4

The Cluster Decomposition Principle

Up to this point we have not had much to say about the detailed structure of the Hamiltonian operator H. This operator can be defined by giving all its matrix elements between states with arbitrary numbers of particles. Equivalently, as we shall show here, any such operator may be expressed as a function of certain operators that create and destroy single particles. We saw in Chapter 1 that such creation and annihilation operators were first encountered in the canonical quantization of the electromagnetic field and other fields in the early days of quantum mechanics. They provided a natural formalism for theories in which massive particles as well as photons can be produced and destroyed, beginning in the early 1930s with Fermi's theory of beta decay.

However, there is a deeper reason for constructing the Hamiltonian out of creation and annihilation operators, which goes beyond the need to quantize any pre-existing field theory like electrodynamics, and has nothing to do with whether particles can actually be produced or destroyed. The great advantage of this formalism is that if we express the Hamiltonian as a sum of products of creation and annihilation operators, with suitable non-singular coefficients, then the S-matrix will automatically satisfy a crucial physical requirement, the cluster decomposition principle,[1] which says in effect that distant experiments yield uncorrelated results. Indeed, it is for this reason that the formalism of creation and annihilation operators is widely used in non-relativistic quantum statistical mechanics, where the number of particles is typically fixed. In relativistic quantum theories, the cluster decomposition principle plays a crucial part in making field theory inevitable. There have been many attempts to formulate a relativistically invariant theory that would not be a local field theory, and it is indeed possible to construct theories that are not field theories and yet yield a Lorentz-invariant S-matrix for two-particle scattering,[2] but such efforts have always run into trouble in sectors with more than two particles: either the three-particle S-matrix is not Lorentz-invariant, or else it violates the cluster decomposition principle.

In this chapter we will first discuss the basis of states containing ar-

bitrary numbers of bosons and fermions, then define the creation and annihilation operators, and finally show how their use facilitates the construction of Hamiltonians that yield S-matrices satisfying the cluster decomposition condition.

4.1 Bosons and Fermions

The Hilbert space of physical states is spanned by states containing 0, 1, 2, $\cdots$ free particles. These can be free-particle states, or 'in' states, or 'out' states; for definiteness we shall deal here with the free-particle states $\Phi_{\mathbf{p}_1 \sigma_1 n_1, \mathbf{p}_2 \sigma_2 n_2, \cdots}$, but all our results will apply equally to 'in' or 'out' states. As usual, σ labels spin z-components (or helicities, for massless particles) and n labels particle species.

We must now go into a matter that has been passed over in Chapter 3; the symmetry properties of these states. As far as we know, all particles are either *bosons* or *fermions*, the difference being that a state is unchanged by the interchange of two identical bosons, and changes sign under the interchange of two identical fermions. That is

$$\Phi_{\cdots \mathbf{p}\sigma n \cdots \mathbf{p}'\sigma' n \cdots} = \pm \Phi_{\cdots \mathbf{p}'\sigma' n \cdots \mathbf{p}\sigma n \cdots} \tag{4.1.1}$$

with an upper or lower sign if n is a boson or a fermion, respectively, and dots representing other particles that may be present in the state. (Equivalently, this could be stated as a condition on the 'wave functions,' the coefficients of these multi-particle basis vectors in physically allowable state-vectors.) These two cases are often referred to as Bose or Fermi 'statistics'. We will see in the next chapter that Bose and Fermi statistics are only possible for particles that have integer or half-integer spins, respectively, but we shall not need this information in the present chapter. In this section we shall offer a non-rigorous argument that all particles must be either bosons or fermions, and then set up normalization conditions for multi-boson or multi-fermion states.

First note that if two particles with spins and momenta $\mathbf{p}, \sigma$ and $\mathbf{p}', \sigma'$ belong to identical species n, then the state-vectors $\Phi_{\cdots \mathbf{p}\sigma n \cdots \mathbf{p}'\sigma' n \cdots}$ and $\Phi_{\cdots \mathbf{p}'\sigma' n \cdots \mathbf{p}\sigma n \cdots}$ represent the same physical state; if this were not the case then the particles would be distinguished by their order in the labelling of the state-vector, and the first listed would not be identical with the second. Since the two state-vectors are physically indistinguishable, they must belong to the same ray, and so

$$\Phi_{\cdots \mathbf{p}\sigma n \cdots \mathbf{p}'\sigma' n \cdots} = \alpha_n \Phi_{\cdots \mathbf{p}'\sigma' n \cdots \mathbf{p}\sigma n \cdots}, \tag{4.1.2}$$

where α_n is a complex number of unit absolute value. We may regard this as part of the definition of what we mean by identical particles.

The crux of the matter is to decide on what the phase factor α_n may depend. *If* it depends only on the species index n, then we are nearly done. Interchanging the two particles in Eq. (4.1.2) again, we find

$$\Phi_{\cdots \mathbf{p}\sigma n \cdots \mathbf{p}'\sigma' n \cdots} = \alpha_n^2 \Phi_{\cdots \mathbf{p}\sigma n \cdots \mathbf{p}'\sigma' n \cdots}$$

so that $\alpha_n^2 = 1$, yielding Eq. (4.1.1) as the only two possibilities.

On what else could α_n depend? It might depend on the numbers and species of the other particles in the state (indicated by dots in Eqs. (4.1.1) and (4.1.2)), but this would lead to the uncomfortable result that the symmetry of state-vectors under interchange of particles here on earth may depend on the presence of particles elsewhere in the universe. This is the sort of thing that is ruled out by the cluster decomposition principle, to be discussed later in this chapter. The phase α_n cannot have any non-trivial dependence on the spins of the two particles that are interchanged, because then these spin-dependent phase factors would have to furnish a representation of the rotation group, and there are no non-trivial representations of the three-dimensional rotation group that are one-dimensional – that is, by phase factors. The phase α_n might conceivably depend on the momenta of the two particles that are interchanged, but Lorentz invariance would require α_n to depend only on the scalar $p_1^\mu p_{2\mu}$; this is symmetric under interchange of particles 1 and 2, and therefore such dependence would not change the argument leading to the conclusion that $\alpha_n^2 = 1$.

The logical gap in the above argument is that (although our notation hides the fact) the states $\Phi_{\mathbf{p}_1 \sigma_1 n_1 \mathbf{p}_2 \sigma_2 n_2 \cdots}$ may carry a phase factor that depends on the path through momentum space by which the momenta of the particles are brought to the values $\mathbf{p}_1$, $\mathbf{p}_2$, etc. In this case the interchange of two particles twice might change the state by a phase factor, so that $\alpha_n^2 \neq 1$. We will see in Section 9.7 that this is a real possibility in two-dimensional space, but not for three or more spatial dimensions.

What about interchanges of particles belonging to different species? If we like, we can avoid this question by simply agreeing from the beginning to label the state-vector by listing all photon momenta and helicities first, then all electron momenta and spin z-components, and so on through the table of elementary particle types. Alternatively, we can allow the particle labels to appear in any order, and *define* the state-vectors with particle labels in an arbitrary order as equal to the state-vector with particle labels in some standard order times phase factors, whose dependence on the interchange of particles of different species can be anything we like. In order to deal with symmetries like isospin invariance that relate particles of different species, it is convenient to adopt a convention that generalizes Eq. (4.1.1): the state-vector will be taken to be symmetric

under interchange of any bosons with each other, or any bosons with any fermions, and antisymmetric with respect to interchange of any two fermions with each other, in all cases, whether the particles are of the same species or not.*

The normalization of these states must be defined in consistency with these symmetry conditions. To save writing, we will use a label q to denote all the quantum numbers of a single particle: its momentum, $\mathbf{p}$, spin z-component (or, for massless particles, helicity) σ, and species n. The N-particle states are thus labelled $\Phi q_1 \ldots q_N$ (with $N = 0$ for the vacuum state Φ_0.) For $N = 0$ and $N = 1$ the question of symmetry does not arise: here we have

$$(\Phi_0, \Phi_0) = 1 \tag{4.1.3}$$

and

$$(\Phi_{q'}, \Phi_q) = \delta(q' - q)\,, \tag{4.1.4}$$

where $\delta(q'-q)$ is a product of all the delta functions and Kronecker deltas for the particle's quantum numbers,

$$\delta(q' - q) \equiv \delta^3(\mathbf{p}' - \mathbf{p})\,\delta_{\sigma'\sigma}\delta_{n'n}. \tag{4.1.5}$$

