transform.

Solution: We take $H = p^2/2m$ and find

$$\begin{aligned} \langle q_b | e^{-itH/\hbar} | q_a \rangle &= \langle q_b | e^{-itp^2/(2\hbar m)} \int |p'\rangle\langle p' | q_a \rangle dp' \\ &= \int \langle q_b | e^{-itp'^2/(2\hbar m)} | p' \rangle\langle p' | q_a \rangle dp' \\ &= \int e^{-itp'^2/(2\hbar m) + ip'(q_b - q_a)/\hbar} \frac{dp'}{2\pi\hbar}. \end{aligned}$$

We use the integral (20.1) with

$$a = -\frac{t}{2\hbar m} \quad \text{and} \quad b = \frac{q_b - q_a}{2\hbar}$$

and get

$$\langle q_b | e^{-itH/\hbar} | q_a \rangle = \sqrt{\frac{m}{2\pi i \hbar t}} e^{im(q_b - q_a)^2/(2\hbar t)}.$$

9. Derive the path-integral formula (20.36) for the quadratic action (20.35).

Solution: If $q_c(t)$ is a classical path, then

$$\begin{aligned} 0 = \delta S[q_c] &= \int_{t_a}^{t_b} 2u\dot{q}_c \delta\dot{q}_c + v\delta q_c \dot{q}_c + vq_c \delta\dot{q}_c + 2wq_c \delta q_c + s\delta\dot{q}_c + j\delta q_c dt \\ &= \delta S[q_c] = \int_{t_a}^{t_b} (-2\dot{u}\dot{q}_c - 2u\ddot{q}_c + v\dot{q}_c - \dot{v}q_c - v\dot{q}_c + 2wq_c - \dot{s} + j) \delta q_c dt \end{aligned}$$

Thus the action of the general path $q(t) = q_c(t) + \delta q(t)$ is

$$S[q] = S[q_c] + S[\delta q].$$

But $\delta q(t_a) = \delta q(t_b) = 0$, so δq is a loop from 0 to 0, and so $S[\delta q]$ is independent of $q(t_a)$ and of $q(t_b)$. Also, $Dq = D\delta q$. Thus

$$\begin{aligned} \int e^{iS[q]/\hbar} Dq &= \int e^{iS[q_c]/\hbar} e^{iS[\delta q]/\hbar} Dq \\ &= e^{iS[q_c]/\hbar} \int e^{iS[q_c]/\hbar} e^{iS[\delta q]/\hbar} Dq \\ &= e^{iS[q_c]/\hbar} \int e^{iS[q_c]/\hbar} e^{iS[\delta q]/\hbar} D\delta q \\ &= f(t_a, t_b, \dots) e^{iS[q_c]/\hbar} \end{aligned}$$

in which the dots stand for the functions u, v, w, s, j .

10. Show that for the simple harmonic oscillator (20.44) the action $S[q_c]$ of the classical path from q_a, t_a to q_b, t_b is (20.46).

Solution: To compute the action $S[q_c]$ of the classical path

$$q_c(t) = q_a \cos \omega(t - t_a) + \frac{q_b - q_a \cos \omega(t_b - t_a)}{\sin \omega(t_b - t_a)} \sin \omega(t - t_a)$$

with

$$\dot{q}_c(t) = -\omega q_a \sin \omega(t - t_a) + \omega \frac{q_b - q_a \cos \omega(t_b - t_a)}{\sin \omega(t_b - t_a)} \cos \omega(t - t_a),$$

we need the integrals

$$\begin{aligned}\int_{t_a}^{t_b} \sin(\omega(t - t_a)) \cos(\omega(t - t_a)) dt &= \frac{\sin^2(\omega(t_b - t_a))}{2\omega} \\ \int_{t_a}^{t_b} \sin^2(\omega(t - t_a)) dt &= \frac{2\omega(t_b - t_a) - \sin[2\omega(t_b - t_a)]}{4\omega} \\ \int_{t_a}^{t_b} \cos^2(\omega(t - t_a)) dt &= \frac{2\omega(t_b - t_a) + \sin[2\omega(t_b - t_a)]}{4\omega}.\end{aligned}$$

Thus setting

$$R = \frac{q_b - q_a \cos \omega(t_b - t_a)}{\sin \omega(t_b - t_a)},$$

we find

$$\begin{aligned}\int_{t_a}^{t_b} \omega^2 q_c^2(t) dt &= \frac{1}{4} \omega q_a^2 \left(2\omega(t_b - t_a) + \sin[2\omega(t_b - t_a)] \right) \\ &\quad + \frac{1}{4} \omega R^2 \left(2\omega(t_b - t_a) - \sin[2\omega(t_b - t_a)] \right) \\ &\quad + \omega q_a R \sin^2(\omega(t_b - t_a)),\end{aligned}$$

and

$$\begin{aligned}\int_{t_a}^{t_b} \dot{q}_c^2(t) dt &= \frac{1}{4} \omega q_a^2 \left(2\omega(t_b - t_a) - \sin[2\omega(t_b - t_a)] \right) \\ &\quad + \frac{1}{4} \omega R^2 \left(2\omega(t_b - t_a) + \sin[2\omega(t_b - t_a)] \right) \\ &\quad - \omega q_a R \sin^2(\omega(t_b - t_a))\end{aligned}$$

We then find that its action is

$$\begin{aligned}S[q_c] &= \frac{1}{2} m \int_{t_a}^{t_b} \dot{q}^2(t) - \omega^2 q^2(t) dt \\ &= \frac{m\omega}{4 \sin^2[\omega(t_b - t_a)]} \left\{ 4q_a(q_a \cos[\omega(t_b - t_a)] - q_b) \sin^3[\omega(t_b - t_a)] \right. \\ &\quad + (q_b + q_a(\sin[\omega(t_b - t_a)] - \cos[\omega(t_b - t_a)])) \\ &\quad \times (q_b - q_a(\sin[\omega(t_b - t_a)] + \cos[\omega(t_b - t_a)])) \sin[2\omega(t_b - t_a)] \Big\} \\ &= \frac{m\omega}{4 \sin^2[\omega(t_b - t_a)]} \left\{ 4q_a(q_a \cos[\omega(t_b - t_a)] - q_b) \sin^3[\omega(t_b - t_a)] \right. \\ &\quad + 2(q_b + q_a(\sin[\omega(t_b - t_a)] - \cos[\omega(t_b - t_a)])) \\ &\quad \times (q_b - q_a(\sin[\omega(t_b - t_a)] + \cos[\omega(t_b - t_a)])) \\ &\quad \times \sin[\omega(t_b - t_a)] \cos[\omega(t_b - t_a)] \Big\}.\end{aligned}$$

Combining terms, we get

$$\begin{aligned} S[q_c] &= \frac{m\omega}{2\sin[\omega(t_b - t_a)]} \left\{ 2q_a(q_a \cos[\omega(t_b - t_a)] - q_b) \sin^2[\omega(t_b - t_a)] \right. \\ &\quad + (q_b + q_a(\sin[\omega(t_b - t_a)] - \cos[\omega(t_b - t_a)])) \\ &\quad \times (q_b - q_a(\sin[\omega(t_b - t_a)] + \cos[\omega(t_b - t_a)])) \cos[\omega(t_b - t_a)] \Big\} \\ &= \frac{m\omega}{2\sin[\omega(t_b - t_a)]} \left\{ 2q_a(q_a \cos[\omega(t_b - t_a)] - q_b) \sin^2[\omega(t_b - t_a)] \right. \\ &\quad \left. + \left[(q_b - q_a \cos[\omega(t_b - t_a)])^2 - q_a^2 \sin^2[\omega(t_b - t_a)] \right] \cos[\omega(t_b - t_a)] \right\}. \end{aligned}$$

Expanding the square and using trigonometry, we find

$$\begin{aligned} S[q_c] &= \frac{m\omega}{2\sin[\omega(t_b - t_a)]} \left\{ q_a^2 \cos[\omega(t_b - t_a)] \sin^2[\omega(t_b - t_a)] \right. \\ &\quad - 2q_a q_b \sin^2[\omega(t_b - t_a)] + (q_b - q_a \cos[\omega(t_b - t_a)])^2 \cos[\omega(t_b - t_a)] \Big\} \\ &= \frac{m\omega}{2\sin[\omega(t_b - t_a)]} \left\{ q_a^2 \cos[\omega(t_b - t_a)] \sin^2[\omega(t_b - t_a)] \right. \\ &\quad - 2q_a q_b \sin^2[\omega(t_b - t_a)] \\ &\quad + q_b^2 \cos[\omega(t_b - t_a)] - 2q_a q_b \cos^2[\omega(t_b - t_a)] + q_a^2 \cos^3[\omega(t_b - t_a)] \Big\} \\ &= \frac{m\omega}{2\sin[\omega(t_b - t_a)]} \left\{ q_a^2 \cos[\omega(t_b - t_a)] - 2q_a q_b \sin^2[\omega(t_b - t_a)] \right. \\ &\quad + q_b^2 \cos[\omega(t_b - t_a)] - 2q_a q_b \cos^2[\omega(t_b - t_a)] \Big\} \\ &= \frac{m\omega}{2\sin[\omega(t_b - t_a)]} \left\{ q_a^2 \cos[\omega(t_b - t_a)] - 2q_a q_b + q_b^2 \cos[\omega(t_b - t_a)] \right\} \\ &= \frac{m\omega}{2\sin[\omega(t_b - t_a)]} \left\{ (q_a^2 + q_b^2) \cos[\omega(t_b - t_a)] - 2q_a q_b \right\} \end{aligned}$$

which is (20.46).

