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SI Units and Natural Units

23.1 Standard International (SI) Units

In Standard International (SI) units, the definitions, values, and dimensions of some basic physical quantities are

$$\begin{aligned}
c &= 2.997\,924\,58 \times 10^{8}\ \mathrm{m\,s^{-1}}\ \text{(exactly)} \\
h &= 6.626\,070\,15 \times 10^{-34}\ \mathrm{J\ s}\ \text{(exactly)} \\
\hbar &\equiv \frac{h}{2\pi} = 1.054\,571\,817\ldots \times 10^{-34}\ \mathrm{J\ s} \\
e &= 1.602\,176\,634 \times 10^{-19}\ \mathrm{C}\ \text{(exactly)} \\
k_{\text{B}} &= 1.380649 \times 10^{-23}\mathrm{J\,K^{-1}}\ \text{(exactly)} \\
N_{\text{A}} &= 6.02214076 \times 10^{23}\ \mathrm{mol^{-1}}\ \text{(exactly)} \\
\Delta\nu_{\text{Cs}} &= 9.192\,631\,770 \times 10^{9}\ \mathrm{Hz}\ \text{(exactly)} \\
G &= 6.67430\,(15) \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}} \\
\mu_0 &= (4\pi) \times 1.000\,000\,000\,55(15) \times 10^{7}\ \mathrm{N\ A^{-2}} \\
\epsilon_0 &\equiv 1/(\mu_0 c^2) = 8.854\,187\,8128(13) \times 10^{-12}\ \mathrm{F\ m^{-1}} \\
\alpha &\equiv \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137.035\,999\,084(21)} = 7.297\,352\,5693(11) \times 10^{-3} \\
0\,\mathrm{K} &= -273.15\,^{\circ}\mathrm{C} = -459.67\,^{\circ}\mathrm{F}
\end{aligned} \tag{23.1}$$

in which $\mathrm{N} = \mathrm{J\ m^{-1}}$, $\mathrm{Hz} = \mathrm{s^{-1}}$, $\mathrm{A} = \mathrm{C\ s^{-1}}$, $\mathrm{F} = \mathrm{kg^{-1}\ m^{-2}\ s^{2}\ C^{2}}$, and $\Delta\nu_{\text{Cs}}$ is a hyperfine transition frequency of Cesium-133.

Numerical prefixes

yotta $\mathrm{Y} = 10^{24}$
zetta $\mathrm{Z} = 10^{21}$
exa $\mathrm{E} = 10^{18}$
peta $\mathrm{P} = 10^{15}$
tera $\mathrm{T} = 10^{12}$
giga $\mathrm{G} = 10^{9}$
mega $\mathrm{M} = 10^{6}$
kilo $\mathrm{k} = 10^{3}$
milli $\mathrm{m} = 10^{-3}$
micro $\mu = 10^{-6}$
nano $\mathrm{n} = 10^{-9}$
pico $\mathrm{p} = 10^{-12}$
femto $\mathrm{f} = 10^{-15}$
atto $\mathrm{a} = 10^{-18}$
zepto $\mathrm{z} = 10^{-21}$
yocto $\mathrm{y} = 10^{-24}$

The frequency $\Delta\nu_{\text{Cs}}$ defines the Hz and therefore the second. The speed of light c defines the meter m in terms of the second s. Planck's constant h defines the Joule J in terms of the second s, which gives us the kilogram kg since we already know the meter and the second. Boltzmann's constant k_{B} defines the degree Kelvin K in terms of the Joule J. Finally the absolute value e of the charge of the electron defines the Coulomb C. To remember that the fine-structure constant is $\alpha \sim .007$, think of James Bond.

23.2 Natural Units

The symbols $\hbar$, c, G, k_{B}, m_e and so forth clutter our formulas. To avoid such distractions, Planck and Hartree invented systems of **natural units** that replace such constants by unity. Planck's system is used in cosmology and in nuclear and particle physics; Hartree's is used in atomic and molecular physics.