On the other hand, for $N = 2$ the states $\Phi_{q_1'q_2'}$ and $\Phi_{q_2'q_1'}$ are physically the same, so here we must take

$$\left(\Phi_{q_1'q_2'}\,,\ \Phi_{q_1q_2}\right) = \delta(q_1' - q_1)\,\delta(q_2' - q_2) \pm \delta(q_2' - q_1)\,\delta(q_1' - q_2) \tag{4.1.6}$$

the sign $\pm$ being $-$ if both particles are fermions and $+$ otherwise. This obviously is consistent with the above stated symmetry properties of the states. More generally,

$$\left(\Phi_{q_1'q_2'\cdots q_M'}, \Phi_{q_1q_2\cdots q_N}\right) = \delta_{NM} \sum_{\mathscr{P}} \delta_{\mathscr{P}} \prod_i \delta(q_i - q'_{\mathscr{P}i}). \tag{4.1.7}$$

The sum here is over all permutations $\mathscr{P}$ of the integers $1, 2, \cdots, N$. (For instance, in the first term in Eq. (4.1.6), $\mathscr{P}$ is the identity, $\mathscr{P}1 = 1$, $\mathscr{P}2 = 2$, while in the second term $\mathscr{P}1 = 2$, $\mathscr{P}2 = 1$.) Also, $\delta_{\mathscr{P}}$ is a sign factor equal to -1 if $\mathscr{P}$ involves an odd permutation of fermions (an odd number of fermion interchanges) and $+1$ otherwise. It is easy to see that Eq. (4.1.7) has the desired symmetry or antisymmetry properties under interchange of the q_i, and also under interchange of the q_j'.

* In fact, by the same reasoning, the symmetry or antisymmetry of the state-vector under interchange of particles of the same species but different helicities or spin z-components is purely conventional, because we could have agreed from the beginning to list first the momenta of photons of helicity $+1$, then the momenta of all photons of helicity -1, then the momenta of all electrons of spin z-component $+\frac{1}{2}$, and so on. We adopt the *convention* that the state-vector is symmetric or antisymmetric under interchange of identical bosons or fermions of different helicities or spin z-components in order to facilitate the use of rotational invariance.

4.2 Creation and Annihilation Operators

Creation and annihilation operators may be defined in terms of their effect on the normalized multi-particle states discussed in the previous section. The *creation operator* $a^\dagger(q)$ (or in more detail, $a^\dagger(\mathbf{p}, \sigma, n)$) is defined as the operator that simply adds a particle with quantum numbers q at the front of the list of particles in the state

$$a^\dagger(q)\Phi_{q_1 q_2 \cdots q_n} \equiv \Phi_{q q_1 q_2 \cdots q_N} \,. \tag{4.2.1}$$

In particular, the N-particle state can be obtained by acting on the vacuum with N creation operators

$$a^\dagger(q_1)a^\dagger(q_2)\cdots a^\dagger(q_N)\Phi_0 = \Phi_{q_1 \cdots q_N} \,. \tag{4.2.2}$$

It is conventional for this operator to be called $a^\dagger(q)$; *its* adjoint, which is then called $a(q)$, may be calculated from Eq. (4.1.7). As we shall now show, $a(q)$ removes a particle from any state on which it acts, and is therefore known as an *annihilation operator*. In particular, when the particles $q\, q_1 \cdots q_N$ are either all bosons or all fermions, we have

$$a(q)\Phi_{q_1 q_2 \cdots q_N} = \sum_{r=1}^{N} (\pm)^{r+1} \delta(q - q_r)\Phi_{q_1 \cdots q_{r-1} q_{r+1} \cdots q_N} \,, \tag{4.2.3}$$

with a $+1$ or -1 sign for bosons or fermions, respectively . (Here is the proof. We want to calculate the scalar product of $a(q)\Phi_{q_1 q_2 \cdots q_N}$ with an arbitrary state $\Phi_{q_1' \cdots q_M'}$. Using Eq. (4.2.1), this is

$$\left(\Phi_{q_1' \cdots q_M'},\ a(q)\Phi_{q_1 \cdots q_N}\right) \equiv \left(a^\dagger(q)\Phi_{q_1' \cdots q_M'},\ \Phi_{q_1 \cdots q_N}\right) = \left(\Phi_{q q_1' \cdots q_M'},\ \Phi_{q_1 \cdots q_N}\right) .$$

We now use Eq. (4.1.7). The sum over permutations $\mathscr{P}$ of $1, 2, \cdots, N$ can be written as a sum over the integer r that is permuted into the first place, i.e. $\mathscr{P}r = 1$, and over mappings $\bar{\mathscr{P}}$ of the remaining integers $1, \cdots, r-1, r+1, \cdots, N$ into $1, \cdots, N-1$. Furthermore, the sign factor is

$$\delta_{\mathscr{P}} = (\pm)^{r-1}\delta_{\bar{\mathscr{P}}}$$

with upper and lower signs for bosons and fermions, respectively. Hence, using Eq. (4.1.7) twice,

$$\begin{aligned}\left(\Phi_{q_1' \cdots q_M'},\ a(q)\Phi_{q_1 \cdots q_N}\right) &= \delta_{N,M+1} \\ &\quad \times \sum_{r=1}^{N} \sum_{\bar{\mathscr{P}}} (\pm)^{r-1} \delta_{\bar{\mathscr{P}}} \delta(q - q_r) \prod_{i=1}^{M} \delta(q_i' - q_{\mathscr{P}i}) \\ &= \delta_{N,M+1} \sum_{r=1}^{N} (\pm)^{r-1} \left(\Phi_{q_1' \cdots q_M'},\ \Phi_{q_1 \cdots q_{r-1} q_{r+1} \cdots q_N}\right) .\end{aligned}$$

Both sides of Eq. (4.2.3) thus have the same matrix element with any state $\Phi_{q_1'\cdots q_M'}$, and are therefore equal, as was to be shown.) As a special case of Eq. (4.2.3), we note that for both bosons and fermions, $a(q)$ annihilates the vacuum

$$a(q)\Phi_0 = 0\,. \tag{4.2.4}$$

As defined here, the creation and annihilation operators satisfy an important commutation or anticommutation relation. Applying the operator $a(q')$ to Eq. (4.2.1) and using Eq. (4.2.3) gives

$$a(q')a^\dagger(q)\Phi_{q_1\cdots q_N} = \delta(q'-q)\Phi_{q_1\cdots q_N} + \sum_{r=1}^{N}(\pm)^{r+2}\delta(q'-q_r)\Phi_{qq_1\cdots q_{r-1}q_{r+1}\cdots q_N}\,.$$

(The sign in the second term is $(\pm)^{r+2}$ because q_r is in the $(r+1)$-th place in $\Phi_{qq_1\cdots q_N}$.) On the other hand, applying the operator $a^\dagger(q)$ to Eq. (4.2.3) gives

$$a^\dagger(q)a(q')\Phi_{q_1\cdots q_N} = \sum_{r=1}^{N}(\pm)^{r+1}\delta(q'-q_r)\Phi_{qq_1\cdots q_{r-1}q_{r+1}\cdots q_N}\,.$$

Subtracting or adding, we have then

$$\left[a(q')a^\dagger(q) \mp a^\dagger(q)a(q')\right]\ \Phi_{q_1\cdots q_N} = \delta(q'-q)\Phi_{q_1\cdots q_N}\,.$$

But this holds for all states $\Phi_{q_1\cdots q_N}$ (and may easily be seen to hold also for states containing both bosons and fermions) and therefore implies the operator relation

$$a(q')a^\dagger(q) \mp a^\dagger(q)a(q') = \delta(q'-q)\,. \tag{4.2.5}$$

In addition, Eq. (4.2.2) gives immediately

$$a^\dagger(q')a^\dagger(q) \mp a^\dagger(q)a^\dagger(q') = 0 \tag{4.2.6}$$

and so also

$$a(q')a(q) \mp a(q)a(q') = 0\,. \tag{4.2.7}$$

As always, the top and bottom signs apply for bosons and fermions, respectively. According to the conventions discussed in the previous section, the creation and/or annihilation operators for particles of two different species commute if either particle is a boson, and anticommute if both are fermions.