In retrospect, I regret making this long computation an exercise.

11. Show that the determinants $|C_n(y)| = \det C_n(y)$ of the tridiagonal matrices (20.52) satisfy the recursion relation (20.53) and have the initial values $|C_1(y)| = y$ and $|C_2(y)| = y^2 - 1$. Incidentally, the Chebyshev polynomials (9.68) of the second kind $U_n(y/2)$ obey the same recursion relation and have the same initial values, so $|C_n(y)| = U_n(y/2)$.

Solution: Define the determinant of the zero-dimensional matrix to be $C_0(y) = 1$. Let

$$C_1(y) = \det y = y,$$

and

$$C_2(y) = \det \begin{pmatrix} y & -1 \\ -1 & y \end{pmatrix} = y^2 - 1.$$

Then the determinant of the $(n+1) \times (n+1)$ matrix

$$\det C_{n+1} = \det \begin{pmatrix} y & -1 & 0 & 0 & \cdots \\ -1 & y & -1 & 0 & \cdots \\ 0 & -1 & y & -1 & \cdots \\ 0 & 0 & -1 & y & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

is

$$\det C_{n+1} = y \det C_n + \det D$$

where D is the $n \times n$ matrix whose determinant is

$$\det D = \det \begin{pmatrix} -1 & 0 & 0 & \cdots \\ -1 & y & -1 & \cdots \\ 0 & -1 & y & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = -\det C_{n-1}$$

which gives us the recursion relation

$$\det C_{n+1} = y \det C_n - \det C_{n-1}.$$

12. (a) Show that the functions $S_n(y) = \sin(n+1)\theta / \sin \theta$ with $y = 2 \cos \theta$ satisfy the Toeplitz recursion relation (20.53) which after a cancellation simplifies to $\sin(n+2)\theta = 2 \cos \theta \sin(n+1)\theta - \sin n\theta$. (b) Derive the initial conditions $S_0(y) = 1$, $S_1(y) = 2y$, and $S_2(y) = 4y^2 - 2$.

Solution: (a) The Toeplitz recursion relation is

$$S_{n+1}(y) = y S_n(y) - S_{n-1}(y)$$

or

$$\frac{\sin(n+2)\theta}{\sin \theta} = 2 \cos \theta \frac{\sin(n+1)\theta}{\sin \theta} - \frac{\sin n\theta}{\sin \theta} \quad (23.482)$$

which is

$$\begin{aligned} e^{i(n+2)\theta} - e^{-i(n+2)\theta} &= (e^{i\theta} + e^{-i\theta})(e^{i(n+1)\theta} - e^{-i(n+1)\theta}) - (e^{in\theta} - e^{-in\theta}) \\ &= e^{i(n+2)\theta} - e^{-in\theta} + e^{-in\theta} - e^{-i(n+2)\theta} - (e^{in\theta} - e^{-in\theta}) \\ &= e^{i(n+2)\theta} - e^{-i(n+2)\theta}. \end{aligned}$$

(b). Clearly $S_0(y) = \sin \theta / \sin \theta = 1$. Also, $S_1(y) = \sin 2\theta / \sin \theta = 2 \cos \theta =$

y . And finally the identity (23.482) lets us write $S_2(y) = \sin 3\theta / \sin \theta = 2 \cos \theta S_1(y) - S_0(y) = y^2 - 1$.

13. Do the q_2 integral (20.75).

Solution: We complete the square in the exponent

$$\begin{aligned} 2(q_3 - q_2)^2 + (q_2 - q_a)^2 &= 3 \left[q_2 - \frac{1}{3}(2q_3 + q_a) \right]^2 + 2q_3^2 + q_a^2 - \frac{1}{3}(2q_3 + q_a)^2 \\ &= 3 \left[q_2 - \frac{1}{3}(q_3 + q_a) \right]^2 + \frac{2}{3}(q_3 - q_a)^2. \end{aligned}$$

So by shifting q_2 , we find

$$\int e^{-m(q_3-q_2)^2/(2\hbar^2\epsilon)-m(q_2-q_a)^2/(4\hbar^2\epsilon)} dq_2 = \int e^{-m[3q_2^2+2(q_3-q_a)^2]/3/(4\hbar^2\epsilon)}.$$

Using the integral formula (20.2) with $a = 3m/(4\hbar^2\epsilon)$ and $b = 0$, we get

$$\int e^{-m(q_3-q_2)^2/(2\hbar^2\epsilon)-m(q_2-q_a)^2/(4\hbar^2\epsilon)} dq_2 = \sqrt{\frac{4\pi\hbar^2\epsilon}{3m}} e^{-m(q_3-q_a)^2/(2\hbar^23\epsilon)}.$$

So multiplying by the prefactor, we have

$$\left(\frac{m}{4\pi\hbar^2\epsilon}\right)^{1/2} \int e^{-m(q_3-q_2)^2/(2\hbar^2\epsilon)-m(q_2-q_a)^2/(4\hbar^2\epsilon)} dq_2 = \frac{e^{-m(q_3-q_a)^2/(2\hbar^23\epsilon)}}{\sqrt{3}}$$

which is (20.75).

14. Show that the euclidian action (20.88) is stationary if the path $q_{ec}(u)$ obeys the euclidian equation of motion $\ddot{q}_{ec}(u) = \omega^2 q_{ec}(u)$.

Solution: For changes $\delta q(u)$ that vanish at the endpoints $\delta q(0) = 0$ and $\delta q(\hbar\beta) = 0$, the first-order change in the action is

$$\begin{aligned} \delta S &= \int_0^{\hbar\beta} m\dot{q}_{ec}(u)\dot{\delta q}_{ec}(u) + m\omega^2 q_{ec}(u)\delta q_{ec}(u) du \\ &= \int_0^{\hbar\beta} [-m\ddot{q}_{ec}(u) + m\omega^2 q_{ec}(u)] \delta q_{ec}(u) du. \end{aligned}$$

Thus $\delta S = 0$ if $\ddot{q}_{ec}(u) = \omega^2 q_{ec}(u)$.

15. By using (20.20) for each of the three exponentials in (20.102), derive (20.103) from (20.102). Hint: From (20.20), one has

$$qe^{-i(t_b-t_a)H/\hbar}q = \int q_b|q_b\rangle e^{iS[q]/\hbar} \langle q_a|q_a Dq dq_a dq_b$$

in which $q_a = q(t_a)$ and $q_b = q(t_b)$.

Solution: We can write (20.20) as

$$\begin{aligned} q e^{-i(t_b-t_a)H/\hbar} q &= \int q_b |q_b\rangle e^{iS[q]/\hbar} \langle q_a | q_a Dq dq_a dq_b \\ &= \int q(t_b) |q(t_b)\rangle e^{iS[q,t_b,t_a]/\hbar} \langle q(t_a) | q(t_a) Dq(t_b, t_a) dq(t_a) dq(t_b) \end{aligned}$$

in which $Dq(t_b, t_a)$ runs from t_a to t_b . So we have

$$\begin{aligned} \langle b | e^{-itH/\hbar} \mathcal{T}[q(t_1)q(t_2)] e^{-itH/\hbar} | a \rangle &= \langle b | e^{-i(t-t_>)H/\hbar} q e^{-i(t_>-t_<)H/\hbar} q e^{-i(t+t_<)H/\hbar} | a \rangle \\ &= \int \langle b | q_b \rangle e^{iS[q,t,t_>]/\hbar} \langle q_a | Dq(t, t_>) dq_a dq_b \\ &\quad \times \int q(t_>) |q(t_>)\rangle e^{iS[q,t_>,t_<]/\hbar} \langle q(t_<) | q(t_<) Dq(t_>, t_<) dq(t_>) dq(t_<) \\ &\quad \times \int |q_c\rangle e^{iS[q,t_<,-t]/\hbar} \langle q_d | Dq(t_<, -t) dq_c dq_d. \end{aligned}$$

Now $\langle q_a | q(t_>) \rangle = \delta(q_a - q(t_>))$ and $\langle q(t_<) | q_c \rangle = \delta(t_< - q_c)$. So this integral is

$$\begin{aligned} \langle b | e^{-itH/\hbar} \mathcal{T}[q(t_1)q(t_2)] e^{-itH/\hbar} | a \rangle &= \int \langle b | q(t) \rangle e^{iS[q,t,t_>]/\hbar} Dq(t, t_>) dq(t) dq(t_>) \\ &\quad \times \int q(t_>) e^{iS[q,t_>,t_<]/\hbar} q(t_<) Dq(t_>, t_<) dq(t_>) dq(t_<) \\ &\quad \times \int e^{iS[q,t_<,-t]/\hbar} \langle q_a | a \rangle Dq(t_<, -t) dq(t_<) dq(-t). \end{aligned}$$

So if $D(q, t, -t)$ denotes an integration over all paths that run from $-t$ to t , then

$$\begin{aligned} \langle b | e^{-itH/\hbar} \mathcal{T}[q(t_1)q(t_2)] e^{-itH/\hbar} | a \rangle &= \int \langle b | q(t) \rangle e^{iS[q,t,-t]/\hbar} q(t_>) q(t_<) \langle q(-t) | a \rangle D(q, t, -t) \end{aligned}$$

which is (20.103).

