our formula (7.469) for $f'(x)$ gives

$$f'(a) = \frac{u'(a)}{A} \int_a^b v(y)\, g(y)\, dy = 0 = f'(b) = \frac{v'(b)}{A} \int_a^b u(y)\, g(y)\, dy. \quad (7.473)$$

For instance, to solve the equation $-f''(x) - f(x) = \exp x$, with the mixed boundary conditions $f(-\pi) = 0$ and $f'(\pi) = 0$, we choose from among the solutions $\alpha \cos x + \beta \sin x$ of the homogeneous equation $-f'' - f = 0$, the functions $u(x) = \sin x$ and $v(x) = \cos x$. Substituting them into the formula (7.468) and setting $p(x) = 1$ and $A = -W(x_0) = \sin^2(x_0) + \cos^2(x_0) = 1$, we find as the Green's function

$$G(x, y) = \theta(x - y) \sin y \cos x + \theta(y - x) \sin x \cos y. \quad (7.474)$$

The solution $f(x) = \int_{-\pi}^{\pi} G(x, y)\, e^y\, dy$ then is

$$\begin{aligned} f(x) &= \int_{-\pi}^{\pi} [\theta(x - y) \sin y \cos x + \theta(y - x) \sin x \cos y]\, e^y\, dy \\ &= -\frac{1}{2} \left(e^{-\pi} \cos x + e^{\pi} \sin x + e^x \right). \end{aligned} \quad (7.475)$$

□

7.44 Principle of Stationary Action in Field Theory

If $\phi(x)$ is a scalar field, and $L(\phi)$ is its action density, then its action $S[\phi]$ is the integral over all of spacetime

$$S[\phi] = \int L(\phi(x), \partial_a \phi(x))\, d^4x. \quad (7.476)$$

The principle of least (or stationary) action says that the field $\phi(x)$ that satisfies the classical equation of motion is the one for which the first-order change in the action due to any tiny variation $\delta\phi(x)$ in the field vanishes, $\delta S[\phi] = 0$. To keep things simple, we'll assume that the action (or Lagrange) density $L(\phi)$ is a function only of the field ϕ and its first derivatives $\partial_a \phi = \partial \phi / \partial x^a$. The first-order change in the action then is

$$\delta S[\phi] = \int \left[\frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial(\partial_a \phi)} \delta(\partial_a \phi) \right] d^4x = 0 \quad (7.477)$$

in which we sum over the repeated index a from 0 to 3. Now $\delta(\partial_a \phi) = \partial_a(\phi + \delta\phi) - \partial_a \phi = \partial_a \delta\phi$. So we may integrate by parts and drop the surface terms because we set $\delta\phi = 0$ on the surface at infinity

$$\delta S[\phi] = \int \left[\frac{\partial L}{\partial \phi} \delta\phi + \frac{\partial L}{\partial(\partial_a \phi)} \partial_a(\delta\phi) \right] d^4x = \int \left[\frac{\partial L}{\partial \phi} - \partial_a \frac{\partial L}{\partial(\partial_a \phi)} \right] \delta\phi \, d^4x.$$

This first-order variation is zero for arbitrary $\delta\phi$ only if the field $\phi(x)$ satisfies Lagrange's equation

$$\partial_a \left(\frac{\partial L}{\partial(\partial_a \phi)} \right) \equiv \frac{\partial}{\partial x^a} \left[\frac{\partial L}{\partial(\partial\phi/\partial x^a)} \right] = \frac{\partial L}{\partial \phi} \tag{7.478}$$

which is the classical equation of motion.

Example 7.66 (Theory of a scalar field) The action density of a single scalar field ϕ of mass m is $L = \frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2} m^2 \phi^2$ or equivalently $L = -\frac{1}{2} \partial_a \phi \, \partial^a \phi - \frac{1}{2} m^2 \phi^2$. Lagrange's equation (7.478) is then

$$\nabla^2 \phi - \ddot{\phi} = \partial_a \, \partial^a \phi = m^2 \phi \tag{7.479}$$

which is the Klein-Gordon equation (7.41). □

In a theory of several fields $\phi_1, \ldots, \phi_n$ with action density $L(\phi_k, \partial_a \phi_k)$, the fields obey n copies of Lagrange's equation one for each field ϕ_k

$$\partial_a \left(\frac{\partial L}{\partial(\partial_a \phi_k)} \right) = \frac{\partial}{\partial x^a} \left(\frac{\partial L}{\partial(\partial_a \phi_k)} \right) = \frac{\partial L}{\partial \phi_k}. \tag{7.480}$$

Application of the principle of stationary action to the action $\int R \sqrt{g} \, d^4x$ gives Einstein's equations as shown in section 13.37.

7.45 Symmetries and Conserved Quantities in Field Theory

A transformation of the coordinates x^a or of the fields ϕ_i and their derivatives $\partial_a \phi_i$ that leaves the action density $L(\phi_i, \partial_a \phi_i)$ invariant is a **symmetry** of the theory. Such a symmetry implies that something is conserved or time independent.