The above discussion could have been presented in reverse order (and in most textbooks usually is). That is, we could have started with the commutation or anticommutation relations Eqs. (4.2.5)–(4.2.7), derived from the canonical quantization of some given field theory. Multi-particle

states would have then been defined by Eq. (4.2.2), and their scalar products Eq. (4.1.7) derived from the commutation or anticommutation relations. In fact, as discussed in Chapter 1, such a treatment would be much closer to the way that this formalism developed historically. We have followed an unhistorical approach here because we want to free ourselves from any dependence on pre-existing field theories, and rather wish to understand why field theories are the way they are.

We will now prove the fundamental theorem quoted at the beginning of this chapter: *any* operator $\mathcal{O}$ may be expressed as a sum of products of creation and annihilation operators

$$\begin{aligned}\mathcal{O} = \sum_{N=0}^{\infty}\sum_{M=0}^{\infty}\int dq'_1\cdots dq'_N dq_1\cdots dq_M \\ \times a^\dagger(q'_1)\cdots a^\dagger(q'_N)a(q_M)\cdots a(q_1) \\ \times C_{NM}(q'_1\cdots q'_N q_1\cdots q_M)\,. \end{aligned} \tag{4.2.8}$$

That is, we want to show that the C_{NM} coefficients can be chosen to give the matrix elements of this expression any desired values. We do this by mathematical induction. First, it is trivial that by choosing C_{00} properly, we can give $(\Phi_0, \mathcal{O}\Phi_0)$ any desired value, irrespective of the values of C_{NM} with $N > 0$ and/or $M > 0$. We need only use Eq. (4.2.4) to see that Eq. (4.2.8) has the vacuum expectation value

$$(\Phi_0, \mathcal{O}\Phi_0) = C_{00}\,.$$

Now suppose that the same is true for all matrix elements of $\mathcal{O}$ between N- and M-particle states, with $N < L, M \leq K$ or $N \leq L, M < K$; that is, that these matrix elements have been given some desired values by an appropriate choice of the corresponding coefficients C_{NM}. To see that the same is then also true of matrix elements of $\mathcal{O}$ between any L- and K-particle states, use Eq. (4.2.8) to evaluate

$$\begin{aligned}\left(\Phi_{q'_1\cdots q'_L}, \mathcal{O}\Phi_{q_1\cdots q_K}\right) = L!K!\,C_{LK}(q'_1\cdots q'_L q_1\cdots q_K) \\ +\ \text{terms involving } C_{NM} \text{ with } N < L, M \leq K \text{ or } N \leq L, M < K\,.\end{aligned}$$

Whatever values have already been given to C_{NM} with $N < L, M \leq K$ or $N \leq L, M < K$, there is clearly some choice of C_{LK} which gives this matrix element any desired value.

Of course, an operator need not be expressed in the form (4.2.8), with all creation operators to the left of all annihilation operators. (This is often called the 'normal' order of the operators.) However, if the formula for some operator has the creation and annihilation operators in some other order, we can always bring the creation operators to the left of the annihilation operators by repeated use of the commutation or

anticommutation relations, picking up new terms from the delta function in Eq. (4.2.5).

For instance, consider any sort of additive operator F (like momentum, charge, etc.) for which

$$F\Phi_{q_1\cdots q_N} = (f(q_1) + \cdots + f(q_N))\,\Phi_{q_1\cdots q_N} \,. \tag{4.2.9}$$

Such an operator can be written as in Eq. (4.2.8), but using only the term with $N = M = 1$:

$$F = \int dq\, a^\dagger(q) a(q) f(q) \,. \tag{4.2.10}$$

In particular, the free-particle Hamiltonian is always

$$H_0 = \int dq\, a^\dagger(q) a(q) E(q) \tag{4.2.11}$$

where $E(q)$ is the single-particle energy

$$E(\mathbf{p}, \sigma, n) = \sqrt{\mathbf{p}^2 + m_n^2} \,.$$

We will need the transformation properties of the creation and annihilation operators for various symmetries. First, let's consider inhomogeneous proper orthochronous Lorentz transformations. Recall that the N-particle states have the Lorentz transformation property

$$\begin{aligned} U_0(\Lambda,\alpha)\Phi_{\mathbf{p}_1\sigma_1 n_1,\mathbf{p}_2\sigma_2 n_2,\cdots} &= e^{-i(\Lambda p_1)\cdot\alpha}\, e^{-i(\Lambda p_2)\cdot\alpha} \\ &\cdots \sqrt{\frac{(\Lambda p_1)^0 (\Lambda p_2)^0 \cdots}{p_1^0 p_2^0 \cdots}} \\ &\quad \times \sum_{\bar\sigma_1\bar\sigma_2\cdots} D^{(j_1)}_{\bar\sigma_1\sigma_1}\Big(W(\Lambda,p_1)\Big) D^{(j_2)}_{\bar\sigma_2\sigma_2}\Big(W(\Lambda,p_2)\Big) \\ &\cdots \Phi_{\mathbf{p}_{1\Lambda}\bar\sigma_1 n_1,\mathbf{p}_{2\Lambda}\bar\sigma_2 n_2,\cdots} \,. \end{aligned}$$

Here $\mathbf{p}_\Lambda$ is the three-vector part of Λp, $D^{(j)}_{\bar\sigma\sigma}(R)$ is the same unitary spin-j representation of the three-dimensional rotation group as used in Section 2.5, and $W(\Lambda, p)$ is the particular rotation

$$W(\Lambda, p) \equiv L^{-1}(\Lambda p)\Lambda L(p),$$

where $L(p)$ is the standard 'boost' that takes a particle of mass m from rest to four-momentum p^μ. (Of course, m and j depend on the species label n. This is all for $m \neq 0$; we will return to the massless particle case in the following chapter.) Now, these states can be expressed as in Eq. (4.2.2)

$$\Phi_{\mathbf{p}_1\sigma_1 n_1,\mathbf{p}_2\sigma_2 n_2,\cdots} = a^\dagger(\mathbf{p}_1\sigma_1 n_1) a^\dagger(\mathbf{p}_2\sigma_2 n_2)\cdots\Phi_0 \,,$$

where Φ_0 is the Lorentz-invariant vacuum state

$$U_0(\Lambda,\alpha)\Phi_0 = \Phi_0 \,.$$

In order that the state (4.2.2) should transform properly, it is necessary and sufficient that the creation operator have the transformation rule

$$U_0(\Lambda,\alpha)a^\dagger(\mathbf{p}\sigma n)U_0^{-1}(\Lambda,\alpha) = e^{-i(\Lambda p)\cdot\alpha}\sqrt{(\Lambda p)^0/p^0} \times \sum_{\bar{\sigma}} D^{(j)}_{\bar{\sigma}\sigma}\Big(W(\Lambda,p)\Big)a^\dagger(\mathbf{p}_\Lambda\,\sigma\,n)\,. \quad (4.2.12)$$

In the same way, the operators C, P, and T, that induce charge-conjugation, space inversion, and time-reversal transformations on free particle states* transform the creation operators as:

$$\mathsf{C}a^\dagger(\mathbf{p}\,\sigma\,n)\mathsf{C}^{-1} = \xi_n\,a^\dagger(\mathbf{p}\,\sigma\,n^c)\,, \quad (4.2.13)$$

$$\mathsf{P}a^\dagger(\mathbf{p}\,\sigma\,n)\mathsf{P}^{-1} = \eta_n\,a^\dagger(-\mathbf{p}\,\sigma\,n)\,, \quad (4.2.14)$$

$$\mathsf{T}a^\dagger(\mathbf{p}\,\sigma\,n)\mathsf{T}^{-1} = \zeta_n(-1)^{j-\sigma}\,a^\dagger(-\mathbf{p}\,-\sigma\,n)\,, \quad (4.2.15)$$

As mentioned in the previous section, although we have been dealing with operators that create and annihilate particles in free-particle states, the whole formalism can be applied to 'in' and 'out' states, in which case we would introduce operators a_{in} and a_{out} defined in the same way by their action on these states. These operators satisfy a Lorentz transformation rule just like Eq. (4.2.12), but with the true Lorentz transformation operator $U(\Lambda,\alpha)$ instead of the free-particle operator $U_0(\Lambda,\alpha)$.