16. Derive the path-integral formula (20.139) from (20.129–20.132).

Solution: For $t_2 \geq t_1$, the matrix element on the left-hand side of equation 20.139 is by equation (20.138)

$$\begin{aligned} \langle n | \phi(x_2) \phi(x_1) | n \rangle &= \langle n | e^{it_2 H / \hbar} \phi(\mathbf{x}, 0) e^{-i(t_2 - t_1)H/\hbar} \phi(\mathbf{x}, 0) e^{-it_1 H / \hbar} | n \rangle \\ &= e^{i(t_2 - t_1)E_n / \hbar} \langle n | \phi(\mathbf{x}_2, 0) e^{-i(t_2 - t_1)H/\hbar} \phi(\mathbf{x}_1, 0) | n \rangle. \end{aligned}$$

If we write the exponential as a path integral (20.132), then we get

$$\begin{aligned}\langle n|\phi(x_2)\phi(x_1)|n\rangle &= e^{i(t_2-t_1)E_n/\hbar} \langle n|\phi(\mathbf{x}_2, 0) \\ &\quad \times \int |\phi_b\rangle e^{iS[\phi]/\hbar} \langle \phi_a| D\phi D\phi_a D\phi_b \phi(\mathbf{x}_1, 0) |n\rangle\end{aligned}$$

in which the integration is over all fields that go from t_1 to t_2 . So since $|\phi_b\rangle$ is $|\phi, t_2\rangle$, which is an eigenstate of the field $\phi(\mathbf{x}_2, 0)$, we have $\phi(\mathbf{x}_2, 0)|\phi_b\rangle = \phi(x_2)|\phi_b\rangle$. Similarly, $\langle \phi_a|\phi(\mathbf{x}_1, 0) = \langle \phi_a|\phi(x_1)$, and we have

$$\begin{aligned}\langle n|\phi(x_2)\phi(x_1)|n\rangle &= e^{i(t_2-t_1)E_n/\hbar} \int \langle n|\phi_b\rangle \phi(x_2)\phi(x_1) \langle \phi_a|n\rangle e^{iS[\phi]/\hbar} \\ &\quad \times D\phi D\phi_a D\phi_b.\end{aligned}$$

This formula implies the simpler one

$$\begin{aligned}\langle n|e^{-i(t_2-t_1)H/\hbar}|n\rangle &= e^{-i(t_2-t_1)E_n/\hbar} \langle n|n\rangle = e^{-i(t_2-t_1)E_n/\hbar} \\ &= \int \langle n|\phi_b\rangle \langle \phi_a|n\rangle e^{iS[\phi]/\hbar} D\phi D\phi_a D\phi_b\end{aligned}$$

or

$$e^{i(t_2-t_1)E_n/\hbar} = 1 \Big/ \int \langle n|\phi_b\rangle \langle \phi_a|n\rangle e^{iS[\phi]/\hbar} D\phi D\phi_a D\phi_b$$

So we have for $t_2 \geq t_1$,

$$\begin{aligned}\langle n|\phi(x_2)\phi(x_1)|n\rangle &= \int \langle n|\phi_b\rangle \phi(x_2)\phi(x_1) \langle \phi_a|n\rangle e^{iS[\phi]/\hbar} D\phi D\phi_a D\phi_b \\ &\quad \Big/ \int \langle n|\phi_b\rangle \langle \phi_a|n\rangle e^{iS[\phi]/\hbar} D\phi D\phi_a D\phi_b.\end{aligned}$$

If $t_1 \geq t_2$, then

$$\begin{aligned}\langle n|\phi(x_1)\phi(x_2)|n\rangle &= \int \langle n|\phi_b\rangle \phi(x_1)\phi(x_2) \langle \phi_a|n\rangle e^{iS[\phi]/\hbar} D\phi D\phi_a D\phi_b \\ &\quad \Big/ \int \langle n|\phi_b\rangle \langle \phi_a|n\rangle e^{iS[\phi]/\hbar} D\phi D\phi_a D\phi_b \\ &= \int \langle n|\phi_b\rangle \phi(x_2)\phi(x_1) \langle \phi_a|n\rangle e^{iS[\phi]/\hbar} D\phi D\phi_a D\phi_b \\ &\quad \Big/ \int \langle n|\phi_b\rangle \langle \phi_a|n\rangle e^{iS[\phi]/\hbar} D\phi D\phi_a D\phi_b.\end{aligned}$$

Putting the two together, we have

$$\begin{aligned} \langle n | \mathcal{T}[\phi(x_2)\phi(x_1)] | n \rangle &= \int \langle n | \phi_b \rangle \phi(x_2) \phi(x_1) \langle \phi_a | n \rangle e^{iS[\phi]/\hbar} D\phi D\phi_a D\phi_b \\ &\quad / \int \langle n | \phi_b \rangle \langle \phi_a | n \rangle e^{iS[\phi]/\hbar} D\phi D\phi_a D\phi_b . \end{aligned} \quad (23.483)$$

Suppose that $T_2 > t_2 \geq t_1 > T_1$. We could have computed

$$\begin{aligned} \langle n | e^{-i(T_2-t_2)H/\hbar} \phi(x_2) \phi(x_1) e^{-i(t_1-T_1)H/\hbar} | n \rangle &= e^{-i(T_2-t_2+t_1-T_1)E_n/\hbar} \\ &\quad \times \langle n | \phi(x_2) \phi(x_1) | n \rangle \end{aligned} \quad (23.484)$$

and

$$\langle n | e^{-i(T_2-T_1)H/\hbar} | n \rangle = e^{-i(T_2-t_2+t_1-T_1)E_n/\hbar} \langle n | e^{-i(t_2-t_1)H/\hbar} | n \rangle . \quad (23.485)$$

When we form the ratio of (23.484) to 23.485), the extra common phase factor

$$e^{-i(T_2-t_2+t_1-T_1)E_n/\hbar}$$

cancels, and we get the ratio (7.380) expressed as a ratio of path integrals over fields that run from T_1 to T_2 .

17. Derive the path-integral formula (20.153) from (20.147–20.150).

Solution: The derivation of (20.153) from (20.147–20.150) is almost the same as that of the preceding problem. Let's assume that $u_1 \geq u_2$. Then

$$\begin{aligned} \langle n | \mathcal{T}[\phi_e(x_1)\phi_e(x_2)] | n \rangle &= \langle n | \phi_e(\mathbf{x}_1) \phi_e(\mathbf{x}_2) | n \rangle \\ &= \langle n | e^{u_1 H/\hbar} \phi_e(\mathbf{x}_1, 0) e^{-(u_1-u_2)H/\hbar} \phi_e(\mathbf{x}_2, 0) e^{-u_2 H/\hbar} | n \rangle . \end{aligned}$$

Next we use (20.150) for the exponential

$$\begin{aligned} \langle n | \mathcal{T}[\phi_e(x_1)\phi_e(x_2)] | n \rangle &= \langle n | e^{u_1 H/\hbar} \phi_e(\mathbf{x}_1, 0) \\ &\quad \times \int |\phi_b\rangle e^{-S_e[\phi]/\hbar} \langle \phi_a | D\phi D\phi_a D\phi_b \\ &\quad \times \phi_e(\mathbf{x}_2, 0) e^{-u_2 H/\hbar} | n \rangle . \end{aligned}$$