Suppose that due to a symmetry a Lagrange density $L(\phi_i, \partial_a \phi_i)$ is unchanged when the fields ϕ_i and their derivatives $\partial_a \phi_i$ change by $\delta\phi_i$ and by $\delta(\partial_a \phi_i) = \partial_a(\delta\phi_i)$ which need not be small:

$$0 = \delta L = \sum_i \frac{\partial L}{\partial \phi_i} \delta\phi_i + \frac{\partial L}{\partial \partial_a \phi_i} \partial_a \delta\phi_i. \tag{7.481}$$

Then using Lagrange's equations (7.480), we can rewrite $\partial L / \partial \phi_i$ and get

$$0 = \sum_i \left(\partial_a \frac{\partial L}{\partial \partial_a \phi_i} \right) \delta\phi_i + \frac{\partial L}{\partial \partial_a \phi_i} \partial_a \delta\phi_i = \partial_a \sum_i \frac{\partial L}{\partial \partial_a \phi_i} \, \delta\phi_i \tag{7.482}$$

which says that the **current**

$$J^a = \sum_i \frac{\partial L}{\partial \partial_a \phi_i} \, \delta\phi_i \tag{7.483}$$

has zero divergence, $\partial_a J^a = 0$. Thus the time derivative of the volume integral of the charge density J^0

$$Q_V = \int_V J^0 \, d^3x \tag{7.484}$$

is the flux of current $\boldsymbol{J}$ entering through the boundary S of the volume V

$$\dot{Q}_V = \int_V \partial_0 J^0 \, d^3x = -\int_V \nabla \cdot \boldsymbol{J} \, d^3x = -\int_S \boldsymbol{J} \cdot d^2\boldsymbol{S}. \tag{7.485}$$

If no current enters V, then the charge Q inside V is conserved. When the volume V is the whole universe, the charge is the integral over all of space

$$Q = \int J^0 \, d^3x = \int \sum_i \frac{\partial L}{\partial \dot{\phi}_i} \, \delta\phi_i \, d^3x = \int \sum_i \pi_i \, \delta\phi_i \, d^3x \tag{7.486}$$

in which π_i is the momentum conjugate to the field ϕ_i

$$\pi_i = \frac{\partial L}{\partial \dot{\phi}_i}. \tag{7.487}$$

Example 7.67 ($O(n)$ symmetry and its charge) Suppose the action density L is the sum of n copies of the quadratic action density

$$L = \sum_{i=1}^{n} \tfrac{1}{2} \, (\dot{\phi}_i)^2 - \tfrac{1}{2} \, (\nabla \phi_i)^2 - \tfrac{1}{2} \, m^2 \phi_i^2 = -\tfrac{1}{2} \, \partial_a \phi \, \partial^a \phi - \tfrac{1}{2} \, m^2 \phi^2. \tag{7.488}$$

Let A_{ij} be any constant antisymmetric matrix, $A_{ij} = -A_{ji}$. Then if the fields change by $\delta\phi_i = \epsilon \sum_j A_{ij} \, \phi_j$, the change (7.481) in the action density

$$\delta L = -\epsilon \sum_{i,j=1}^{n} \partial^a \phi_i A_{ij} \partial_a \phi_j + m^2 \phi_i A_{ij} \phi_j = 0 \tag{7.489}$$

vanishes. Thus the charge (7.486) associated with the matrix A

$$Q_A = \int \sum_{i,j} \pi_i \, \delta\phi_i \, d^3x = \epsilon \int \sum_{i,j} \pi_i \, A_{ij} \, \phi_j \, d^3x \tag{7.490}$$

is conserved. There are $n(n-1)/2$ antisymmetric $n \times n$ imaginary matrices; they generate the group $O(n)$ of $n \times n$ orthogonal matrices (example 11.3). □

If an action density $L(\phi_i, \partial_a\phi_i)$ depends upon the spacetime coordinate x^a only through the fields ϕ_i and their derivatives $\partial_a\phi_i$, then its spacetime derivative is

$$\frac{\partial L}{\partial x^a} = \sum_i \left(\frac{\partial L}{\partial \phi_i}\frac{\partial \phi_i}{\partial x^a} + \frac{\partial L}{\partial \partial_b \phi_i}\frac{\partial^2 \phi_i}{\partial x^b \partial x^a} \right). \tag{7.491}$$

Using Lagrange's equations (7.480) to rewrite $\partial L/\partial\phi_i$, we find

$$\begin{aligned} 0 &= \left(\sum_i \partial_b \left(\frac{\partial L}{\partial \partial_b \phi_i} \right) \partial_a \phi_i + \frac{\partial L}{\partial \partial_b \phi_i}\frac{\partial^2 \phi_i}{\partial x^b \partial x^a} \right) - \frac{\partial L}{\partial x^a} \\ 0 &= \partial_b \left[\left(\sum_i \frac{\partial L}{\partial \partial_b \phi_i}\frac{\partial \phi_i}{\partial x^a} \right) - \delta^b_a L \right] \end{aligned} \tag{7.492}$$

which says that the **energy-momentum tensor** of the theory

$$T^b{}_a = \sum_i \frac{\partial L}{\partial \partial_b \phi_i}\frac{\partial \phi_i}{\partial x^a} - \delta^b_a L \tag{7.493}$$

has zero divergence, $\partial_b T^b{}_a = 0$.