4.3 Cluster Decomposition and Connected Amplitudes

It is one of the fundamental principles of physics (indeed, of all science) that experiments that are sufficiently separated in space have unrelated results. The probabilities for various collisions measured at Fermilab should not depend on what sort of experiments are being done at CERN at the same time. If this principle were not valid, then we could never make any predictions about any experiment without knowing everything about the universe.

In S-matrix theory, the cluster decomposition principle states that if multi-particle processes $\alpha_1 \to \beta_1, \alpha_2 \to \beta_2, \cdots, \alpha_{\mathcal{N}} \to \beta_{\mathcal{N}}$ are studied in $\mathcal{N}$ very distant laboratories, then the S-matrix element for the overall process

* We omit the subscript '0' on these operators, because in virtually all cases where C, P, and/or T are conserved, the operators that induce these transformations on 'in' and 'out' states are the same as those defined by their action on free-particle states. This is not the case for continuous Lorentz transformations, for which it is necessary to distinguish between the operators $U(\Lambda,a)$ and $U_0(\Lambda,a)$.

factorizes. That is,*

$$S_{\beta_1+\beta_2+\cdots+\beta_{N'},\,\alpha_1+\alpha_2+\cdots+\alpha_N} \;\to\; S_{\beta_1\alpha_1}\,S_{\beta_2\alpha_2}\cdots S_{\beta_{N'}\alpha_N} \tag{4.3.1}$$

if for all $i \neq j$, *all* of the particles in states α_i and β_i are at a great spatial distance from *all* of the particles in states α_j and β_j. This factorization of S-matrix elements will ensure a factorization of the corresponding transition probabilities, corresponding to uncorrelated experimental results.

There is a combinatoric trick that allows us to rewrite Eq. (4.3.1) in a more transparent way. Suppose we define the *connected* part of the S-matrix, $S^{\rm C}_{\beta\alpha}$, by the formula**

$$S_{\beta\alpha} = \sum_{\rm PART} (\pm)\, S^{\rm C}_{\beta_1\alpha_1} S^{\rm C}_{\beta_2\alpha_2}\cdots . \tag{4.3.2}$$

Here the sum is over all different ways of partitioning the particles in the state α into clusters $\alpha_1, \alpha_2, \cdots$, and likewise a sum over all ways of partitioning the particles in the state β into clusters $\beta_1, \beta_2, \cdots$, not counting as different those that merely arrange particles within a given cluster or permute whole clusters. The sign is $+$ or $-$ according to whether the rearrangements $\alpha \to \alpha_1\alpha_2\cdots$ and $\beta \to \beta_1\beta_2\cdots$ involve altogether an even or an odd number of fermion interchanges, respectively. The term 'connected' is used because of the interpretation of $S^{\rm C}_{\beta\alpha}$ in terms of diagrams representing different contributions in perturbation theory, to be discussed in the next section.

This is a recursive definition. For each α and β, the sum on the right-hand side of Eq. (4.3.2) consists of a term $S^{\rm C}_{\beta\alpha}$, plus a sum Σ' over products of two or more $S^{\rm C}$-matrix elements, with a total number of particles in each of the states α_j and β_j that is *less* than the number of particles in

* We are here returning to the notation used in Chapter 3; Greek letters α or β stand for a collection of particles, including for each particle a specification of its momentum, spin, and species. Also, $\alpha_1 + \alpha_2 + \cdots + \alpha_{N'}$ is the state formed by combining all the particles in the states $\alpha_1, \alpha_2, \cdots$, and $\alpha_{N'}$, and likewise for $\beta_1 + \beta_2 + \cdots + \beta_{N'}$.

** This decomposition has been used in classical statistical mechanics by Ursell, Mayer, and others, and in quantum statistical mechanics by Lee and Yang and others.[3] It has also been used to calculate many-body ground state energies by Goldstone[4] and Hugenholtz.[5] In all of these applications the purpose of isolating the connected parts of Green's functions, partition functions, resolvents, etc., is to deal with objects with a simple volume dependence. This is essentially our purpose too, because as we shall see, the crucial property of the connected parts of the S-matrix is that they are proportional to a single momentum-conservation delta function, and in a box the delta function becomes a Kronecker delta times the volume. The cluster decomposition is also the same formal device as that used in the theory of noise[6] to decompose the correlation function of several random variables into its 'cumulants'; if the random variable receives contributions from a large number N of independent fluctuations, then each cumulant is proportional to N.

the states α and β

$$S_{\beta\alpha} = S^{\rm C}_{\beta\alpha} + \sum_{\rm PART}{}' (\pm)\, S^{\rm C}_{\beta_1\alpha_1} S^{\rm C}_{\beta_2\alpha_2} \cdots .$$

Suppose that the $S^{\rm C}$-matrix elements in this sum have already been chosen in such a way that Eq. (4.3.2) is satisfied for states β, α containing together fewer than, say, N particles. Then no matter what values are found in this way for the S-matrix elements appearing in the sum Σ', we can always choose the remaining term $S^{\rm C}_{\beta\alpha}$ so that Eq. (4.3.2) is also satisfied for states α, β containing a total of N particles.† Thus Eq. (4.3.2) contains no information in itself; it is merely a definition of $S^{\rm C}$.

If the states α and β each consist of just a single particle, say with quantum numbers q and q' respectively, then the only term on the right-hand-side of Eq. (4.3.2) is just $S^{\rm C}_{\beta\alpha}$ itself, so for one-particle states

$$S^{\rm C}_{q'q} \equiv S_{q'q} = \delta(q' - q)\,. \tag{4.3.3}$$

(Apart from possible degeneracies, the fact that $S_{q'q}$ is proportional to $\delta(q' - q)$ follows from conservation laws. The absence of any proportionality factor in Eq. (4.3.3) is based on a suitable choice of the relative phase of 'in' and 'out' states.) We are here assuming that single-particle states are stable, so that there are no transitions between single-particle states and any others, such as the vacuum.

For transitions between two-particle states, Eq. (4.3.2) reads

$$S_{q_1'q_2',q_1q_2} = S^{\rm C}_{q_1'q_2',q_1q_2} + \delta(q_1' - q_1)\delta(q_2' - q_2) \pm \delta(q_1' - q_2)\delta(q_2' - q_1)\,. \tag{4.3.4}$$

(We are here using Eq. (4.3.3).) The sign $\pm$ is $-$ if both particles are fermions, and otherwise $+$. We recognize that the two delta function terms just add up to the norm (4.1.6), so here $S^{\rm C}_{\beta\alpha}$ is just $(S-1)_{\beta\alpha}$. But the general case is more complicated.