The state $|\phi_b\rangle$ is an eigenstate of the operator $\phi_e(\mathbf{x}_1, 0)$ with eigenvalue $\phi(x_1)$, where $x_1 = (\mathbf{x}_1, u_b)$ and the state $|\phi_a\rangle$ is an eigenstate of the operator $\phi_e(\mathbf{x}_2)$ with eigenvalue $\phi(x_2)$, where $x_2 = (\mathbf{x}_2, u_a)$. And the state $|n\rangle$ is an eigenstate of the hamiltonian H with eigenvalue E_n , so

$$\langle n | \mathcal{T}[\phi_e(x_1)\phi_e(x_2)] | n \rangle = e^{(u_1-u_2)E_n/\hbar} \int \langle n | \phi_b \rangle \phi(x_1) \phi(x_2) e^{-S_e[\phi]/\hbar} \langle \phi_a | n \rangle D\phi$$

and the integration is over all fields that run from before u_2 to after u_1 . As in the preceding example, we use (20.147) with $|\phi_b\rangle = |n\rangle$ and $|\phi_a\rangle = |n\rangle$ to write

$$\begin{aligned} e^{(u_1-u_2)E_n/\hbar} &= \left[e^{-(u_1-u_2)E_n/\hbar} \right]^{-1} \\ &= \left[\int \langle n| \phi_b \rangle e^{-S_e[\phi]/\hbar} \langle \phi_a | n \rangle D\phi \right]^{-1}. \end{aligned}$$

The last two equations now give

$$\begin{aligned} \langle n | \mathcal{T}[\phi_e(x_1)\phi_e(x_2)] | n \rangle &= \int \langle n | \phi_b \rangle \phi(x_1) \phi(x_2) e^{-S_e[\phi]/\hbar} \langle \phi_a | n \rangle D\phi \\ &\quad \left/ \int \langle n | \phi_b \rangle e^{-S_e[\phi]/\hbar} \langle \phi_a | n \rangle D\phi. \right. \end{aligned}$$

18. Derive the free action in momentum space (20.161) from the free action in position space (20.159) and the four-dimensional Fourier transforms (20.160).

Solution: Substituting the transforms (20.160) into the action (20.159), we get

$$\begin{aligned} S_0[\phi] &= \frac{1}{2} \int -\partial_a \phi(x) \partial^a \phi(x) - m^2 \phi^2(x) d^4x \\ &= \frac{1}{2} \int \left[-\partial_a \int e^{ipx} \tilde{\phi}(p) \frac{d^4p}{(2\pi)^4} \partial^a \int e^{ip'x} \tilde{\phi}(p') \frac{d^4p'}{(2\pi)^4} \right. \\ &\quad \left. - m^2 \int e^{ipx} \tilde{\phi}(p) \frac{d^4p}{(2\pi)^4} \int e^{ip'x} \tilde{\phi}(p') \frac{d^4p'}{(2\pi)^4} \right] d^4x. \end{aligned} \quad (23.486)$$

Differentiating, we have

$$\begin{aligned} S_0[\phi] &= \frac{1}{2} \int \left[- \int ip_a e^{ipx} \tilde{\phi}(p) \frac{d^4p}{(2\pi)^4} \int ip'^a e^{ip'x} \tilde{\phi}(p') \frac{d^4p'}{(2\pi)^4} \right. \\ &\quad \left. - m^2 \int e^{ipx} \tilde{\phi}(p) \frac{d^4p}{(2\pi)^4} \int e^{ip'x} \tilde{\phi}(p') \frac{d^4p'}{(2\pi)^4} \right] d^4x. \end{aligned} \quad (23.487)$$

The space-time integral is a four-dimensional delta function

$$\int \frac{d^4x}{(2\pi)^4} e^{i(p+p')x} = \delta^4(p + p') \quad (23.488)$$

which reduces the eight-dimensional integral over momentum to a four-

dimensional integral

$$\begin{aligned} S_0[\phi] &= -\frac{1}{2} \int \delta^4(p + p') ip_a e^{ipx} \tilde{\phi}(p) \frac{d^4p}{(2\pi)^4} \int ip'^a e^{ip'x} \tilde{\phi}(p') d^4p' \\ &\quad - m^2 \int \delta^4(p + p') e^{ipx} \tilde{\phi}(p) \frac{d^4p}{(2\pi)^4} \int e^{ip'x} \tilde{\phi}(p') d^4p' \\ &= -\frac{1}{2} \int (p_a p^a + m^2) \tilde{\phi}(p) \tilde{\phi}(-p) \frac{d^4p}{(2\pi)^4}. \end{aligned} \quad (23.489)$$

Since the field $\phi(x)$ is real, its Fourier transform obeys the reality condition $\tilde{\phi}(-p) = \tilde{\phi}^*(p)$ and so we find that

$$S_0[\phi] = -\frac{1}{2} \int |\tilde{\phi}(p)|^2 (p^2 + m^2) \frac{d^4p}{(2\pi)^4} \quad (23.490)$$

which is (20.161).

19. By taking the nonrelativistic limit of the formula (12.55) for the action of a relativistic particle of mass m and charge q , derive the expression (20.41) for the action a nonrelativistic particle in an electromagnetic field with no scalar potential.

Solution: For $A_0 = 0$, the nonrelativistic limit of the formula (12.55) is

$$\begin{aligned} S &= \int_{t_1}^{t_2} \left(-mc^2 \sqrt{1 - \mathbf{v}^2/c^2} dt + q\mathbf{A} \cdot d\mathbf{x} \right) \\ &= -mc^2(t_2 - t_1) + \int_{t_1}^{t_2} \left(\frac{1}{2}m\mathbf{v}^2 dt + q\mathbf{A} \cdot d\mathbf{x} \right) \\ &= -mc^2(t_2 - t_1) + \int_{\mathbf{x}_1}^{\mathbf{x}_2} \left(\frac{1}{2}m\mathbf{v} + q\mathbf{A} \right) \cdot d\mathbf{x} \end{aligned}$$

which is (20.41) together with the “rest action.”

20. Work out the path-integral formula for the amplitude for a mass m initially at rest to fall to the ground from height h in a gravitational field of local acceleration g to lowest order and then including loops up to an overall constant. Hint: use the technique of section 20.4.

Solution: The classical path is

$$q_c(t) = h - \frac{1}{2}gt^2$$

and the time to fall to the ground is $t = \sqrt{2h/g} \equiv T$. The action of the

classical path is

$$\begin{aligned}
 S[q_c] &= \int_0^T \frac{m}{2} (\dot{q}_c(t))^2 - mgq_c(t) dt \\
 &= m \int_0^T \frac{1}{2} (-gt)^2 - g(h - \frac{1}{2}gt^2) dt \\
 &= m \int_0^T g^2 t^2 - gh dt \\
 &= mgT \left(\frac{1}{3}gT^2 - h \right) = -\frac{1}{3}mghT = -\frac{mh\sqrt{2gh}}{3}.
 \end{aligned}$$

So to lowest order, the path integral is

$$\langle 0 | e^{-itH/\hbar} | h \rangle = e^{iS[q_c]/\hbar} = e^{-imh\sqrt{2gh}/3\hbar}.$$

We can use as the loop

$$\delta q(t) = \sum_{j=1}^{n-1} a_j \sin \frac{j\pi t}{T}.$$

Its time derivative is

$$\delta \dot{q}(t) = \sum_{j=1}^{n-1} \frac{j\pi a_j}{T} \cos \frac{j\pi t}{T}.$$

So the loop's action is

$$\begin{aligned}
 S[\delta q] &= m \int_0^T \left[\frac{1}{2}(\delta \dot{q})^2 - g \delta q \right] dt \\
 &= m \int_0^T \left[\frac{1}{2} \left(\sum_{j=1}^{n-1} \frac{j\pi a_j}{T} \cos \frac{j\pi t}{T} \right)^2 - g \sum_{j=1}^{n-1} a_j \sin \frac{j\pi t}{T} \right] dt.
 \end{aligned}$$

The integral formulas

$$\begin{aligned}
 \int_0^T \cos \frac{j\pi t}{T} \cos \frac{j'\pi t}{T} dt &= \frac{T}{2} \delta_{jj'} \\
 \int_0^T \sin \frac{j\pi t}{T} dt &= \frac{T}{j\pi} [1 - (-1)^j]
 \end{aligned}$$

give for the action

$$\begin{aligned} S[\delta q] &= \sum_{j=1}^{\infty} \left\{ \frac{m(j\pi a_j)^2}{4T} - mg a_j \frac{T}{j\pi} [1 - (-1)^j] \right\} \\ &= \sum_{k=1}^{\infty} \frac{m\pi^2 k^2 a_{2k}^2}{T} + \sum_{k=1}^{\infty} \left[\frac{m\pi^2 (2k+1)^2 a_{2k+1}^2}{4T} - 2 \frac{mga_{2k+1}T}{(2k+1)\pi} \right]. \end{aligned}$$

The path integral is then the product of a path integral over the even loops and one over the odd ones. With J a constant jacobian, the path integral over N even loops is

$$\begin{aligned} \int_e e^{iS[\delta q]} D\delta q &= J_e \left(\frac{Nm}{2\pi iT} \right)^{N/2} \int \exp \left\{ i \sum_{k=1}^N \frac{m\pi^2 k^2}{T} a_{2k}^2 \right\} \prod_{k=1}^N da_{2k} \\ &= J \left(\frac{Nm}{2\pi iT} \right)^{N/2} \prod_{k=1}^N \left(\frac{iT}{m\pi k^2} \right)^{1/2} \\ &= J \left(\frac{N}{2\pi^2} \right)^{N/2} \prod_{k=1}^N \frac{1}{k} \end{aligned}$$

