## Complex-Variable Theory

### 6.1 Analytic Functions

A complex-valued function $f(z)$ of a complex variable $z$ is differentiable at $z$ with derivative $f^{\prime}(z)$ if the limit

$$
\begin{equation*}
f^{\prime}(z)=\lim _{z^{\prime} \rightarrow z} \frac{f\left(z^{\prime}\right)-f(z)}{z^{\prime}-z} \tag{6.1}
\end{equation*}
$$

exists and is unique as $z^{\prime}$ approaches $z$ from any direction in the complex plane. The limit must exist no matter how or from what direction $z^{\prime}$ approaches $z$.

If the function $f(z)$ is differentiable in a small disk around a point $z_{0}$, then $f(z)$ is said to be analytic (or equivalently holomorphic) at $z_{0}$ (and at all points inside the disk).

Example 6.1 (Polynomials) If $f(z)=z^{n}$ for some integer $n$, then for tiny $d z$ and $z^{\prime}=z+d z$, the difference $f\left(z^{\prime}\right)-f(z)$ is

$$
\begin{equation*}
f\left(z^{\prime}\right)-f(z)=(z+d z)^{n}-z^{n} \approx n z^{n-1} d z \tag{6.2}
\end{equation*}
$$

and so the limit

$$
\begin{equation*}
\lim _{z^{\prime} \rightarrow z} \frac{f\left(z^{\prime}\right)-f(z)}{z^{\prime}-z}=\lim _{d z \rightarrow 0} \frac{n z^{n-1} d z}{d z}=n z^{n-1} \tag{6.3}
\end{equation*}
$$

exists and is $n z^{n-1}$ independently of how $z^{\prime}$ approaches $z$. Thus the function $z^{n}$ is analytic at $z$ for all $z$ with derivative

$$
\begin{equation*}
\frac{d z^{n}}{d z}=n z^{n-1} . \tag{6.4}
\end{equation*}
$$

A function that is analytic everywhere is entire. All polynomials

$$
\begin{equation*}
P(z)=\sum_{k=0}^{n} c_{k} z^{k} \tag{6.5}
\end{equation*}
$$

are entire.
Example 6.2 (A function that's not analytic) The function $f(x, y)=$ $x^{2}+y^{2}=z \bar{z}$ for $z=x+i y$. If we compute its derivative at $(x, y)=(1,0)$ by setting $x=1+\epsilon$ and $y=0$, then the limit is

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0} \frac{f(1+\epsilon, 0)-f(1,0)}{\epsilon}=\lim _{\epsilon \rightarrow 0} \frac{(1+\epsilon)^{2}-1}{\epsilon}=2 \tag{6.6}
\end{equation*}
$$

while if we instead set $x=1$ and $y=\epsilon$, then the limit is

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0} \frac{f(1, \epsilon)-f(1,0)}{i \epsilon}=\lim _{\epsilon \rightarrow 0} \frac{1+\epsilon^{2}-1}{i \epsilon}=-i \lim _{\epsilon \rightarrow 0} \epsilon=0 . \tag{6.7}
\end{equation*}
$$

So the derivative depends upon the direction through which $z \rightarrow 1$.

### 6.2 Cauchy-Riemann Conditions

When is a complex function of two real variables $x$ and $y f(x, y)=u(x, y)+$ $i v(x, y)$ whose real and imaginary parts are $u(x, y)$ and $v(x, y)$ is analytic? We apply the criterion (6.1) of analyticity and require that the change $d f$ in the function $f(x, y)$ be proportional to the change $d z=d x+i d y$ so that the ratio $d f / d z$ is independent of how $d z$ approaches 0

$$
\begin{equation*}
\left(\frac{\partial u}{\partial x}+i \frac{\partial v}{\partial x}\right) d x+\left(\frac{\partial u}{\partial y}+i \frac{\partial v}{\partial y}\right) d y=f^{\prime}(z)(d x+i d y) . \tag{6.8}
\end{equation*}
$$

Setting first $d y$ and then $d x$ equal to zero, we have

$$
\begin{equation*}
\left(\frac{\partial u}{\partial x}+i \frac{\partial v}{\partial x}\right)=f^{\prime}(z)=\frac{1}{i}\left(\frac{\partial u}{\partial y}+i \frac{\partial v}{\partial y}\right) . \tag{6.9}
\end{equation*}
$$

This complex equation implies the two real equations

$$
\begin{equation*}
\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text { and } \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \tag{6.10}
\end{equation*}
$$

which are the Cauchy-Riemann conditions. In a notation in which partial derivatives are labeled by subscripts, the Cauchy-Riemann conditions are $u_{x}=v_{y}$ and $v_{x}=-u_{y}$ (Augustin-Louis Cauchy, 1789-1857; Bernhard Riemann, 1826-1866).

Example 6.3 Is the function $f(x, y)=u(x, y)+i v(x, y)$ with $u(x, y)=x^{2} y$ and $v(x, y)=x y^{2}$ analytic? Well $u_{x}=2 x y=v_{y}$, but $v_{x}=y^{2} \neq-u_{x}=$ $-2 x y$. So no, $f(x, y)=x^{2} y+i x y^{2}$ is not analytic.

What if $u(x, y)=v(x, y)=x^{2} y^{2}$ ? Now $u_{x}=2 x y^{2}$, but $v_{y}=2 x^{2} y$. So no, $f(x, y)=(1+i) x^{2} y^{2}$ is not analytic.

What if $u(x, y)=x^{2}-y^{2}$ and $v(x, y)=2 x y$ ? Now $u_{x}=2 x=v_{y}$, and $v_{x}=2 y=-u_{y}$. So, yes, $f(x, y)=(x+i y)^{2}=z^{2}$ is analytic.