For transitions between three-particle or four-particle states, Eq. (4.3.2) reads

$$\begin{aligned} S_{q_1'q_2'q_3',q_1q_2q_3} &= S^{\rm C}_{q_1'q_2'q_3',q_1q_2q_3} \\ &+ \delta(q_1' - q_1) S^{\rm C}_{q_2'q_3',q_2q_3} \pm \text{permutations} \\ &+ \delta(q_1' - q_1)\delta(q_2' - q_2)\delta(q_3' - q_3) \pm \text{permutations} \end{aligned} \tag{4.3.5}$$

† A technicality should be mentioned here. This argument works only if we neglect the possibility that for one or more of the connected S-matrix elements in Eq. (4.3.2), the states α_j and β_j both contain no particles at all. We must therefore define the connected vacuum–vacuum element $S^{\rm C}_{0,0}$ to be zero. We do not use Eq. (4.3.2) for the vacuum vacuum S-matrix $S_{0,0}$, which in the absence of time-varying external fields is simply defined to be unity, $S_{0,0} = 1$. We will have more to say about the vacuum vacuum amplitude in the presence of external fields in Volume II.

and

$$
\begin{aligned}
S_{q_1'q_2'q_3'q_4',q_1q_2q_3q_4} &= S^{\mathrm{C}}_{q_1'q_2'q_3'q_4',q_1q_2q_3q_4} \\
&+ S^{\mathrm{C}}_{q_1'q_2',q_1q_2} S^{\mathrm{C}}_{q_3'q_4',q_3q_4} \pm \text{permutations} \\
&+ \delta(q_1'-q_1) S^{\mathrm{C}}_{q_2'q_3'q_4',q_2q_3q_4} \pm \text{permutations} \\
&+ \delta(q_1'-q_1)\delta(q_2'-q_2) S^{\mathrm{C}}_{q_3'q_4',q_3q_4} \pm \text{permutations} \\
&+ \delta(q_1'-q_1)\delta(q_2'-q_2)\delta(q_3'-q_3)\delta(q_4'-q_4) \pm \text{permutations} . \quad (4.3.6)
\end{aligned}
$$

(Taking account of all permutations, there are a total of $1+9+6=16$ terms in Eq. (4.3.5) and $1+18+16+72+24=131$ terms in Eq. (4.3.6). If we had not assumed that one-particle states are stable, there would be even more terms.) As explained previously, the definition of $S^{\mathrm{C}}_{\beta\alpha}$ is recursive: we use Eq. (4.3.4) to define $S^{\mathrm{C}}_{\beta\alpha}$ for two-particle states, then use this definition in Eq. (4.3.5) when we define $S^{\mathrm{C}}_{\beta\alpha}$ for three-particle states, then use both of these definitions in Eq. (4.3.6) to obtain the definition of $S^{\mathrm{C}}_{\beta\alpha}$ for four-particle states, and so on.

The point of this definition of the connected part of the S-matrix is that the cluster decomposition principle is equivalent to the requirement that $S^{\mathrm{C}}_{\beta\alpha}$ must vanish when any one or more of the particles in the states β and/or α are far away in space from the others.‡ To see this, suppose that the particles in the states β and α are grouped into clusters $\beta_1, \beta_2, \cdots$ and $\alpha_1, \alpha_2, \cdots$, and that all particles in the set $\alpha_i + \beta_i$ are far from all particles in the set $\alpha_j + \beta_j$ for any $j \neq i$. Then if $S^{\mathrm{C}}_{\beta'\alpha'}$ vanishes if any particles in β' or α' are far from the others, it vanishes if any particles in these states are in different clusters, so the definition (4.3.2) yields

$$
S_{\beta\alpha} \to \sum{}^{(1)}(\pm) S^{\mathrm{C}}_{\beta_{11}\alpha_{11}} S^{\mathrm{C}}_{\beta_{12}\alpha_{12}} \cdots \times \sum{}^{(2)}(\pm) S^{\mathrm{C}}_{\beta_{21}\alpha_{21}} S^{\mathrm{C}}_{\beta_{22}\alpha_{22}} \cdots \times \cdots , \quad (4.3.7)
$$

where $\Sigma^{(j)}$ is a sum over all different ways of partitioning the clusters β_j and α_j into subclusters $\beta_{j1}, \beta_{j2}, \cdots$ and $\alpha_{j1}, \alpha_{j2}, \cdots$. But referring back to Eq. (4.3.2), this is just the desired factorization property (4.3.1).

For instance, suppose that in the four-particle reaction $1234 \to 1'2'3'4'$, we let particles $1, 2, 1'$, and $2'$ be very far from $3, 4, 3'$, and $4'$. Then if $S^{\mathrm{C}}_{\beta\alpha}$ vanishes when any particles in β and/or α are far from the others, the only terms in Eq. (4.3.6) that survive (in an even more abbreviated notation) are

‡ In order to give a meaning to 'far', we will have to Fourier transform S^{C}, so that each three-momentum label $\mathbf{p}$ is replaced with a spatial coordinate three-vector $\mathbf{x}$.

$$\begin{aligned} S_{1'2'3'4',1234} \to\ & S^{\rm C}_{1'2',12} S^{\rm C}_{3'4',34} \\ & + (\delta_{1'1}\delta_{2'2} \pm \delta_{1'2}\delta_{2'1}) S^{\rm C}_{3'4',34} \\ & + (\delta_{3'3}\delta_{4'4} \pm \delta_{3'4}\delta_{4'3}) S^{\rm C}_{1'2',12} \\ & + (\delta_{1'1}\delta_{2'2} \pm \delta_{1'2}\delta_{2'1})(\delta_{3'3}\delta_{4'4} \pm \delta_{3'4}\delta_{4'3}) \,. \end{aligned}$$

Comparison with Eq. (4.3.4) shows that this is just the required factorization condition (4.3.1)

$$S_{1'2'3'4',1234} \to S_{1'2',12} S_{3'4',34} \,.$$

We have formulated the cluster decomposition principle in coordinate space, as the condition that $S^{\rm C}_{\beta\alpha}$ vanishes if any particles in the states β or α are far from any others. It is convenient for us to reexpress this in momentum space. The coordinate space matrix elements are defined as a Fourier transform

$$\begin{aligned} S^{\rm C}_{\mathbf{x}'_1\mathbf{x}'_2\cdots,\mathbf{x}_1\mathbf{x}_2\cdots} \equiv & \int d^3\mathbf{p}'_1 d^3\mathbf{p}'_2 \cdots d^3\mathbf{p}_1 d^3\mathbf{p}_2 \cdots \; S^{\rm C}_{\mathbf{p}'_1\mathbf{p}'_2\cdots\mathbf{p}_1\mathbf{p}_2\cdots} \\ & \times e^{i\mathbf{p}'_1\cdot\mathbf{x}'_1} \, e^{i\mathbf{p}'_2\cdot\mathbf{x}'_2} \cdots e^{-i\mathbf{p}_1\cdot\mathbf{x}_1} \, e^{-i\mathbf{p}_2\cdot\mathbf{x}_2\cdots} \,. \end{aligned} \tag{4.3.8}$$

(We are here temporarily dropping spin and species labels, which just go along with the momentum or coordinate labels.) If $|S^{\rm C}_{\mathbf{p}'_1\mathbf{p}'_2\cdots\mathbf{p}_1\mathbf{p}_2\cdots}|$ were sufficiently well behaved (to be specific, if it were Lebesgue integrable) then according to the Riemann–Lebesgue theorem[7] the integral (4.3.8) would vanish when any combination of spatial coordinates goes to infinity. Now, this is certainly too strong a requirement. Translational invariance tells us that the connected part of the S-matrix, like the S-matrix itself, can only depend on differences of coordinate vectors, and therefore does not change at all if all of the x_i and x'_j vary together, with their differences held constant. This requires that the elements of $S^{\rm C}$ in a momentum basis must, like those of S, be proportional to a three-dimensional delta function that ensures momentum conservation (and makes $|S^{\rm C}_{\mathbf{p}'_1\mathbf{p}'_2\cdots\mathbf{p}_1\mathbf{p}_2\cdots}|$ *not* Lebesgue integrable), as well as the energy-conservation delta function required by scattering theory. That is, we can write

$$\begin{aligned} S^{\rm C}_{\mathbf{p}'_1\mathbf{p}'_2\cdots,\mathbf{p}_1\mathbf{p}_2\cdots} = & \, \delta^3(\mathbf{p}'_1 + \mathbf{p}'_2 + \cdots - \mathbf{p}_1 - \mathbf{p}_2 - \cdots) \\ & \times \delta(E'_1 + E'_2 + \cdots - E_1 - E_2 - \cdots) C_{\mathbf{p}'_1\mathbf{p}'_2\cdots,\mathbf{p}_1\mathbf{p}_2\cdots} \,. \end{aligned} \tag{4.3.9}$$