The path integral over N odd loops is

$$\begin{aligned} \int_o e^{iS[\delta q]} D\delta q &= J_o \left(\frac{Nm}{2\pi iT} \right)^{N/2} \\ &\quad \times \int \exp \left\{ i \sum_{k=1}^N \left[\frac{m\pi^2 (2k+1)^2 a_{2k+1}^2}{4T} - \frac{2mga_{2k+1}T}{(2k+1)\pi} \right] \right\} \prod_{k=1}^N da_{2k+1} \\ &= J_o \left(\frac{Nm}{2\pi iT} \right)^{N/2} \prod_{k=1}^N \left(\frac{4iT}{m\pi(2k+1)^2} \right)^{1/2} \exp \left[-i \frac{4mg^2 T^3}{(2k+1)^4 \pi^4} \right] \\ &= J_o \left(\frac{N}{2\pi^2} \right)^{N/2} \prod_{k=1}^N \frac{2}{2k+1} \exp \left[-i \frac{4mg^2 T^3}{\pi^4 (2k+1)^4} \right]. \end{aligned}$$

So the loop integral is

$$\int e^{iS[\delta q]} D\delta q = J \left(\frac{N}{2\pi^2} \right)^N \left[\prod_{k=1}^N \frac{1}{k} \right] \prod_{k=1}^N \frac{2}{2k+1} \exp \left[-i \frac{4mg^2 T^3}{\pi^4 (2k+1)^4} \right].$$

21. Show that the euclidian action of the stationary solution (20.87) is (20.88).

Solution: The kinetic action is

$$\begin{aligned} S_{ek}[q_e] &= \int_0^\beta \frac{1}{2}m (A\omega e^{\omega t} - B\omega e^{-\omega t})^2 dt \\ &= \frac{1}{2}m\omega^2 \int_0^\beta [A^2 e^{2\omega t} + B^2 e^{-2\omega t} - 2AB] dt \\ &= \frac{1}{2}m\omega^2 \left[A^2 \frac{e^{2\omega\beta} - 1}{2\omega} + B^2 \frac{1 - e^{-2\omega\beta}}{2\omega} - 2AB\beta \right]. \end{aligned}$$

The potential action is

$$\begin{aligned} S_{ev}[q_e] &= \frac{1}{2}m\omega^2 \int_0^\beta [A^2 e^{2\omega t} + B^2 e^{-2\omega t} + 2AB] dt \\ &= \frac{1}{2}m\omega^2 \left[A^2 \frac{e^{2\omega\beta} - 1}{2\omega} + B^2 \frac{1 - e^{-2\omega\beta}}{2\omega} + 2AB\beta \right] \end{aligned}$$

So the action of the stationary solution (20.87) is the sum

$$S_e[q_e] = \frac{1}{2}m\omega \left[A^2 (e^{2\omega\beta} - 1) + B^2 (1 - e^{-2\omega\beta}) \right].$$

22. Derive formula (20.161) for the action $S_0[\phi]$ from (20.159 & 20.160).

Solution: Differentiating (20.160), we have

$$\begin{aligned} \partial^a \phi(x) &= \partial^a \int e^{ipx} \tilde{\phi}(p) \frac{d^4 p}{(2\pi)^4} \\ &= \int i p^a e^{ipx} \tilde{\phi}(p) \frac{d^4 p}{(2\pi)^4} \end{aligned}$$

so

$$\begin{aligned} S_0[\phi] &= \int \frac{1}{2} [-\partial_a \phi(x) \partial^a \phi(x) - m^2 \phi^2(x)] d^4 x \\ &= \frac{1}{2} \int \left[-ip_a e^{ipx} \tilde{\phi}(p) ip'^a e^{ip'x} \tilde{\phi}(p') - m^2 e^{ipx} \tilde{\phi}(p) e^{ip'x} \tilde{\phi}(p') \right] \\ &\quad \times d^4 x \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \\ &= \frac{1}{2} \int \left[p_a \tilde{\phi}(p) p'^a \tilde{\phi}(p') - m^2 \tilde{\phi}(p) \tilde{\phi}(p') \right] \delta(p + p') \frac{d^4 p d^4 p'}{(2\pi)^4} \\ &= \frac{1}{2} \int \left[-p_a p^a \tilde{\phi}(p) \tilde{\phi}(-p) - m^2 \tilde{\phi}(p) \tilde{\phi}(-p) \right] \frac{d^4 p}{(2\pi)^4} \\ &= \frac{1}{2} \int \left[-p_a p^a |\tilde{\phi}(p)|^2 - m^2 |\tilde{\phi}(p)|^2 \right] \frac{d^4 p}{(2\pi)^4} \end{aligned}$$

which is (20.161).

23. Derive identity (20.165). Split the time integral at $t = 0$ into two halves, use $\epsilon e^{\pm\epsilon t} = \pm de^{\pm\epsilon t}/dt$ and then integrate each half by parts.

Solution: We assume that the limits $\lim_{t \rightarrow \pm\infty} f(t) = f(\pm\infty)$ exist. We split the integral and integrate by parts:

$$\begin{aligned}\epsilon \int_{-\infty}^{\infty} f(t) e^{-\epsilon|t|} dt &= \int_{-\infty}^0 f(t) \epsilon e^{\epsilon t} dt + \int_0^{\infty} f(t) \epsilon e^{-\epsilon t} dt \\ &= \int_{-\infty}^0 f(t) \frac{de^{\epsilon t}}{dt} dt - \int_0^{\infty} f(t) \frac{de^{-\epsilon t}}{dt} dt \\ &= \int_{-\infty}^0 \left[\frac{d(f(t)e^{\epsilon t})}{dt} - e^{\epsilon t} \frac{df(t)}{dt} \right] dt \\ &\quad - \int_0^{\infty} \left[\frac{d(f(t)e^{-\epsilon t})}{dt} - e^{-\epsilon t} \frac{df(t)}{dt} \right] dt \\ &= f(0) - \int_{-\infty}^0 e^{\epsilon t} \dot{f}(t) dt + f(0) + \int_0^{\infty} e^{-\epsilon t} \dot{f}(t) dt\end{aligned}$$

in which the dot means time derivative. We now let $\epsilon \rightarrow 0$ and get

$$\begin{aligned}\epsilon \int_{-\infty}^{\infty} f(t) e^{-\epsilon|t|} dt &= 2f(0) - \int_{-\infty}^0 \dot{f}(t) dt + \int_0^{\infty} \dot{f}(t) dt \\ &= 2f(0) - f(0) + f(-\infty) + f(\infty) - f(0)\end{aligned}$$

which is the desired identity (20.165)

$$\epsilon \int_{-\infty}^{\infty} f(t) e^{-\epsilon|t|} dt = f(\infty) + f(-\infty).$$

24. Derive the third term in equation (20.167) from the second term.

Solution: The Fourier transforms (20.160) tell us that

$$\tilde{\phi}(\mathbf{p}, t) = \int e^{-ip^0 t} \tilde{\phi}(\mathbf{p}, p^0) \frac{dp^0}{2\pi}$$

so we have

$$\begin{aligned}\int |\tilde{\phi}(\mathbf{p}, t)|^2 dt &= \int e^{-ip^0 t} \tilde{\phi}(\mathbf{p}, p^0) \frac{dp^0}{2\pi} e^{ip'^0 t} \tilde{\phi}^*(\mathbf{p}, p'^0) \frac{dp'^0}{2\pi} dt \\ &= \int \tilde{\phi}(\mathbf{p}, p^0) \tilde{\phi}^*(\mathbf{p}, p'^0) \delta(p^0 - p'^0) \frac{dp^0 dp'^0}{2\pi} \\ &= \int \tilde{\phi}(\mathbf{p}, p^0) \tilde{\phi}^*(\mathbf{p}, p^0) \frac{dp^0}{2\pi} = \int |\tilde{\phi}(\mathbf{p}, p^0)|^2 \frac{dp^0}{2\pi}.\end{aligned}$$

Thus,

$$\int \sqrt{\mathbf{p}^2 + m^2} \int_{-\infty}^{\infty} |\tilde{\phi}(\mathbf{p}, t)|^2 dt \frac{d^3 p}{(2\pi)^3} = \int \sqrt{\mathbf{p}^2 + m^2} |\tilde{\phi}(\mathbf{p})|^2 \frac{d^4 p}{(2\pi)^4}.$$

25. Use (20.171) and the Fourier transform (20.172) of the external current j to derive the formula (20.173) for the modified action $S_0[\phi, \epsilon, j]$.