Example 6.4 (A function analytic except at a point) The real and imaginary parts of the function

$$
\begin{equation*}
f(z)=\frac{1}{z-z_{0}}=\frac{z^{*}-z_{0}^{*}}{\left|z-z_{0}\right|^{2}}=\frac{x-x_{0}-i\left(y-y_{0}\right)}{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}} \tag{6.11}
\end{equation*}
$$

are

$$
\begin{equation*}
u(x, y)=\frac{x-x_{0}}{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}} \quad \text { and } \quad v(x, y)=\frac{-\left(y-y_{0}\right)}{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}} \tag{6.12}
\end{equation*}
$$

They satisfy the Cauchy-Riemann conditions (6.10)

$$
\begin{equation*}
\frac{\partial u(x, y)}{\partial x}=\frac{\left(y-y_{0}\right)^{2}-\left(x-x_{0}\right)^{2}}{\left[\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}\right]^{2}}=\frac{\partial v(x, y)}{\partial y} \tag{6.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial v(x, y)}{\partial x}=\frac{2\left(x-x_{0}\right)\left(y-y_{0}\right)}{\left[\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}\right]^{2}}=-\frac{\partial u(x, y)}{\partial y} \tag{6.14}
\end{equation*}
$$

except at the point $z=z_{0}$ where $x=x_{0}$ and $y=y_{0}$.

### 6.3 Cauchy's Integral Theorem

The Cauchy-Riemann conditions imply that the integral of a function along a closed contour (one that ends where it starts) vanishes if the function is analytic on the contour and everywhere inside it. To keep the notation simple, let's consider a rectangle $R$ of length $\ell$ and height $h$ with one corner at the origin and edges running along the $x$ and $y$ axes of the $z$
plane. The integral along the four sides of the rectangle is

$$
\begin{align*}
\oint_{R} f(z) d z & =\oint_{R}(u(x, y)+i v(x, y))(d x+i d y) \\
& =\int_{0}^{\ell}[u(x, 0)+i v(x, 0)] d x+\int_{0}^{h}[u(\ell, y)+i v(\ell, y)] i d y  \tag{6.15}\\
& +\int_{\ell}^{0}[u(x, h)+i v(x, h)] d x+\int_{h}^{0}[u(0, y)+i v(0, y)] i d y .
\end{align*}
$$

The real and imaginary parts of this contour integral are

$$
\begin{align*}
& \operatorname{Re}\left(\oint_{R} f(z) d z\right)=\int_{0}^{\ell}[u(x, 0)-u(x, h)] d x-\int_{0}^{h}[v(\ell, y)-v(0, y)] d y \\
& \operatorname{Im}\left(\oint_{R} f(z) d z\right)=\int_{0}^{\ell}[v(x, 0)-v(x, h)] d x+\int_{0}^{h}[u(\ell, y)-u(0, y)] d y . \tag{6.16}
\end{align*}
$$

The differences $u(x, 0)-u(x, h)$ and $v(\ell, y)-v(0, y)$ in the real part are integrals of the $y$ derivative $u_{y}(x, y)$ and of the $x$ derivative $v_{x}(x, y)$

$$
\begin{align*}
u(x, 0)-u(x, h) & =-\int_{0}^{h} u_{y}(x, y) d y \\
v(\ell, y)-v(0, y) & =\int_{0}^{\ell} v_{x}(x, y) d x \tag{6.17}
\end{align*}
$$

The real part of the contour integral therefore vanishes due to the second $v_{x}=-u_{y}$ of the Cauchy-Riemann conditions (6.10)

$$
\begin{align*}
\operatorname{Re}\left(\oint_{R} f(z) d z\right) & =-\int_{0}^{\ell} \int_{0}^{h} u_{y}(x, y) d y d x-\int_{0}^{h} \int_{0}^{\ell} v_{x}(x, y) d x d y  \tag{6.18}\\
& =-\int_{0}^{\ell} \int_{0}^{h}\left[u_{y}(x, y)+v_{x}(x, y)\right] d y d x=0
\end{align*}
$$

Similarly, differences $v(x, 0)-v(x, h)$ and $u(\ell, y)-u(0, y)$ in the imaginary part are integrals of the $y$ derivative $v_{y}(x, y)$ and of the $x$ derivative $u_{x}(x, y)$

$$
\begin{align*}
& v(x, 0)-v(x, h)=-\int_{0}^{h} v_{y}(x, y) d y \\
& u(\ell, y)-u(0, y)=\int_{0}^{\ell} u_{x}(x, y) d x \tag{6.19}
\end{align*}
$$

Thus the imaginary part of the contour integral vanishes due to the first
$u_{x}=v_{y}$ of the Cauchy-Riemann conditions (6.10)

$$
\begin{align*}
\operatorname{Im}\left(\oint_{R} f(z) d z\right) & =-\int_{0}^{\ell} \int_{0}^{h} v_{y}(x, y) d y d x+\int_{0}^{h} \int_{0}^{\ell} u_{x}(x, y) d x d y  \tag{6.20}\\
& =\int_{0}^{\ell} \int_{0}^{h}\left[-v_{y}(x, y)+u_{x}(x, y)\right] d y d x=0
\end{align*}
$$

A similar argument shows that the contour integral along the four sides of any rectangle vanishes as long as the function $f(z)$ is analytic on and within the rectangle whether or not the rectangle has one corner at the origin $z=0$.