This is no problem: the cluster decomposition principle only requires that Eq. (4.3.8) vanish when the *differences* among some of the $\mathbf{x}_i$ and/or $\mathbf{x}'_i$ become large. However, if C itself in Eq. (4.3.9) contained additional delta functions of linear combinations of the three-momenta, then this principle would not be satisfied. For instance, suppose that there were a delta function in C that required that the sum of the $\mathbf{p}'_i$ and $-\mathbf{p}_j$ for some subset of the particles vanished. Then Eq. (4.3.8) would not vary if all of the $\mathbf{x}'_i$ and

$\mathbf{x}_j$ for the particles in that subset moved together (with constant differences) away from all the other $\mathbf{x}'_k$ and $\mathbf{x}_\ell$, in contradiction to the cluster decomposition principle. Loosely speaking then, the cluster decomposition principle simply says that *the connected part of the S-matrix, unlike the S-matrix itself, contains just a single momentum-conservation delta function.*

In order to put this a bit more precisely, we can say that the coefficient function $C_{\mathbf{p}'_1\mathbf{p}'_2\cdots,\mathbf{p}_1\mathbf{p}_2\cdots}$ in Eq. (4.3.9) is a smooth function of its momentum labels. But how smooth? It would be most straightforward if we could simply require that $C_{\mathbf{p}'_1\mathbf{p}'_2\cdots,\mathbf{p}_1\mathbf{p}_2\cdots}$ be analytic in all of the momenta at $\mathbf{p}'_1 = \mathbf{p}'_2 = \cdots = \mathbf{p}_1 = \mathbf{p}_2 = \cdots = 0$. This requirement would indeed guarantee that $S^{\rm C}_{\mathbf{x}'_1\mathbf{x}'_2\cdots,\mathbf{x}_1\mathbf{x}_2\cdots}$ vanishes exponentially fast when any of the $\mathbf{x}$ and $\mathbf{x}'$ is very distant from any of the other $\mathbf{x}$ and $\mathbf{x}'$. However, an exponential fall-off of $S^{\rm C}$ is not an essential part of the cluster decomposition principle, and, in fact, the requirement of analyticity is not met in all theories. Most notably, in theories with massless particles, $S^{\rm C}$ can have poles at certain values of the $\mathbf{p}$ and $\mathbf{p}'$. For instance, as we will see in Chapter 10, if a massless particle can be emitted in the transition $1 \to 3$ and absorbed in the transition $2 \to 4$, then $S^{\rm C}_{34,12}$ will have a term proportional to $1/(p_1 - p_3)^2$. After Fourier transforming, such poles yield terms in $S^{\rm C}_{\mathbf{x}'_1\mathbf{x}'_2\cdots,\mathbf{x}_1\mathbf{x}_2\cdots}$ that fall off only as negative powers of coordinate differences.[1] There is no need to formulate the cluster decomposition principle so stringently that such behavior is ruled out. Thus the 'smoothness' condition on $S^{\rm C}$ should be understood to allow various poles and branch-cuts at certain values of the $\mathbf{p}$ and $\mathbf{p}'$, but not singularities as severe as delta functions.

4.4 Structure of the Interaction

We now ask, what sort of Hamiltonian will yield an S-matrix that satisfies the cluster decomposition principle? It is here that the formalism of creation and annihilation operators comes into its own. The answer is contained in the theorem that the S-matrix satisfies the cluster decomposition principle if (and as far as I know, only if) the Hamiltonian can be expressed as in Eq. (4.2.8):

$$\begin{aligned} H = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \int & dq'_1 \cdots dq'_N \, dq_1 \cdots dq_M \\ & \times a^\dagger(q'_1) \cdots a^\dagger(q'_N) a(q_M) \cdots a(q_1) \\ & \times h_{NM}(q'_1 \cdots q'_N \,,\, q_1 \cdots q_M) \end{aligned} \tag{4.4.1}$$

with coefficient functions h_{NM} that contain just a *single* three-dimensional momentum-conservation delta function (returning here briefly to a more

explicit notation)

$$h_{NM}(\mathbf{p}'_1\sigma'_1n'_1\cdots\mathbf{p}'_N\sigma'_Nn'_N\,,\ \mathbf{p}_1\sigma_1n_1\cdots\mathbf{p}_M\sigma_Mn_M)$$
$$=\delta^3(\mathbf{p}'_1+\cdots+\mathbf{p}'_N-\mathbf{p}_1-\cdots-\mathbf{p}_N)$$
$$\times\tilde{h}_{NM}(\mathbf{p}'_1\sigma'_1n'_1\cdots\mathbf{p}'_N\sigma'_Nn'_N\,,\ \mathbf{p}_1\sigma_1n_1\cdots\mathbf{p}_M\sigma_Mn_M)\,, \qquad (4.4.2)$$

where $\tilde{h}_{NM}$ contains no delta function factors. Note that Eq. (4.4.1) by itself has no content — we saw in Section 4.2 that *any* operator can be put in this form. It is only Eq. (4.4.1) combined with the requirement that h_{NM} has only the single delta function shown in Eq. (4.4.2) that guarantees that the S-matrix satisfies the cluster decomposition principle.

The validity of this theorem in perturbation theory will become obvious when we develop the Feynman diagram formalism in Chapter 6. The trusting reader may prefer to skip the rest of the present chapter, and move on to consider the implications of this theorem in Chapter 5. However, the proof has some instructive features, and will help to clarify in what sense the field theory of the next chapter is inevitable.

To prove this theorem, we make use of perturbation theory in its time-dependent form. (One of the advantages of time-dependent perturbation theory is that it makes the combinatorics underlying the cluster decomposition principle much more transparent; if E is a sum of one-particle energies then e^{-iEt} is a product of functions of the individual energies, while $[E-E_\alpha+i\epsilon]^{-1}$ is not.) The S-matrix is given by Eq. (3.5.10) as*

$$S_{\beta\alpha}=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}dt_1\cdots dt_n\Big(\Phi_\beta,\,T\Big\{V(t_1)\cdots V(t_n)\Big\}\Phi_\alpha\Big)\,, \qquad (4.4.3)$$

where the Hamiltonian is split into a free-particle part H_0 and an interaction V, and

$$V(t)\equiv\exp(iH_0t)V\exp(-iH_0t)\,. \qquad (4.4.4)$$

Now, the states Φ_α and Φ_β may be expressed as in Eq. (4.2.2) as products of creation operators acting on the vacuum Φ_0, and $V(t)$ is itself a sum of products of creation and annihilation operators, so each term in the sum (4.4.3) may be written as a sum of vacuum expectation values of products of creation and annihilation operators. By using the commutation or anticommutation relations (4.2.5) we may move each annihilation operator in turn to the right past all the creation operators. For each annihilation operator moved to the right past a creation operator we have two terms, as shown by writing Eq. (4.2.5) in the form

$$a(q')a^\dagger(q)=\pm a^\dagger(q)a(q')+\delta(q'-q)\,.$$

* We are now adopting the convention that for $n=0$, the time-ordered product in Eq. (4.4.3) is taken as the unit operator, so the $n=0$ term in the sum just yields the term $\delta(\beta-\alpha)$ in $S_{\beta\alpha}$.

Moving other creation operators past the annihilation operator in the first term generates yet more terms. But Eq. (4.2.4) shows that any annihilation operator that moves all the way to the right and acts on Φ_0 gives zero, so in the end all we have left is the delta functions. In this way, the vacuum expectation value of a product of creation and annihilation operators is given by a sum of different terms, each term equal to a product of delta functions and $\pm$ signs from the commutators or anticommutators. It follows that each term in Eq. (4.4.3) may be expressed as a sum of terms, each term equal to a product of delta functions and $\pm$ signs from the commutators or anticommutators and whatever factors are contributed by $V(t)$, integrated over all the times and integrated and summed over the momenta, spins, and species in the arguments of the delta functions.