Solution: Using the Fourier transforms

$$\phi(x) = \int e^{ipx} \tilde{\phi}(p) \frac{d^4 p}{(2\pi)^4}$$

and (20.172)

$$\tilde{j}(p) = \int e^{-ipx} j(x) d^4 x,$$

we have the inverse transform

$$j(x) = \int e^{ipx} \tilde{j}(p) \frac{d^4 p}{(2\pi)^4}$$

and the desired relation

$$\begin{aligned} \int j(x) \phi(x) d^4 x &= \int e^{ipx} \tilde{j}(p) e^{ip'x} \tilde{\phi}(p') d^4 x \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \\ &= \int \tilde{j}(p) \tilde{\phi}(p') \delta(p + p') \frac{d^4 p d^4 p'}{(2\pi)^4} = \int \tilde{j}(p) \tilde{\phi}(-p) \frac{d^4 p}{(2\pi)^4} \\ &= \frac{1}{2} \int [\tilde{j}(p) \tilde{\phi}(-p) + \tilde{j}(-p) \tilde{\phi}(p)] \frac{d^4 p}{(2\pi)^4} \\ &= \frac{1}{2} \int [\tilde{j}(p) \tilde{\phi}^*(p) + \tilde{j}^*(p) \tilde{\phi}(p)] \frac{d^4 p}{(2\pi)^4} \end{aligned}$$

since $j(x)$ and $\phi(x)$ are real.

26. Derive equation (20.174) from equation (20.173).

Solution: Changing variables to

$$\tilde{\psi}(p) = \tilde{\phi}(p) - \tilde{j}(p)/(p^2 + m^2 - i\epsilon), \quad (23.491)$$

we can write the action

$$\begin{aligned} S_0[\phi, \epsilon, j] &= -\frac{1}{2} \int \left[|\tilde{\phi}(p)|^2 (p^2 + m^2 - i\epsilon) \right. \\ &\quad \left. - \tilde{j}^*(p) \tilde{\phi}(p) - \tilde{\phi}^*(p) \tilde{j}(p) \right] \frac{d^4 p}{(2\pi)^4} \end{aligned}$$

as

$$\begin{aligned}
 S_0[\phi, \epsilon, j] &= -\frac{1}{2} \int \left[\left| \tilde{\psi}(p) + \frac{\tilde{j}(p)}{p^2 + m^2 - i\epsilon} \right|^2 (p^2 + m^2 - i\epsilon) \right. \\
 &\quad \left. - \tilde{j}^*(p) \left(\tilde{\psi}(p) + \frac{\tilde{j}(p)}{p^2 + m^2 - i\epsilon} \right) \right. \\
 &\quad \left. - \left(\tilde{\psi}^*(p) + \frac{\tilde{j}^*(p)}{p^2 + m^2 + i\epsilon} \right) \tilde{j}(p) \right] \frac{d^4 p}{(2\pi)^4} \\
 &= -\frac{1}{2} \int \left[\left| \tilde{\psi}(p) \right|^2 (p^2 + m^2 - i\epsilon) - \frac{|\tilde{j}(p)|^2}{p^2 + m^2 - i\epsilon} \right] \frac{d^4 p}{(2\pi)^4} \\
 &= S_0[\psi, \epsilon] + \frac{1}{2} \int \frac{|\tilde{j}(p)|^2}{p^2 + m^2 - i\epsilon} \frac{d^4 p}{(2\pi)^4}.
 \end{aligned}$$

27. Derive the formula (20.175) for $Z_0[j]$ from the formula for $S_0[\phi, \epsilon, j]$.

Solution: Our formula for $Z_0[j]$ is

$$\begin{aligned}
 Z_0[j] &= \frac{\int e^{i \int j(x) \phi(x) d^4 x} e^{i S_0[\phi, \epsilon]} D\phi}{\int e^{i S_0[\phi, \epsilon]} D\phi} \\
 &= \frac{\int e^{i S_0[\phi, \epsilon, j]} D\phi}{\int e^{i S_0[\phi, \epsilon]} D\phi} = \frac{\int e^{i S_0[\psi, \epsilon]} D\psi}{\int e^{i S_0[\psi, \epsilon]} D\psi} \exp \left[\frac{1}{2} \int \frac{|\tilde{j}(p)|^2}{p^2 + m^2 - i\epsilon} \frac{d^4 p}{(2\pi)^4} \right] \\
 &= \exp \left[\frac{1}{2} \int \frac{|\tilde{j}(p)|^2}{p^2 + m^2 - i\epsilon} \frac{d^4 p}{(2\pi)^4} \right]
 \end{aligned}$$

since $D\phi = D\psi$.

28. Derive equations (20.176 & 20.177) from formula (20.175).

Solution: From our formula (20.175), we have

$$\begin{aligned} Z_0[j] &= \exp \left(\frac{i}{2} \int \frac{|\tilde{j}(p)|^2}{p^2 + m^2 - i\epsilon} \frac{d^4 p}{(2\pi)^4} \right) \\ &= \exp \left(\frac{i}{2} \int \frac{j(x)j(x')}{p^2 + m^2 - i\epsilon} \frac{d^4 p}{(2\pi)^4} e^{ipx} e^{-ipx'} d^4 x d^4 x' \right) \\ &= \exp \left(\frac{i}{2} \int j(x) \int \frac{e^{ip(x-x')}}{p^2 + m^2 - i\epsilon} \frac{d^4 p}{(2\pi)^4} j(x') d^4 x d^4 x' \right) \\ &= \exp \left[\frac{i}{2} \int j(x) \Delta(x - x') j(x') d^4 x d^4 x' \right] \end{aligned}$$

where

$$\Delta(x - x') = \int \frac{e^{ip(x-x')}}{p^2 + m^2 - i\epsilon} \frac{d^4 p}{(2\pi)^4}.$$

29. Derive equation (20.181) from the formula (20.176) for $Z_0[j]$.

Solution: From our definition (20.170) of the functional $Z_0[j]$ and from our formula (20.176) for it, we have

$$\begin{aligned} Z_0[j] &= \langle 0 | \mathcal{T} \left\{ \exp \left[i \int j(x) \phi(x) d^4 x \right] \right\} | 0 \rangle \\ &= \exp \left[\frac{i}{2} \int j(x) \Delta(x - x') j(x') d^4 x d^4 x' \right]. \end{aligned}$$

Thus, we find

$$\begin{aligned} \langle 0 | \mathcal{T} [\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)] | 0 \rangle &= \frac{1}{i^4} \frac{\delta^4 Z_0[j]}{\delta j(x_1)\delta j(x_2)\delta j(x_3)\delta j(x_4)} \Big|_{j=0} \\ &= \frac{1}{i^4} \frac{\delta^4 \exp \left[\frac{i}{2} \int j(x) \Delta(x - x') j(x') d^4 x d^4 x' \right]}{\delta j(x_1)\delta j(x_2)\delta j(x_3)\delta j(x_4)} \Big|_{j=0} \\ &= \frac{1}{i^3} \frac{\delta^3 \int \Delta(x_4 - x) j(x) d^4 x e^{\frac{i}{2} \int j(x) \Delta(x - x') j(x') d^4 x d^4 x'}}{\delta j(x_1)\delta j(x_2)\delta j(x_3)} \Big|_{j=0} \\ &= \frac{1}{i^3} \frac{\delta^2}{\delta j(x_1)\delta j(x_2)} \left\{ \left[\Delta(x_4 - x_3) \right. \right. \\ &\quad \left. \left. + \int \Delta(x_4 - x) j(x) d^4 x i \int \Delta(x_3 - x') j(x') d^4 x' \right] \right. \\ &\quad \left. \times e^{\frac{i}{2} \int j(x) \Delta(x - x') j(x') d^4 x d^4 x'} \right\} \Big|_{j=0}. \end{aligned}$$

Continuing in this way, we get

$$\begin{aligned} \langle 0 | \mathcal{T}[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)] | 0 \rangle = & \\ & \frac{1}{i^2} \frac{\delta}{\delta j(x_1)} \left\{ \left[\Delta(x_4 - x_3) \int \Delta(x_2 - x) j(x) d^4x \right. \right. \\ & + \Delta(x_4 - x_2) \int \Delta(x_3 - x) j(x) d^4x \\ & + \Delta(x_3 - x_2) \int \Delta(x_4 - x) j(x) d^4x \\ & + i \int \Delta(x_4 - x) j(x) d^4x \Delta(x_3 - x') j(x') d^4x' \Delta(x_2 - x'') j(x'') d^4x'' \\ & \left. \times e^{\frac{i}{2} \int j(x) \Delta(x-x') j(x') d^4x d^4x'} \right] \Big|_{j=0} \end{aligned}$$

and finally

$$\begin{aligned} \langle 0 | \mathcal{T}[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)] | 0 \rangle = & -\Delta(x_4 - x_3)\Delta(x_2 - x_1) \\ & -\Delta(x_4 - x_2)\Delta(x_3 - x_1) - \Delta(x_3 - x_2)\Delta(x_4 - x_1) \end{aligned}$$

which is (20.181) since $\Delta(-x) = \Delta(x)$.