Suppose a function $f(z)$ is analytic along a closed contour $C$ and also at every point inside it. We can tile the inside area $A$ with a suitable collection of contiguous rectangles some of which might be very small. The integral of $f(z)$ along the perimeter of each rectangle will vanish because each rectangle lies entirely within the region in which $f(z)$ is analytic. Now consider two adjacent rectangles like the two squares in Fig. 6.1. The sum of the two contour integrals around the two adjacent squares is equal to the contour integral around the perimeter of the two squares because the up integral along the right side (dots) of the left square cancels the down integral along the left side of the right square. Thus the sum of the contour integrals around the perimeters of all the rectangles that tile the inside area $A$ amounts to just the integral along the outer contour $C$. The integral around each rectangle vanishes. So the integral of $f(z)$ along the contour $C$ also must vanish because it is the sum of these vanishing integrals around the rectangles that tile the inside area $A$. This is Cauchy's Integral Theorem: The integral of a function $f(z)$ along a closed contour vanishes

$$
\begin{equation*}
\oint_{C} f(z) d z=0 \tag{6.21}
\end{equation*}
$$

if the function $f(z)$ is analytic on the contour and at every point inside it.
What could go wrong? The area $A$ inside the contour might have a hole in it in which the function $f(z)$ is not analytic. To exclude this possibility, we require that the area $A$ inside the contour be simply connected, that is, we insist that we be able to shrink every loop in $A$ to a point while keeping the loop inside $A$. A slice of American cheese is simply connected, a slice of Swiss Emmental is not. A dime is simply connected, a washer isn't. The surface of a sphere is simply connected, the surface of a bagel isn't. So another version of Cauchy's integral theorem is that the integral of a function $f(z)$ along a closed contour vanishes if the contour lies within a simply connected region in which $f(z)$ is analytic (Augustin-Louis Cauchy, 1789-1857).
Example 6.5 (Tiny circular contour) If $f(z)$ is analytic at $z_{0}$, then the

## Cauchy's Integral Theorem



Figure 6.1 The sum of two contour integrals around two adjacent squares is equal to the contour integral around the perimeter of the two squares because the up integral along the right side (dots) of the left square cancels the down integral along the left side (dots) of the right square. A contour integral around a big square is equal to the sum of the contour integrals around the smaller interior squares that tile the big square. Matlab scripts for this chapter's figures are in Complex-variable_theory at github.com/ kevinecahill.
definition (6.1) of the derivative $f^{\prime}(z)$ shows that $f(z) \approx f\left(z_{0}\right)+f^{\prime}\left(z_{0}\right)\left(z-z_{0}\right)$ near $z_{0}$ to first order in $z-z_{0}$. The points of a small circle of radius $\epsilon$ and center $z_{0}$ are $z=z_{0}+\epsilon e^{i \theta}$. Since $z-z_{0}=\epsilon e^{i \theta}$ and $d z=i \epsilon e^{i \theta} d \theta$, the closed contour integral around the circle is

$$
\begin{align*}
\oint_{O} f(z) d z & =\int_{0}^{2 \pi}\left[f\left(z_{0}\right)+f^{\prime}\left(z_{0}\right)\left(z-z_{0}\right)\right] i \epsilon e^{i \theta} d \theta  \tag{6.22}\\
& =f\left(z_{0}\right) \int_{0}^{2 \pi} i \epsilon e^{i \theta} d \theta+f^{\prime}\left(z_{0}\right) \int_{0}^{2 \pi} \epsilon e^{i \theta} i \epsilon e^{i \theta} d \theta
\end{align*}
$$

which vanishes because the $\theta$-integrals are zero. Thus the contour integral of an analytic function $f(z)$ around a tiny circle, lying within the region in which $f(z)$ is analytic, vanishes.

Example 6.6 (Tiny square contour) The analyticity of $f(z)$ at $z=z_{0}$ lets us expand $f(z)$ near $z_{0}$ as $f(z) \approx f\left(z_{0}\right)+f^{\prime}\left(z_{0}\right)\left(z-z_{0}\right)$. A tiny square
contour consists of four complex segments $d z_{1}=\epsilon, d z_{2}=i \epsilon, d z_{3}=-\epsilon$, and $d z_{4}=-i \epsilon$. The integral of the constant $f\left(z_{0}\right)$ around the square vanishes

$$
\begin{equation*}
\oint_{\square} f\left(z_{0}\right) d z=f\left(z_{0}\right) \oint_{\square} d z=f\left(z_{0}\right)[\epsilon+i \epsilon+(-\epsilon)+(-i \epsilon)]=0 . \tag{6.23}
\end{equation*}
$$

The integral of the second term $f^{\prime}\left(z_{0}\right)\left(z-z_{0}\right)$ also vanishes. It is the sum of four integrals along the four sides of the tiny square. Like the integral of the constant $f\left(z_{0}\right)$, the integral of the constant $-f^{\prime}\left(z_{0}\right) z_{0}$ also vanishes. Dropping that term, we are left with the integral of $f^{\prime}\left(z_{0}\right) z$ along the four sides of the tiny square.

The integral from left to right along the bottom of the square where $z=x-i \epsilon / 2$ is

$$
\begin{equation*}
I_{1}=f^{\prime}(0) \int_{-\epsilon / 2}^{\epsilon / 2}\left(x-i \frac{\epsilon}{2}\right) d x=-\frac{i \epsilon^{2}}{2} f^{\prime}(0) . \tag{6.24}
\end{equation*}
$$

The integral up the right side of the square where $z=\epsilon / 2+i y$ is

$$
\begin{equation*}
I_{2}=f^{\prime}(0) \int_{-\epsilon / 2}^{\epsilon / 2}\left(\frac{\epsilon}{2}+i y\right) i d y=\frac{i \epsilon^{2}}{2} f^{\prime}(0) . \tag{6.25}
\end{equation*}
$$

The integral backwards along the top of the square where $z=x+i \epsilon / 2$ is

$$
\begin{equation*}
I_{3}=f^{\prime}(0) \int_{\epsilon / 2}^{-\epsilon / 2}\left(x+i \frac{\epsilon}{2}\right) d x=-\frac{i \epsilon^{2}}{2} f^{\prime}\left(z_{0}\right) . \tag{6.26}
\end{equation*}
$$

Finally, the integral down the left side where $z=-\epsilon / 2+i y$ is

$$
\begin{equation*}
I_{4}=f^{\prime}(0) \int_{\epsilon / 2}^{-\epsilon / 2}\left(-\frac{\epsilon}{2}+i y\right) i d y=\frac{i \epsilon^{2}}{2} f^{\prime}(0) . \tag{6.27}
\end{equation*}
$$

These integrals cancel in pairs. Thus the contour integral of an analytic function $f(z)$ around a tiny square of side $\epsilon$ is zero to order $\epsilon^{2}$ as long as the square lies inside the region in which $f(z)$ is analytic.