Each of the terms generated in this way may be symbolized by a diagram. (This is not yet the full Feynman diagram formalism, because we are not yet going to associate numerical quantities with the ingredients in the diagrams; we are using the diagrams here only as a way of keeping track of three-momentum delta functions.) Draw n points, called *vertices*, one for each $V(t)$ operator. For each delta function produced when an annihilation operator in one of these $V(t)$ operators moves past a creation operator in the initial state Φ_α, draw a line coming into the diagram from below that ends at the corresponding vertex. For each delta function produced when an annihilation operator in the adjoint of the final state Φ_β moves past a creation operator in one of the $V(t)$, draw a line from the corresponding vertex upwards out of the diagram. For each delta function produced when an annihilation operator in one $V(t)$ moves past a creation operator in another $V(t)$ draw a line between the two corresponding vertices. Finally, for each delta function produced when an annihilation operator in the adjoint of the final state moves past a creation operator in the initial state, draw a line from bottom to top, right through the diagrams. Each of the delta functions associated with one of these lines enforces the equality of the momentum arguments of the pair of creation and annihilation operators represented by the line. There is also at least one delta function contributed by each of the vertices, which enforces the conservation of the total three-momentum at the vertex.

Such a diagram may be connected (every point connected to every other by a set of lines) and if not connected, it breaks up into a number of connected pieces. The $V(t)$ operator associated with a vertex in one connected component effectively commutes with the $V(t)$ associated with any vertex in any other connected component, because for this diagram, we are not including any terms in which an annihilation operator in one vertex destroys a particle that is produced by a creation operator in the other vertex — if we did, the two vertices would be in the same connected

component. Thus the matrix element in Eq. (4.4.3) can be expressed as a sum over *products* of contributions, one from each connected component:

$$\left(\Phi_\beta, T\left\{V(t_1)\cdots V(t_n)\right\}\Phi_\alpha\right)$$
$$= \sum_{\text{clusterings}} (\pm) \prod_{j=1}^{v} \left(\Phi_{\beta_j}, T\left\{V(t_{j1})\cdots V(t_{jn_j})\right\}\Phi_{\alpha_j}\right)_C . \qquad (4.4.5)$$

Here the sum is over all ways of splitting up the incoming and outgoing particles and $V(t)$ operators into v clusters (including a sum over v from 1 to n) with the n_j operators $V(t_{j_1})\cdots V(t_{jn_j})$ and the subsets of initial particles α_j and final particles β_j all in the jth cluster. Of course, this means that

$$n = n_1 + \cdots + n_v$$

and also the set α is the union of all the particles in the subsets $\alpha_1, \alpha_2, \cdots \alpha_v$, and likewise for the final state. Some of the clusters in Eq. (4.4.5) may contain no vertices at all, i.e., $n_j = 0$; for these factors, we must take the matrix element factor in Eq. (4.4.5) to vanish unless β_j and α_j are both one-particle states (in which case it is just a delta function $\delta(\alpha_j - \beta_j)$), because the only connected diagrams without vertices consist of a single line running through the diagram from bottom to top. Most important, the subscript C in Eq. (4.4.5) means that we exclude any contributions corresponding to disconnected diagrams, that is, any contributions in which any $V(t)$ operator or any initial or final particle is not connected to every other by a sequence of particle creations and annihilations.

Now let us use Eq. (4.4.5) in the sum (4.4.3). Every time variable is integrated from $-\infty$ to $+\infty$, so it makes no difference which of the $t_1, \cdots t_n$ are sorted out into each cluster. The sum over clusterings therefore yields a factor $n!/n_1!n_2!\cdots n_v!$, equal to the number of ways of sorting out n vertices into v clusters, each containing $n_1, n_2, \cdots$ vertices:

$$\int_{-\infty}^{\infty} dt_1 \cdots dt_n \left(\Phi_\beta,\ T\left\{V(t_1)\cdots V(t_n)\right\}\Phi_\alpha\right)$$
$$= \frac{n!}{n_1!n_2!\cdots n_v!} \sum_{\text{PART}} (\pm) \sum_{\substack{n_1 \cdots n_v \\ n_1+\cdots+n_v=n}} \prod_{j=1}^{v} \int_{-\infty}^{\infty} dt_{j_1} \cdots dt_{jn_j}$$
$$\times \left(\Phi_{\beta_j}, T\left\{V(t_{j_1})\cdots V(t_{jn_j})\right\}\Phi_{\alpha_j}\right)_C .$$

The first sum here is over all ways of partitioning the particles in the initial and final states into clusters $\alpha_1 \cdots \alpha_v$ and $\beta_1 \cdots \beta_v$ (including a sum over the number v of clusters). The factor $n!$ here cancels the $1/n!$ in Eq. (4.4.3), and the factor $(-i)^n$ in the perturbation series for (4.4.5) can be written as a product $(-i)^{n_1} \cdots (-i)^{n_v}$, so instead of summing over n and

then summing separately over $n_1, \cdots n_v$ constrained by $n_1 + \cdots + n_v = n$, we can simply sum independently over each $n_1, \cdots n_v$. This gives finally

$$S_{\beta\alpha} = \sum_{\text{PART}} (\pm) \prod_{j=1}^{v} \sum_{n_j=0}^{\infty} \frac{(-i)^{n_j}}{n_j!} \int_{-\infty}^{\infty} dt_{j_1} \cdots dt_{jn_j} \times \left(\Phi_{\beta_j},\ T\left\{ V(t_{j_1}) \cdots V(t_{jn_j}) \right\} \Phi_{\alpha_j} \right)_{\text{C}} .$$

Comparing this with the definition (4.2.2) of the connected matrix elements $S^{\text{C}}_{\beta\alpha}$, we see that these matrix elements are just given by the factors in the product here

$$S^{\text{C}}_{\beta\alpha} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \cdots dt_n \left(\Phi_\beta, T\left\{ V(t_1) \cdots V(t_n) \right\} \Phi_\alpha \right)_{\text{C}} . \qquad (4.4.6)$$

(The subscript j is dropped on all the ts and ns, as these are now mere integration and summation variables.) We see that $S^{\text{C}}_{\beta\alpha}$ is calculated by a very simple prescription: *$S^{\text{C}}_{\beta\alpha}$ is the sum of all contributions to the S-matrix that are connected, in the sense that we drop all terms in which any initial or final particle or any operator $V(t)$ is not connected to all the others by a sequence of particle creations and annihilations.* This justifies the adjective 'connected' for S^{C}.

As we have seen, momentum is conserved at each vertex and along every line, so the connected parts of the S-matrix individually conserve momentum: $S^{\text{C}}_{\beta\alpha}$ contains a factor $\delta^3(\mathbf{p}_\beta - \mathbf{p}_\alpha)$. What we want to prove is that $S^{\text{C}}_{\beta\alpha}$ contains no other delta functions.

We now make the assumption that the coefficient fractions h_{NM} in the expansion (4.4.1) of the Hamiltonian in terms of creation and annihilation operators are proportional to a *single* three-dimensional delta function, that ensures momenta conservation. This is automatically true for the free-particle Hamiltonian H_0, so it is also then true separately for the interaction V. Returning to the graphical interpretation of the matrix elements that we have been using, this means that each vertex contributes one three-dimensional delta function. (The other delta functions in matrix elements $V_{\gamma\delta}$ simply keep the momentum of any particle that is not created or annihilated at the corresponding vertex unchanged.) Now, most of these delta functions simply go to fix the momentum of intermediate particles. The only momenta that are left unfixed by such delta functions are those that circulate in loops of internal lines. (Any line which if cut leaves the diagram disconnected carries a momentum that is fixed by momentum conservation as some linear combination of the momenta of the lines coming into or going out of the diagram. If the diagram has L lines that can all be cut at the same time without the diagram becoming disconnected, then we say it has L independent loops, and there are L

momenta that are not fixed by momentum conservation.) With V vertices, I internal lines, and L loops, there are V delta functions, of which $I - L$ go to fix internal momenta, leaving $V - I + L$ delta functions relating the momenta of incoming or outgoing particles. But a well-known topological identity** tells that for any graph consisting of C connected pieces, the numbers of vertices, internal lines, and loops are related by

$$V - I + L = C \,. \tag{4.4.7}$$

Hence for a connected matrix element like $S^{\rm C}_{\beta\alpha}$, which arises from graphs with $C = 1$, we find just a single three-dimensional delta function $\delta^3(\mathbf{p}_\beta - \mathbf{p}_\alpha)$, as was to be proved.