23.3 Planck Units

Planck set his constant $\hbar$, the speed of light c, Newton's constant G, and Boltzmann's constant k_{B} equal to unity. In his units, $c = 1$, $G = 1$, $k_{\text{B}} = 1$, and $\hbar \equiv h/2\pi = 1$. The Planck length $L_{\text{P}} = \sqrt{\hbar G/c^3} \sim 1.6 \times 10^{-35}$ m, mass $M_{\text{P}} = \sqrt{\hbar c/G} \sim 2.1 \times 10^{-8}$ kg, time $T_{\text{P}} = \sqrt{\hbar G/c^5} \sim 5.4 \times 10^{-44}$ s, and temperature $T_{\text{P, temp}} = \sqrt{\hbar\, c^5/G k_{\text{B}}^2} \sim 1.4 \times 10^{32}$ K are all unity in Planck units.

In a calculation done in Planck's units, a quantity represented as a mass m can have the dimensions of inverse time $= 1/T$, inverse length $= 1/L$, mass $= M$, energy E, temperature K, inverse time $= 1/T$, inverse length $= 1/L$, or inverse temperature K. A quantity represented as a mass m can be a momentum or an energy

$$[mc] = \frac{ML}{T} \quad \text{or} \quad \left[mc^2\right] = \frac{ML^2}{T^2} \tag{23.2}$$

or an inverse length or an inverse time

$$\left[\frac{mc}{\hbar}\right] = \frac{MLT}{TML^2} = \frac{1}{L} \quad \text{or} \quad \left[\frac{mc^2}{\hbar}\right] = \frac{E}{ET} = \frac{1}{T} \tag{23.3}$$

or a length or a time

$$\left[\frac{Gm}{c^2}\right] = \frac{L^3}{MT^2}\,\frac{MT^2}{L^2} = L \quad \text{or} \quad \left[\frac{Gm}{c^3}\right] = \frac{L^3}{MT^2}\,\frac{MT^3}{L^3} = T. \tag{23.4}$$

or a temperature or an inverse temperature

$$\left[\frac{m\,c^2}{k_{\text{B}}}\right] = \frac{E\,K}{E} = K \quad \text{or} \quad \left[\frac{m\,k_{\text{B}}G}{c^3\hbar}\right] = \frac{ME}{K}\,\frac{L^3}{MT^2}\,\frac{T^3}{L^3ET} = \frac{1}{K}. \tag{23.5}$$

Example 23.1 (Mass as inverse length, energy, and length) The range of the nuclear force is given approximately in terms of the mass of the pion as $r \approx 1/m_\pi = \hbar c/m_\pi = 197\,\text{MeV fm}/139\,\text{MeV} = 1.4\,\text{fm} = 1.4 \times 10^{-15}$ m. The energy levels of a hydrogen atom are $E_n = -\frac{1}{2}\alpha^2 m_e/n^2 = -13.6/n^2$ eV. The Schwarzschild radius of a point mass m as massive as the Earth is $r_{\text{S}} = 2m_E = 2Gm_E/c^2 = 2(6.67 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}) \times 5.97 \times 10^{24}\text{kg}/(3 \times 10^8\text{m/s})^2 = 8.87$ mm. □

23.4 Hartree Units

Hartree set Planck's constant $\hbar$, the mass of the electron m_e, the charge of the proton e, and Coulomb's constant $k_e = 1/(4\pi\epsilon_0)$ equal to unity. The Hartree length $L_{\text{H}} = 4\pi\epsilon_0\,\hbar^2/m_e e^2 \sim 5.3 \times 10^{-11}$ m, mass $M_{\text{H}} = m_e \sim 9.1 \times 10^{-31}$ kg, time $T_{\text{H}} = (4\pi\epsilon_0)^2\hbar^2/m_e e^4 \sim 2.4 \times 10^{-17}$ s, and charge $Q_{\text{H}} = e \sim 1.6 \times 10^{-19}$ C are all unity in Hartree units. In his units, the Bohr radius $a_0 = L_{\text{H}} = 1$, and the energy levels of a hydrogen atom are $E_n = 1/(2n^2)$.