Suppose a function $f(z)$ is analytic in a simply connected region $R$ and that $C$ and $C^{\prime}$ are two contours that lie inside $R$ and that both run from $z_{1}$ to $z_{2}$. The difference of the two contour integrals is an integral along a closed contour $C^{\prime \prime}$ that runs from $z_{1}$ to $z_{2}$ and back to $z_{1}$ and that vanishes by Cauchy's theorem

$$
\begin{equation*}
\int_{z_{1} C}^{z_{2}} f(z) d z-\int_{z_{1} C^{\prime}}^{z_{2}} f(z) d z=\int_{z_{1} C}^{z_{2}} f(z) d z+\int_{z_{2} C^{\prime}}^{z_{1}} f(z) d z=\oint_{C^{\prime \prime}} f(z) d z=0 . \tag{6.28}
\end{equation*}
$$

It follows that any two contour integrals that lie within a simply connected


Figure 6.2 As long as the four contours are within the domain of analyticity of $f(z)$ and have the same endpoints, the four contour integrals of that function are all equal.
region in which $f(z)$ is analytic are equal if they start at the same point $z_{1}$ and end at the same point $z_{2}$. Thus we may continuously deform the contour of an integral of an analytic function $f(z)$ from $C$ to $C^{\prime}$ without changing the value of the contour integral as long as long as these contours and all the intermediate contours lie entirely within the region $R$ and have the same fixed endpoints $z_{1}$ and $z_{2}$ as in Fig. 6.2

$$
\begin{equation*}
\int_{z_{1} C}^{z_{2}} f(z) d z=\int_{z_{1} C^{\prime}}^{z_{2}} f(z) d z \tag{6.29}
\end{equation*}
$$

So a contour integral depends upon its end points and upon the function $f(z)$ but not upon the actual contour as long as the contour stays within the region $R$ in which $f(z)$ is analytic as the contour is deformed from $C$ to some other contour $C^{\prime}$.

If the end points $z_{1}$ and $z_{2}$ are the same, then the contour $\mathcal{C}$ is closed, and we write the integral as

$$
\begin{equation*}
\oint_{z_{1} \mathcal{C}}^{z_{1}} f(z) d z \equiv \oint_{\mathcal{C}} f(z) d z \tag{6.30}
\end{equation*}
$$

with a little circle to denote that the contour is a closed loop. The value
of that integral is independent of the contour as long as our deformations of the contour keep it within the domain of analyticity of the function and as long as the contour starts and ends at $z_{1}=z_{2}$. Now suppose that the function $f(z)$ is analytic along the contour and at all points within it. Then we can shrink the contour, staying within the domain of analyticity of the function, until the area enclosed is zero and the contour is of zero length-all this without changing the value of the integral. But the value of the integral along such a null contour of zero length is zero. Thus the value of the original contour integral also must be zero

$$
\begin{equation*}
\oint_{z_{1} C}^{z_{1}} f(z) d z=0 \tag{6.31}
\end{equation*}
$$

And so we again arrive at Cauchy's integral theorem: The contour integral of a function $f(z)$ around a closed contour $C$ lying entirely within the domain of analyticity of the function vanishes

$$
\begin{equation*}
\oint_{C} f(z) d z=0 \tag{6.32}
\end{equation*}
$$

as long as the function $f(z)$ is analytic at all points within the contour.
Example 6.7 (Polynomials) Since $d z^{n+1}=(n+1) z^{n} d z$, the integral of the entire function $z^{n}$ along any contour $C$ that ends and starts at the same point $z_{0}$ must vanish for any integer $n \geq 0$

$$
\begin{equation*}
\oint_{C} z^{n} d z=\frac{1}{n+1} \oint_{C} d z^{n+1}=\frac{1}{n+1}\left(z_{0}^{n+1}-z_{0}^{n+1}\right)=0 \tag{6.33}
\end{equation*}
$$

Thus the integral of any polynomial $P(z)=c_{0}+c_{1} z+c_{2} z^{2}+\ldots$ along any closed contour $C$ also vanishes

$$
\begin{equation*}
\oint_{C} P(z) d z=\oint_{C} \sum_{n=0}^{m} c_{n} z^{n} d z=0 \tag{6.34}
\end{equation*}
$$

Example 6.8 (A pole) The function $f(z)=1 /\left(z-z_{0}\right)$ is analytic in a region that is not simply connected because its derivative

$$
\begin{equation*}
f^{\prime}(z)=\lim _{d z \rightarrow 0}\left(\frac{1}{z+d z-z_{0}}-\frac{1}{z-z_{0}}\right) \frac{1}{d z}=-\frac{1}{\left(z-z_{0}\right)^{2}} \tag{6.35}
\end{equation*}
$$

exists in the whole complex plane except for the point $z=z_{0}$.