30. Show that the time integral of the Coulomb term (20.186) is the term that is quadratic in j^0 in the number F defined by (20.190).

Solution: Since

$$-\Delta \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} = \delta(\mathbf{x} - \mathbf{y}),$$

we can write

$$-\Delta^{-1} = \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}.$$

Thus, by integrating successively by parts we find that the term in F that is quadratic in j^0 is

$$\begin{aligned} \frac{1}{2} \int (\nabla \Delta^{-1} j^0)^2 d^4x &= \frac{1}{2} \int (\nabla \Delta^{-1} j^0(x)) d^4x \nabla \frac{-1}{4\pi|\mathbf{x} - \mathbf{y}|} j^0(\mathbf{y}, x^0) d^3y \\ &= \frac{1}{2} \int (\Delta^{-1} j^0(x)) d^4x \nabla \cdot \nabla \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} j^0(\mathbf{y}, x^0) d^3y \\ &= \frac{1}{2} \int j^0(x) d^4x \Delta^{-1} \Delta \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} j^0(\mathbf{y}, x^0) d^3y \\ &= \frac{1}{2} \int j^0(x) \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} j^0(\mathbf{y}, x^0) d^4x d^3y = \int V_C dt \end{aligned}$$

the time integral of the Coulomb term (20.186).

31. By following steps analogous to those led to (20.177), derive the formula (20.201) for the photon propagator in Feynman's gauge.

Solution: In the Feynman gauge, $\alpha = 1$, the main terms in the action S_α are

$$\begin{aligned} \int -\frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}(\partial_b A^b)^2 d^4x &= \int \frac{1}{2}A_b \square A^b d^4x \\ &= \int \frac{1}{2}A_a \eta^{ab} \square A_b d^4x. \end{aligned}$$

This theory is therefore equivalent to four copies of the scalar theory (20.159) with action $S_0[\phi]$ and $m = 0$ multiplied by η^{ab} . Thus, following the steps of exercises 20.24 and 20.25, we have

$$\begin{aligned} \langle 0 | \mathcal{T} \left\{ \exp \left[i \int j^b(x) A_b(x) d^4x \right] \right\} \\ = \exp \left[\frac{i}{2} \int j^a(x) \eta_{ab} \Delta(x - x') j^b(x') d^4x d^4x' \right], \end{aligned}$$

and so

$$\langle 0 | \mathcal{T} [A_\mu(x) A_\nu(y)] | 0 \rangle = -i \Delta_{\mu\nu}(x - y) = -i \int \frac{\eta_{\mu\nu}}{q^2 - i\epsilon} e^{iq \cdot (x-y)} \frac{d^4q}{(2\pi)^4}$$

in which

$$\Delta_{\mu\nu}(x - y) \equiv \eta_{\mu\nu} \Delta(x - y).$$

32. Derive expression (20.215) for the inner product $\langle \zeta | \theta \rangle$.

Solution: We have

$$\begin{aligned} \langle \zeta | \theta \rangle &= \langle 0 | \left(1 + \zeta^* \psi - \frac{1}{2} \zeta^* \zeta \right) \left(1 + \psi^\dagger \theta - \frac{1}{2} \theta^* \theta \right) | 0 \rangle \\ &= \langle 0 | \zeta^* \psi \left(1 - \frac{1}{2} \theta^* \theta \right) + \left(1 - \frac{1}{2} \zeta^* \zeta \right) \psi^\dagger \theta | 0 \rangle \\ &\quad + \langle 0 | 1 + \zeta^* \psi \psi^\dagger \theta - \frac{1}{2} \zeta^* \zeta - \frac{1}{2} \theta^* \theta + \frac{1}{4} \zeta^* \zeta \theta^* \theta | 0 \rangle \end{aligned}$$

in which the first four terms vanish because $\psi | 0 \rangle = 0$ and $\langle 0 | \psi^\dagger = 0$. Thus, since $\theta^2 = \zeta^2 = 0$, we find

$$\begin{aligned} \langle \zeta | \theta \rangle &= \langle 0 | 1 + \zeta^* \theta - \frac{1}{2} \zeta^* \zeta - \frac{1}{2} \theta^* \theta + \frac{1}{4} \zeta^* \zeta \theta^* \theta | 0 \rangle \\ &= \exp \left[\zeta^* \theta - \frac{1}{2} (\zeta^* \zeta + \theta^* \theta) \right]. \end{aligned}$$

33. Derive the representation (20.218) of the identity operator I for a single fermionic degree of freedom from the rules (20.204 & 20.208) for Grassmann integration and the anticommutation relations (20.207).

Solution: The formulas (20.210) and (20.214) for fermionic coherent states let us write

$$\begin{aligned} \int |\theta\rangle\langle\theta| d\theta^*d\theta &= \int (1 + \psi^\dagger\theta - \frac{1}{2}\theta^*\theta)|0\rangle\langle 0| (1 + \theta^*\psi - \frac{1}{2}\theta^*\theta) d\theta^*d\theta \\ &= \int (1 + \psi^\dagger\theta - \frac{1}{2}\theta^*\theta)|0\rangle\langle 0| (1 + \theta^*\psi - \frac{1}{2}\theta^*\theta) d\theta^*d\theta \\ &= |0\rangle\langle 0| \int \theta^*d\theta^* \theta d\theta + \psi^\dagger|0\rangle\langle 0|\psi \int \theta^*d\theta^* \theta d\theta \\ &= |0\rangle\langle 0| + |1\rangle\langle 1| = I. \end{aligned}$$

34. Derive the eigenvalue equation $\psi_k|\theta\rangle = \theta_k|\theta\rangle$ from the definitions (20.221 & 20.222) of the eigenstate $|\theta\rangle$.

Solution: For each mode k , we have from (20.213) that

$$\begin{aligned} \psi_k|\theta_k\rangle &= \psi_k \left(1 + \psi_k^\dagger\theta_k - \frac{1}{2}\theta_k^*\theta_k \right) |0\rangle = \theta_k \left(1 + \psi_k^\dagger\theta_k - \frac{1}{2}\theta_k^*\theta_k \right) |0\rangle \\ &= \theta_k|\theta_k\rangle. \end{aligned}$$

The factors in the direct-product state $|\theta\rangle$ are all even, and so

$$\begin{aligned} \psi_k|\theta\rangle &= \psi_k \left[\prod_{\ell=1}^n \left(1 + \psi_\ell^\dagger\theta_\ell - \frac{1}{2}\theta_\ell^*\theta_\ell \right) \right] |0\rangle \\ &= \left[\prod_{\ell=1, \ell \neq k}^n \left(1 + \psi_\ell^\dagger\theta_\ell - \frac{1}{2}\theta_\ell^*\theta_\ell \right) \right] \psi_k|\theta_k\rangle \\ &= \left[\prod_{\ell=1, \ell \neq k}^n \left(1 + \psi_\ell^\dagger\theta_\ell - \frac{1}{2}\theta_\ell^*\theta_\ell \right) \right] \theta_k|\theta_k\rangle \\ &= \theta_k \left[\prod_{\ell=1, \ell \neq k}^n \left(1 + \psi_\ell^\dagger\theta_\ell - \frac{1}{2}\theta_\ell^*\theta_\ell \right) \right] |\theta_k\rangle = \theta_k|\theta\rangle. \end{aligned}$$

35. Derive the eigenvalue relation (20.235) for the Fermi field $\psi_m(\mathbf{x}, t)$ from the anticommutation relations (20.231 & 20.232) and the definitions (20.233 & 20.234).

Solution: This exercise is very similar to exercise 20.33. The principal

difference is that $\delta_{k\ell}$ is replaced by $\delta(\mathbf{x} - \mathbf{x}')$. From the definition (20.233) of the state $|0\rangle$, we see that

$$\psi_m(\mathbf{x})|0\rangle = 0.$$

Let us define

$$\begin{aligned} V(\alpha) &\equiv \exp \left(\alpha \int \sum_m \psi_m^\dagger(\mathbf{x}, 0) \chi_m(\mathbf{x}) - \frac{1}{2} \chi_m^*(\mathbf{x}) \chi_m(\mathbf{x}) d^3x \right) \\ &= \exp \left[\alpha \int Q(\mathbf{x}) d^3x \right], \end{aligned}$$

and

$$\psi_m(\mathbf{x}, \alpha) \equiv V(-\alpha) \psi_m(\mathbf{x}) V(\alpha).$$

Then $\psi_m(\mathbf{x}, 0) = \psi_m(\mathbf{x})$, and

$$\begin{aligned} \partial_\alpha \psi_m(\mathbf{x}, \alpha) &= V(-\alpha) \left[\psi_m(\mathbf{x}), \int Q(\mathbf{x}') d^3x' \right] V(\alpha) \\ &= V(-\alpha) \int \sum_{m'} \{\psi_m(\mathbf{x}), \psi_{m'}^\dagger(\mathbf{x}')\} \chi_{m'}(\mathbf{x}') d^3x' V(\alpha) \\ &= V(-\alpha) \chi_m(\mathbf{x}) V(\alpha) = \chi_m(\mathbf{x}) \end{aligned}$$

since the argument of the exponential $V(\alpha)$ is quadratic in fermionic variables. Thus, we have

$$\begin{aligned} \psi_m(\mathbf{x}, 1) &= V(-1) \psi_m(\mathbf{x}) V(1) = \int_0^1 \partial_\alpha \psi_m(\mathbf{x}, \alpha) d\alpha + \psi_m(\mathbf{x}, 0) \\ &= \psi_m(\mathbf{x}) + \chi_m(\mathbf{x}). \end{aligned}$$