Thus the time derivative of the 4-momentum P_{aV} inside a volume V

$$P_{aV} = \int_V \left(\sum_i \frac{\partial L}{\partial \partial_0 \phi_i}\frac{\partial \phi_i}{\partial x^a} - \delta^0_a L \right) d^3x = \int_V T^0{}_a \, d^3x \tag{7.494}$$

is equal to the flux entering through V's boundary S

$$\partial_0 P_{aV} = \int_V \partial_0 T^0{}_a \, d^3x = -\int_V \partial_k T^k{}_a \, d^3x = -\int_S T^k{}_a \, d^2S_k. \tag{7.495}$$

And so if an action density has no explicit dependence on any of the spacetime coordinates x, then its **energy** P_0 and **momentum** $\vec{P}$ are conserved.

Example 7.68 (Conservation of angular momentum) Under an infinitesimal rotation by $\boldsymbol{\theta}$, spatial coordinates change by $\delta\boldsymbol{x} = \boldsymbol{\theta} \times \boldsymbol{x}$ and the change in a scalar field is $\delta\phi(t, \boldsymbol{x}) = \boldsymbol{\nabla}\phi \cdot \delta\boldsymbol{x} = \boldsymbol{\nabla}\phi \cdot (\boldsymbol{\theta} \times \boldsymbol{x}) = \boldsymbol{\theta} \cdot (\boldsymbol{x} \times \boldsymbol{\nabla}\phi)$. So if the spatial integral of the Lagrange density L is unchanged under rotations

$$0 = \int d^3x \, \frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial \partial_a \phi}\partial_a \delta\phi = \int d^3x \, \partial_a \frac{\partial L}{\partial \partial_a \phi}\delta\phi \tag{7.496}$$

then the time derivative of the angular momentum

$$\boldsymbol{J} = \int d^3x \, \frac{\partial L}{\partial \dot{\phi}} \, \delta\boldsymbol{x} = \int d^3x \, \frac{\partial L}{\partial \dot{\phi}} \, (\boldsymbol{x} \times \boldsymbol{\nabla}\phi) \tag{7.497}$$

vanishes as long as the fields fall to zero as $|\boldsymbol{x}| \to \infty$. If a rotation changes the field component ϕ^ℓ by $\delta\phi^\ell = \boldsymbol{\theta}\cdot(\boldsymbol{x}\times\boldsymbol{\nabla}\phi^\ell - i\boldsymbol{S}^\ell{}_m\phi^m)$, where S represents the spin of the field, then the conserved angular momentum is

$$\boldsymbol{J} = \int d^3x\, \frac{\partial L}{\partial\dot{\phi}^\ell}\,(\boldsymbol{x}\times\boldsymbol{\nabla}\phi^\ell - i\boldsymbol{S}^\ell{}_m\phi^m). \tag{7.498}$$

□

The momentum $\pi_i(x)$ that is canonically conjugate to the field $\phi_i(x)$ is the derivative of the action density L with respect to the time derivative of the field

$$\pi_i = \frac{\partial L}{\partial\dot{\phi}_i}. \tag{7.499}$$

If one can express the time derivatives $\dot{\phi}_i$ of the fields in terms of the fields ϕ_i and their momenta π_i, then the hamiltonian of the theory is the spatial integral of

$$H = P_0 = T^0{}_0 = \left(\sum_{i=1}^{n}\pi_i\,\dot{\phi}_i\right) - L \tag{7.500}$$

in which $\dot{\phi}_i = \dot{\phi}_i(\phi,\pi)$.

Example 7.69 (Hamiltonian of a scalar field) For the lagrangian L of example 7.66, the hamiltonian density (7.500) is $H = \frac{1}{2}\,\pi^2 + \frac{1}{2}\,(\nabla\phi)^2 + \frac{1}{2}\,m^2\phi^2$. □

Example 7.70 (Euler's theorem and the Nambu-Goto string) When the action density is a first-degree homogeneous function (section 7.10) of the time derivatives of the fields, as is that of the Nambu-Goto string

$$L = -\frac{T_0}{c}\sqrt{\left(\dot{X}\cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2}, \tag{7.501}$$

Euler's theorem (7.112) implies that the energy density (7.500) vanishes identically, independently of the equations of motion,

$$E^0 = \frac{\partial L}{\partial\dot{X}^\mu}\,\dot{X}^\mu - L = 0. \tag{7.502}$$

□