### 6.4 Cauchy's Integral Formula

Suppose that $f(z)$ is analytic in a simply connected region $R$ and that $z_{0}$ is a point inside this region. We first will integrate the function $f(z) /(z-$ $z_{0}$ ) along a tiny closed counterclockwise contour around the point $z_{0}$. The contour is a circle of radius $\epsilon$ with center at $z_{0}$ and points $z=z_{0}+\epsilon e^{i \theta}$ for $0 \leq \theta \leq 2 \pi$. Since $z-z_{0}=\epsilon e^{i \theta}$ and $d z=i \epsilon e^{i \theta} d \theta$, the contour integral in the limit $\epsilon \rightarrow 0$ is

$$
\begin{align*}
\oint_{\epsilon} \frac{f(z)}{z-z_{0}} d z & =\int_{0}^{2 \pi} \frac{\left[f\left(z_{0}\right)+f^{\prime}\left(z_{0}\right)\left(z-z_{0}\right)\right]}{z-z_{0}} i \epsilon e^{i \theta} d \theta \\
& =\int_{0}^{2 \pi} \frac{\left[f\left(z_{0}\right)+f^{\prime}\left(z_{0}\right) \epsilon e^{i \theta}\right]}{\epsilon e^{i \theta}} i \epsilon e^{i \theta} d \theta  \tag{6.36}\\
& =\int_{0}^{2 \pi}\left[f\left(z_{0}\right)+f^{\prime}\left(z_{0}\right) \epsilon e^{i \theta}\right] i d \theta=2 \pi i f\left(z_{0}\right)
\end{align*}
$$

since the $\theta$-integral involving $f^{\prime}\left(z_{0}\right)$ vanishes. Thus $f\left(z_{0}\right)$ is the integral

$$
\begin{equation*}
f\left(z_{0}\right)=\frac{1}{2 \pi i} \oint_{\epsilon} \frac{f(z)}{z-z_{0}} d z \tag{6.37}
\end{equation*}
$$

which is a miniature version of Cauchy's integral formula.
Now consider the counterclockwise contour $\mathcal{C}^{\prime}$ in Fig. 6.3 which is a big counterclockwise circle, a small clockwise circle, and two parallel straight lines, all within a simply connected region $\mathcal{R}$ in which $f(z)$ is analytic. As we saw in examples 6.4 and 6.8 , the function $1 /\left(z-z_{0}\right)$ is analytic except at $z=z_{0}$. Thus since the product of two analytic functions is analytic (exercise 6.3), the function $f(z) /\left(z-z_{0}\right)$ is analytic everywhere in $\mathcal{R}$ except at the point $z_{0}$. We can withdraw the contour $\mathcal{C}^{\prime}$ to the left of the point $z_{0}$ and shrink it to a point without having the contour $\mathcal{C}^{\prime}$ cross $z_{0}$. During this process, the integral of $f(z) /\left(z-z_{0}\right)$ does not change. Its final value is zero. So its initial value also is zero

$$
\begin{equation*}
0=\frac{1}{2 \pi i} \oint_{\mathcal{C}^{\prime}} \frac{f(z)}{z-z_{0}} d z \tag{6.38}
\end{equation*}
$$

We let the two straight-line segments approach each other so that they cancel. What remains of contour $\mathcal{C}^{\prime}$ is a big counterclockwise contour $\mathcal{C}$ around $z_{0}$ and a tiny clockwise circle of radius $\epsilon$ around $z_{0}$. The tiny clockwise circle integral is the negative of the counterclockwise integral (6.37), so we have

$$
\begin{equation*}
0=\frac{1}{2 \pi i} \oint_{\mathcal{C}^{\prime}} \frac{f(z)}{z-z_{0}} d z=\frac{1}{2 \pi i} \oint_{\mathcal{C}} \frac{f(z)}{z-z_{0}} d z-\frac{1}{2 \pi i} \oint_{\epsilon} \frac{f(z)}{z-z_{0}} d z . \tag{6.39}
\end{equation*}
$$



Figure 6.3 The full contour is the sum of a big counterclockwise contour $\mathcal{C}^{\prime}$ and a small clockwise contour, both around $z_{0}$, and two straight lines which cancel.

Using the miniature result (6.37), we find

$$
\begin{equation*}
f\left(z_{0}\right)=\frac{1}{2 \pi i} \oint_{\mathcal{C}} \frac{f(z)}{z-z_{0}} d z \tag{6.40}
\end{equation*}
$$

which is Cauchy's integral formula.
We can use this formula to compute the first derivative $f^{\prime}(z)$ of $f(z)$

$$
\begin{align*}
f^{\prime}(z) & =\frac{f(z+d z)-f(z)}{d z} \\
& =\frac{1}{2 \pi i} \frac{1}{d z} \oint d z^{\prime} f\left(z^{\prime}\right)\left(\frac{1}{z^{\prime}-z-d z}-\frac{1}{z^{\prime}-z}\right) \\
& =\frac{1}{2 \pi i} \oint d z^{\prime} \frac{f\left(z^{\prime}\right)}{\left(z^{\prime}-z-d z\right)\left(z^{\prime}-z\right)} \tag{6.41}
\end{align*}
$$

So in the limit $d z \rightarrow 0$, we get

$$
\begin{equation*}
f^{\prime}(z)=\frac{1}{2 \pi i} \oint d z^{\prime} \frac{f\left(z^{\prime}\right)}{\left(z^{\prime}-z\right)^{2}} \tag{6.42}
\end{equation*}
$$

The second derivative $f^{(2)}(z)$ of $f(z)$ then is

$$
\begin{equation*}
f^{(2)}(z)=\frac{2}{2 \pi i} \oint d z^{\prime} \frac{f\left(z^{\prime}\right)}{\left(z^{\prime}-z\right)^{3}} . \tag{6.43}
\end{equation*}
$$

And its $n$th derivative $f^{(n)}(z)$ is

$$
\begin{equation*}
f^{(n)}(z)=\frac{n!}{2 \pi i} \oint d z^{\prime} \frac{f\left(z^{\prime}\right)}{\left(z^{\prime}-z\right)^{n+1}} . \tag{6.44}
\end{equation*}
$$

In these formulas, the contour runs counterclockwise about the point $z$ and lies within the simply connected domain $\mathcal{R}$ in which $f(z)$ is analytic.

Thus a function $f(z)$ that is analytic in a region $\mathcal{R}$ is infinitely differentiable there.