And so,

$$\begin{aligned} \psi_m(\mathbf{x})|\chi\rangle &= \psi_m(\mathbf{x})V(1)|0\rangle = V(1)V(-1)\psi_m(\mathbf{x})V(1)|0\rangle \\ &= V(1)[\psi_m(\mathbf{x}) + \chi_m(\mathbf{x})]|0\rangle = V(1)\chi_m(\mathbf{x})|0\rangle \\ &= \chi_m(\mathbf{x})V(1)|0\rangle = |\chi\rangle. \end{aligned}$$

36. Derive the formula (20.236) for the inner product $\langle \chi' | \chi \rangle$ from the definition (20.234) of the ket $|\chi\rangle$.

Solution: Taking the adjoint of the definition (20.234), we have

$$\langle \chi' | = \langle 0 | \exp \left(\int \chi'^\dagger \psi - \frac{1}{2} \chi'^\dagger \chi' d^3x \right).$$

Thus, since $|\chi\rangle$ is an eigenstate of the field ψ , we find

$$\begin{aligned}\langle\chi'|\chi\rangle &= \langle 0| \exp \left(\int \chi'^{\dagger} \psi - \frac{1}{2} \chi'^{\dagger} \chi' d^3x \right) |\chi\rangle \\ &= \langle 0| \exp \left(\int \chi'^{\dagger} \chi - \frac{1}{2} \chi'^{\dagger} \chi' d^3x \right) |\chi\rangle \\ &= \langle 0| \exp \left(\int \chi'^{\dagger} \chi - \frac{1}{2} \chi'^{\dagger} \chi' d^3x \right) \exp \left(\int \psi^{\dagger} \chi - \frac{1}{2} \chi^{\dagger} \chi d^3x \right) |0\rangle \\ &= \langle 0| \exp \left(\int \chi'^{\dagger} \chi - \frac{1}{2} \chi'^{\dagger} \chi' d^3x \right) \exp \left(\int -\frac{1}{2} \chi^{\dagger} \chi d^3x \right) |0\rangle \\ &= \exp \left(\int \chi'^{\dagger} \chi - \frac{1}{2} \chi'^{\dagger} \chi' - \frac{1}{2} \chi^{\dagger} \chi d^3x \right)\end{aligned}$$

since $\langle 0|\psi^{\dagger} = 0$.

37. Imitate the derivation of the path-integral formula (20.66) and derive its three-dimensional version (20.73).

Solution: Now we have $H = \mathbf{p}^2/2m + V(\mathbf{q})$. We set $k = \beta \mathbf{p}^2/(2m)$ and $v = \beta V(\mathbf{q})$. We use Trotter's formula (20.60) and get as the 3D version of (20.63)

$$\begin{aligned}\langle \mathbf{q}_1 | e^{-\epsilon k} e^{-\epsilon v} | \mathbf{q}_a \rangle &= e^{-\epsilon V(\mathbf{q}_a)} \int_{-\infty}^{\infty} \exp \left(-\epsilon \frac{\mathbf{p}'^2}{2m} + i \epsilon \mathbf{p}' \cdot \dot{\mathbf{q}}_a \right) \frac{d\mathbf{p}'}{2\pi\hbar} \\ &= \left(\frac{m}{2\pi\hbar^2\epsilon} \right)^{3/2} \exp \left[-\epsilon \left(\frac{m\dot{\mathbf{q}}_a^2}{2} + V(\mathbf{q}_a) \right) \right]\end{aligned}$$

in which \mathbf{q}_1 is hidden in the formula $\mathbf{q}_1 - \mathbf{q}_a = \hbar\epsilon \dot{\mathbf{q}}_a$. Putting n such factors together, we get

$$\begin{aligned}\langle \mathbf{q}_b | e^{-n\epsilon H} | \mathbf{q}_a \rangle &= \iiint_{-\infty}^{\infty} \langle \mathbf{q}_b | e^{-\epsilon k} e^{-\epsilon v} | \mathbf{q}_{n-1} \rangle \dots \langle \mathbf{q}_1 | e^{-\epsilon k} e^{-\epsilon v} | \mathbf{q}_a \rangle d\mathbf{q}_{n-1} \dots d\mathbf{q}_1 \\ &= \left(\frac{m}{2\pi\hbar^2\epsilon} \right)^{3n/2} \iiint_{-\infty}^{\infty} \exp \left[-\epsilon \sum_{j=0}^{n-1} \left(\frac{m\dot{\mathbf{q}}_j^2}{2} + V(\mathbf{q}_j) \right) \right] d\mathbf{q}_{n-1} \dots d\mathbf{q}_1.\end{aligned}$$

Setting $du = \hbar\epsilon = \hbar\beta/n$ and taking the limit $n \rightarrow \infty$, we find that the matrix element $\langle \mathbf{q}_b | e^{-\beta H} | \mathbf{q}_a \rangle$ is the path integral

$$\langle \mathbf{q}_b | e^{-\beta H} | \mathbf{q}_a \rangle = \int e^{-S_e[\mathbf{q}]/\hbar} D\mathbf{q}$$

where

$$S_e[\mathbf{q}] = \int_0^{\hbar\beta} \frac{m\dot{\mathbf{q}}^2(u)}{2} + V(\mathbf{q}(u)) du.$$

38. Differentials $d\zeta_i$ of complex linear combinations $\zeta_i = A_{i\ell} \theta_\ell$ of Grassmann variables are defined as $d\zeta_i = d\theta_\ell (A^{-1})_{\ell i}$ and as $d\zeta_1 \cdots d\zeta_n = \det(A^{-1}) d\theta_1 \cdots d\theta_n$. Show that the ζ 's inherit the rules of integration of the θ 's:

$$\delta_{ik} = \int \theta_i d\theta_k \implies \delta_{ik} = \int \zeta_i d\zeta_k. \quad (23.492)$$

Solution: We need to show that the integral

$$D_{ik} = \int \zeta_i d\zeta_k \quad (23.493)$$

is δ_{ik} . Substituting for the ζ 's their definitions $\zeta_i = A_{i\ell} \theta_\ell$ and $d\zeta_k = d\theta_j (A^{-1})_{jk}$, we get

$$\begin{aligned} D_{ik} &= \int \zeta_i d\zeta_k = \int A_{i\ell} \theta_\ell d\theta_j (A^{-1})_{jk} \\ &= A_{i\ell} \delta_{\ell j} (A^{-1})_{jk} = A_{i\ell} (A^{-1})_{jk} = \delta_{ik}. \end{aligned} \quad (23.494)$$

23.20 Solutions to the exercises on the renormalization group

1. Show that for $q^2 = \mu^2 \gg m^2$, the vacuum polarization term (21.10) reduces to (21.12). Hint: Use $\ln ab = \ln a + \ln b$ when integrating.

Solution: For $q^2 = \mu^2 \gg m^2$, the logarithm in the vacuum polarization term

$$\pi(q^2) = \frac{e^2}{2\pi^2} \int_0^1 x(1-x) \ln \left[1 + \frac{q^2 x(1-x)}{m^2} \right] dx$$

is approximately

$$\ln \left[1 + \frac{q^2 x(1-x)}{m^2} \right] \approx \ln \left[\frac{q^2 x(1-x)}{m^2} \right] = \ln(q^2/m^2) + \ln[x(1-x)].$$

So the vacuum polarization term is then for $q^2 = \mu^2 \gg m^2$

$$\begin{aligned} \pi(\mu^2) &\approx \frac{e^2}{2\pi^2} \int_0^1 x(1-x) [\ln(\mu^2/m^2) + \ln[x(1-x)]] dx \\ &\approx \frac{e^2}{2\pi^2} \left[2 \ln \left(\frac{m}{\mu} \right) \int_0^1 x(1-x) dx + \int_0^1 x(1-x) \ln[x(1-x)] dx \right] \\ &\approx \frac{e^2}{2\pi^2} \left[2 \ln \left(\frac{m}{\mu} \right) \frac{1}{6} - \frac{5}{18} \right] = \frac{e^2}{6\pi^2} \left[\ln \left(\frac{m}{\mu} \right) - \frac{5}{6} \right]. \end{aligned}$$