It was not important in the above argument that the time variables were integrated from $-\infty$ to $+\infty$. Thus exactly the same arguments can be used to show that if the coefficients $h_{N,M}$ in the Hamiltonian contain just single delta functions, then $U(t, t_0)$ can also be decomposed into connected parts, each containing a single momentum-conservation delta function factor. On the other hand, the connected part of the S-matrix also contains an energy-conservation delta function, and when we come to Feynman diagrams in Chapter 6 we shall see that $S^{\rm C}_{\beta\alpha}$ contains only a single energy-conservation delta function, $\delta(E_\beta - E_\alpha)$, while $U(t, t_0)$ contains no energy-conservation delta functions at all.

It should be emphasized that the requirement that h_{NM} in Eq. (4.4.1) should have only a single three-dimensional momentum conservation delta function factor is very far from trivial, and has far-reaching implications. For instance, assume that V has non-vanishing matrix elements between two-particle states. Then Eq. (4.4.1) must contain a term with $N = M = 2$, and coefficient

$$v_{2,2}(\mathbf{p}'_1\mathbf{p}'_2, \mathbf{p}_1\,\mathbf{p}_2) = V_{\mathbf{p}'_1\mathbf{p}'_2,\mathbf{p}_1\,\mathbf{p}_2} \,. \tag{4.4.8}$$

(We are here temporarily dropping spin and species labels.) But then the matrix element of the interaction between three-particle states is

$$\begin{aligned} V_{\mathbf{p}'_1\mathbf{p}'_2\mathbf{p}'_3,\mathbf{p}_1\,\mathbf{p}_2\,\mathbf{p}_3} &= v_{3,3}(\mathbf{p}'_1\mathbf{p}'_2\mathbf{p}'_3, \mathbf{p}_1\,\mathbf{p}_2\,\mathbf{p}_3) \\ &\quad + v_{2,2}(\mathbf{p}'_1\mathbf{p}'_2, \mathbf{p}_1\mathbf{p}_2)\,\delta^3(\mathbf{p}'_3 - \mathbf{p}_3) \pm \text{permutations.} \end{aligned} \tag{4.4.9}$$

** A graph consisting of a single vertex has $V = 1$, $L = 0$, and $C = 1$. If we add $V - 1$ vertices with just enough internal lines to keep the graph connected, we have $I = V - 1$, $L = 0$, and $C = 1$. Any additional internal lines attached (without new vertices) to the same connected graph produce an equal number of loops, so $I = V + L - 1$ and $C = 1$. If a disconnected graph consists of C such connected parts, the sums of I, V, and L in each connected part will than satisfy $\sum I = \sum V + \sum L - C$.

As mentioned at the beginning of this chapter, we might try to make a relativistic quantum theory that is not a field theory by choosing $v_{2,2}$ so that the two–body S-matrix is Lorentz-invariant, and adjusting the rest of the Hamiltonian so that there is no scattering in states containing three or more particles. We would then have to take $v_{3,3}$ to cancel the other terms in Eq. (4.4.9)

$$v_{3,3}(\mathbf{p}'_1\mathbf{p}'_2\mathbf{p}'_3, \mathbf{p}_1\,\mathbf{p}_2\,\mathbf{p}_3) = -v_{2,2}(\mathbf{p}'_1\mathbf{p}'_2, \mathbf{p}_1\mathbf{p}_2)\,\delta^3(\mathbf{p}'_3 - \mathbf{p}_3) \mp \text{ permutations.} \tag{4.4.10}$$

However, this would mean that each term in $v_{3,3}$ contains *two* delta function factors (recall that $v_{2,2}(\mathbf{p}'_1\mathbf{p}'_2, \mathbf{p}_1\mathbf{p}_2)$ has a factor $\delta^3(\mathbf{p}'_1 + \mathbf{p}_2 - \mathbf{p}_1 - \mathbf{p}_2)$) and this would violate the cluster decomposition principle. Thus in a theory satisfying the cluster decomposition principle, the existence of scattering processes involving two particles makes processes involving three or more particles inevitable.

* * *

When we set out to solve three-body problems in quantum theories that satisfy the cluster decomposition principle, the term $v_{3,3}$ in Eq. (4.4.9) gives no particular trouble, but the extra delta function in the other terms makes the Lippmann–Schwinger equation difficult to solve directly. The problem is that these delta functions make the kernel $[E_\alpha - E_\beta + i\epsilon]^{-1} V_{\beta\alpha}$ of this equation not square-integrable, even after we factor out an overall momentum conservation delta function. In consequence, it cannot be approximated by a finite matrix, even one of very large rank. To solve problems involving three or more particles, it is necessary to replace the Lippmann–Schwinger equation with one that has a connected right-hand side. Such equations have been developed for the scattering of three or more particles,[8,9] and in non-relativistic scattering problems they can be solved recursively, but they have not turned out to be useful in relativistic theories and so will not be described in detail here.

However, recasting the Lippmann–Schwinger equation in this manner is useful in another way. Our arguments in this section have so far relied on perturbation theory. I do not know of any non-perturbative proof of the main theorem of this section, but it has been shown[9] that these reformulated non-perturbative dynamical equations are *consistent* with the requirement that $U^C(t, t_0)$ (and hence S^C) should also contain only a single momentum-conservation delta function, as required by the cluster decomposition principle, provided that the Hamiltonian satisfies our condition that the coefficient functions $h_{N,M}$ each contain only a single momentum-conservation delta function.

Problems

1. Define generating functionals for the S-matrix and its connected part:

$$F[v] \equiv 1 + \sum_{N=1}^{\infty} \sum_{M=1}^{\infty} \frac{1}{N!M!} \int v^*(q_1') \cdots v^*(q_N') v(q_1) \cdots v(q_M) \times S_{q_1' \cdots q_N', q_1 \cdots q_M} \, dq_1' \cdots dq_N' \, dq_1 \cdots dq_M$$

$$F^{\mathrm{C}}[v] \equiv \sum_{N=1}^{\infty} \sum_{M=1}^{\infty} \frac{1}{N!M!} \int v^*(q_1') \cdots v^*(q_N') v(q_1) \cdots v(q_M) \times S^{\mathrm{C}}_{q_1' \cdots q_N', q_1 \cdots q_M} \, dq_1' \cdots dq_N' \, dq_1 \cdots dq_M .$$

   Derive a formula relating $F[v]$ and $F^{\mathrm{C}}[v]$. (You may consider the purely bosonic case.)

2. Consider an interaction

$$V = g \int d^3\mathbf{p}_1 \, d^3\mathbf{p}_2 \, d^3\mathbf{p}_3 \, d^3\mathbf{p}_4 \, \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \times a^\dagger(\mathbf{p}_1) \, a^\dagger(\mathbf{p}_2) \, a(\mathbf{p}_3) \, a(\mathbf{p}_4) ,$$

   where g is a real constant and $a(\mathbf{p})$ is the annihilation operator of a spinless boson of mass $M > 0$. Use perturbation theory to calculate the S-matrix element for scattering of these particles in the center-of-mass frame to order g^2. What is the corresponding differential cross-section?

3. A *coherent state* Φ_λ is defined to be an eigenstate of the annihilation operators $a(q)$ with eigenvalues $\lambda(q)$. Construct such a state as a superposition of the multi-particle states $\Phi_{q_1 q_2 \cdots q_N}$.

References

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