Example 6.9 (Schlaefli's Formula for the Legendre Polynomials) Rodrigues showed (section 9.2) that the Legendre polynomial $P_{n}(x)$ is the $n$th derivative

$$
\begin{equation*}
P_{n}(x)=\frac{1}{2^{n} n!}\left(\frac{d}{d x}\right)^{n}\left(x^{2}-1\right)^{n} . \tag{6.45}
\end{equation*}
$$

Schlaefli used this expression and Cauchy's integral formula (6.44) to represent $P_{n}(z)$ as the contour integral (exercise 6.9)

$$
\begin{equation*}
P_{n}(z)=\frac{1}{2^{n} 2 \pi i} \oint \frac{\left(z^{\prime 2}-1\right)^{n}}{\left(z^{\prime}-z\right)^{n+1}} d z^{\prime} \tag{6.46}
\end{equation*}
$$

in which the contour encircles the complex point $z$ counterclockwise. This formula tells us that at $z=1$ the Legendre polynomial is

$$
\begin{equation*}
P_{n}(1)=\frac{1}{2^{n} 2 \pi i} \oint \frac{\left(z^{\prime 2}-1\right)^{n}}{\left(z^{\prime}-1\right)^{n+1}} d z^{\prime}=\frac{1}{2^{n} 2 \pi i} \oint \frac{\left(z^{\prime}+1\right)^{n}}{\left(z^{\prime}-1\right)} d z^{\prime}=1 \tag{6.47}
\end{equation*}
$$

in which we applied Cauchy's integral formula (6.40) to $f(z)=(z+1)^{n}$.
Example 6.10 (Bessel Functions of the First Kind) The counterclockwise integral around the unit circle $z=e^{i \theta}$ of the ratio $z^{m} / z^{n}$ in which both $m$ and $n$ are integers is

$$
\begin{equation*}
\frac{1}{2 \pi i} \oint d z \frac{z^{m}}{z^{n}}=\frac{1}{2 \pi i} \int_{0}^{2 \pi} i e^{i \theta} d \theta e^{i(m-n) \theta}=\frac{1}{2 \pi} \int_{0}^{2 \pi} d \theta e^{i(m+1-n) \theta} \tag{6.48}
\end{equation*}
$$

If $m+1-n \neq 0$, this integral vanishes because $\exp 2 \pi i(m+1-n)=1$

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{0}^{2 \pi} d \theta e^{i(m+1-n) \theta}=\frac{1}{2 \pi}\left[\frac{e^{i(m+1-n) \theta}}{i(m+1-n)}\right]_{0}^{2 \pi}=0 \tag{6.49}
\end{equation*}
$$

If $m+1-n=0$, the exponential is unity $\exp i(m+1-n) \theta=1$, and the integral is $2 \pi / 2 \pi=1$. Thus the original integral is the Kronecker delta

$$
\begin{equation*}
\frac{1}{2 \pi i} \oint d z \frac{z^{m}}{z^{n}}=\delta_{m+1, n} \tag{6.50}
\end{equation*}
$$

The generating function (10.5) for Bessel functions $J_{m}$ of the first kind is

$$
\begin{equation*}
e^{t(z-1 / z) / 2}=\sum_{m=-\infty}^{\infty} z^{m} J_{m}(t) \tag{6.51}
\end{equation*}
$$

Applying our integral formula (6.50) to it, we find

$$
\begin{align*}
\frac{1}{2 \pi i} \oint d z e^{t(z-1 / z) / 2} \frac{1}{z^{n+1}} & =\frac{1}{2 \pi i} \oint d z \sum_{m=-\infty}^{\infty} \frac{z^{m}}{z^{n+1}} J_{m}(t)  \tag{6.52}\\
& =\sum_{m=-\infty}^{\infty} \delta_{m+1, n+1} J_{m}(t)=J_{n}(t)
\end{align*}
$$

Thus letting $z=e^{i \theta}$, we have

$$
\begin{equation*}
J_{n}(t)=\frac{1}{2 \pi} \int_{0}^{2 \pi} d \theta \exp \left[\frac{1}{2} t\left(e^{i \theta}-e^{-i \theta}\right)-i n \theta\right] \tag{6.53}
\end{equation*}
$$

or more simply

$$
\begin{equation*}
J_{n}(t)=\frac{1}{2 \pi} \int_{0}^{2 \pi} d \theta e^{i(t \sin \theta-n \theta)}=\frac{1}{\pi} \int_{0}^{\pi} d \theta \cos (t \sin \theta-n \theta) \tag{6.54}
\end{equation*}
$$

(exercise 6.4).

### 6.5 Harmonic Functions

The Cauchy-Riemann conditions (6.10)

$$
\begin{equation*}
u_{x}=v_{y} \quad \text { and } \quad u_{y}=-v_{x} \tag{6.55}
\end{equation*}
$$

tell us about the laplacian of the real and imaginary parts of an analytic function $f=u+i v$. The second $x$-derivative $u_{x x}$ of the real part $u$ is $u_{x x}=v_{y x}=v_{x y}=-u_{y y}$. So the real part $u$ of an analytic function $f$ is a harmonic function

$$
\begin{equation*}
u_{x x}+u_{y y}=0 \tag{6.56}
\end{equation*}
$$

that is, one with a vanishing laplacian. Similarly $v_{x x}=-u_{y x}=-v_{y y}$, so the imaginary part $v$ of an analytic function also is a harmonic function

$$
\begin{equation*}
v_{x x}+v_{y y}=0 \tag{6.57}
\end{equation*}
$$

A harmonic function $h(x, y)$ can have saddle points, but not local minima or maxima because at a local minimum both $h_{x x}>0$ and $h_{y y}>0$, while at a local maximum both $h_{x x}<0$ and $h_{y y}<0$. So in its domain of analyticity, the real and imaginary parts of an analytic function $f$ have neither minima nor maxima.

For static fields, the electrostatic potential $\phi(x, y, z)$ is a harmonic function of the three spatial variables $x, y$, and $z$ in regions that are free of charge because the electric field is $\mathbf{E}=-\nabla \phi$, and its divergence vanishes $\nabla \cdot \mathbf{E}=0$ where the charge density is zero. Thus the laplacian of the electrostatic potential $\phi(x, y, z)$ vanishes

$$
\begin{equation*}
\nabla \cdot \nabla \phi=\phi_{x x}+\phi_{y y}+\phi_{z z}=0 \tag{6.58}
\end{equation*}
$$

and $\phi(x, y, z)$ is harmonic where there is no charge. The location of each positive charge is a local maximum of the electrostatic potential $\phi(x, y, z)$ and the location of each negative charge is a local minimum of $\phi(x, y, z)$. But in the absence of charges, the electrostatic potential has neither local maxima nor local minima. Thus one cannot trap charged particles with an electrostatic potential, a result known as Earnshaw's theorem.

The Cauchy-Riemann conditions imply that the real and imaginary parts of an analytic function are harmonic functions with two-dimensional gradients that are mutually perpendicular

$$
\begin{equation*}
\left(u_{x}, u_{y}\right) \cdot\left(v_{x}, v_{y}\right)=u_{x} v_{x}+u_{y} v_{y}=v_{y} v_{x}-v_{x} v_{y}=0 \tag{6.59}
\end{equation*}
$$

In regions with no charge, the electrostatic potential is a harmonic function. So the real part $u(x, y)$ (or the imaginary part $v(x, y)$ ) of any analytic function $f(z)=u(x, y)+i v(x, y)$ describes the electrostatic potential $\phi(x, y)$ for some electrostatic problem that does not involve the third spatial coordinate $z$. If the surfaces of constant $u(x, y)$ are equipotential surfaces, then since the two gradients are orthogonal, the surfaces of constant $v(x, y)$ are the electric field lines.

Example 6.11 (Two-dimensional potentials) The function

$$
\begin{equation*}
f(z)=u+i v=E z=E x+i E y \tag{6.60}
\end{equation*}
$$

can represent a potential $V(x, y, z)=E x$ for which the electric-field lines $\boldsymbol{E}=-E \hat{\boldsymbol{x}}$ are lines of constant $y$. It also can represent a potential $V(x, y, z)=$ $E y$ in which $\boldsymbol{E}$ points in the negative $y$-direction, which is to say along lines of constant $x$.

Another simple example is the function

$$
\begin{equation*}
f(z)=u+i v=z^{2}=x^{2}-y^{2}+2 i x y \tag{6.61}
\end{equation*}
$$

for which $u=x^{2}-y^{2}$ and $v=2 x y$. This function gives us a potential $V(x, y, z)$ whose equipotentials are the hyperbolas $u=x^{2}-y^{2}=c^{2}$ and whose electric-field lines are the perpendicular hyperbolas $v=2 x y=d^{2}$. Equivalently, we may take these last hyperbolas $2 x y=d^{2}$ to be the equipotentials and the other ones $x^{2}-y^{2}=c^{2}$ to be the lines of the electric field.

For a third example, we write the variable $z$ as $z=r e^{i \theta}=\exp (\log r+i \theta)$ and use the function

$$
\begin{equation*}
f(z)=u(x, y)+i v(x, y)=-\frac{\lambda}{2 \pi \epsilon_{0}} \log z=-\frac{\lambda}{2 \pi \epsilon_{0}}(\log r+i \theta) \tag{6.62}
\end{equation*}
$$

which describes the potential $V(x, y, z)=-\left(\lambda / 2 \pi \epsilon_{0}\right) \log \sqrt{x^{2}+y^{2}}$ due to a line of charge per unit length $\lambda=q / L$. The electric-field lines are the lines of constant $v$

$$
\begin{equation*}
\boldsymbol{E}=\frac{\lambda}{2 \pi \epsilon_{0}} \frac{(x, y, 0)}{x^{2}+y^{2}} \tag{6.63}
\end{equation*}
$$

or equivalently of constant $\theta$.

### 6.6 Taylor Series for Analytic Functions

Let's consider the contour integral of the function $f\left(z^{\prime}\right) /\left(z^{\prime}-z\right)$ along a circle $\mathcal{C}$ inside a simply connected region $\mathcal{R}$ in which $f(z)$ is analytic. For any point $z$ inside the circle, Cauchy's integral formula (6.40) tells us that

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \oint_{\mathcal{C}} \frac{f\left(z^{\prime}\right)}{z^{\prime}-z} d z^{\prime} \tag{6.64}
\end{equation*}
$$

We add and subtract the center $z_{0}$ from the denominator $z^{\prime}-z$

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \oint_{\mathcal{C}} \frac{f\left(z^{\prime}\right)}{z^{\prime}-z_{0}-\left(z-z_{0}\right)} d z^{\prime} \tag{6.65}
\end{equation*}
$$

and then factor the denominator

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \oint_{\mathcal{C}} \frac{f\left(z^{\prime}\right)}{\left(z^{\prime}-z_{0}\right)\left(1-\frac{z-z_{0}}{z^{\prime}-z_{0}}\right)} d z^{\prime} \tag{6.66}
\end{equation*}
$$

From Fig. 6.4, we see that the modulus of the ratio $\left(z-z_{0}\right) /\left(z^{\prime}-z_{0}\right)$ is less than unity, and so the power series

$$
\begin{equation*}
\left(1-\frac{z-z_{0}}{z^{\prime}-z_{0}}\right)^{-1}=\sum_{n=0}^{\infty}\left(\frac{z-z_{0}}{z^{\prime}-z_{0}}\right)^{n} \tag{6.67}
\end{equation*}
$$



Figure 6.4 Contour of integral for the Taylor series (6.69).
by (5.31-5.33) converges absolutely and uniformly on the circle. We therefore are allowed to integrate the series

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \oint_{\mathcal{C}} \frac{f\left(z^{\prime}\right)}{z^{\prime}-z_{0}} \sum_{n=0}^{\infty}\left(\frac{z-z_{0}}{z^{\prime}-z_{0}}\right)^{n} d z^{\prime} \tag{6.68}
\end{equation*}
$$

term by term

$$
\begin{equation*}
f(z)=\sum_{n=0}^{\infty}\left(z-z_{0}\right)^{n} \frac{1}{2 \pi i} \oint_{\mathcal{C}} \frac{f\left(z^{\prime}\right) d z^{\prime}}{\left(z^{\prime}-z_{0}\right)^{n+1}} . \tag{6.69}
\end{equation*}
$$

Cauchy's integral formula (6.44) tells us that the integral is just the $n$th derivative $f^{(n)}(z)$ divided by $n$-factorial. Thus the function $f(z)$ possesses the Taylor series

$$
\begin{equation*}
f(z)=\sum_{n=0}^{\infty} \frac{\left(z-z_{0}\right)^{n}}{n!} f^{(n)}\left(z_{0}\right) \tag{6.70}
\end{equation*}
$$

which converges as long as the point $z$ is inside a circle centered at $z_{0}$ that lies within a simply connected region $\mathcal{R}$ in which $f(z)$ is analytic.

### 6.7 Cauchy's Inequality

Suppose a function $f(z)$ is analytic in a region that includes the disk $|z| \leq R$ and that $f(z)$ is bounded by $|f(z)| \leq M$ on the circle $|z|=R$ that is the perimeter of the disk. Then by using Cauchy's integral formula (6.44), we may bound the $n$th derivative $f^{(n)}(0)$ of $f(z)$ at $z=0$ by

$$
\begin{align*}
\left|f^{(n)}(0)\right| & \leq \frac{n!}{2 \pi} \oint \frac{|f(z)||d z|}{|z|^{n+1}} \\
& \leq \frac{n!M}{2 \pi} \int_{0}^{2 \pi} \frac{R d \theta}{R^{n+1}}=\frac{n!M}{R^{n}} \tag{6.71}
\end{align*}
$$

which is Cauchy's inequality. This inequality bounds the terms of the Taylor series (6.70)

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{\left|z-z_{0}\right|^{n}}{n!}\left|f^{(n)}\left(z_{0}\right)\right| \leq M \sum_{n=0}^{\infty} \frac{\left|z-z_{0}\right|^{n}}{R^{n}} \tag{6.72}
\end{equation*}
$$

showing that it converges (5.33) absolutely and uniformly for $\left|z-z_{0}\right|<R$.

### 6.8 Liouville's Theorem

Suppose now that $f(z)$ is analytic everywhere (entire) and bounded by

$$
\begin{equation*}
|f(z)| \leq M \quad \text { for all } \quad|z| \geq R_{0} \tag{6.73}
\end{equation*}
$$

Then by applying Cauchy's inequality (6.71) at successively larger values of $R$, we have

$$
\begin{equation*}
\left|f^{(n)}(0)\right| \leq \lim _{R \rightarrow \infty} \frac{n!M}{R^{n}}=0 \tag{6.74}
\end{equation*}
$$

which shows that for $n \geq 1$ every derivative $f^{(n)}(z)$ vanishes at $z=0$

$$
\begin{equation*}
f^{(n)}(0)=0 \tag{6.75}
\end{equation*}
$$

But then the Taylor series (5.79) about $z=0$ for the function $f(z)$ consists of only a single term, and $f(z)$ is a constant

$$
\begin{equation*}
f(z)=\sum_{n=0}^{\infty} \frac{z^{n}}{n!} f^{(n)}(0)=f^{(0)}(0)=f(0) \tag{6.76}
\end{equation*}
$$

So every bounded entire function is a constant (Joseph Liouville, 1809-1882).

### 6.9 Fundamental Theorem of Algebra

Gauss applied Liouville's theorem to the function

$$
\begin{equation*}
f(z)=\frac{1}{P_{n}(z)}=\frac{1}{c_{0}+c_{1} z+c_{2} z^{2}+\ldots+c_{n} z^{n}} \tag{6.77}
\end{equation*}
$$

which is the inverse of an arbitrary polynomial of order $n$. Suppose that the polynomial $P_{n}(z)$ had no zero, that is, no root anywhere in the complex plane. Then $f(z)$ would be analytic everywhere. Moreover, for sufficiently large $|z|$, the polynomial $P_{n}(z)$ is approximately $P_{n}(z) \approx c_{n} z^{n}$, and so $f(z)$ would be bounded by something like

$$
\begin{equation*}
|f(z)| \leq \frac{1}{\left|c_{n}\right| R_{0}^{n}} \equiv M \quad \text { for all } \quad|z| \geq R_{0} \tag{6.78}
\end{equation*}
$$

So if $P_{n}(z)$ had no root, then the function $f(z)$ would be a bounded entire function and so would be a constant by Liouville's theorem (6.76). But of course, $f(z)=1 / P_{n}(z)$ is not a constant unless $n=0$. Thus any polynomial $P_{n}(z)$ that is not a constant must have a root, a pole of $f(z)$, so that $f(z)$ is not entire.

If the root of $P_{n}(z)$ is at $z=z_{1}$, then $P_{n}(z)=\left(z-z_{1}\right) P_{n-1}(z)$, in which $P_{n-1}(z)$ is a polynomial of order $n-1$, and we may repeat the argument for its reciprocal $f_{1}(z)=1 / P_{n-1}(z)$. In this way, one arrives at the fundamental theorem of algebra: Every polynomial $P_{n}(z)=c_{0}+c_{1} z+\ldots+c_{n} z^{n}$ has $n$ roots somewhere in the complex plane

$$
\begin{equation*}
P_{n}(z)=c_{n}\left(z-z_{1}\right)\left(z-z_{2}\right) \ldots\left(z-z_{n}\right) . \tag{6.79}
\end{equation*}
$$

### 6.10 Laurent Series

Consider a function $f(z)$ that is analytic in an annulus that contains an outer circle $\mathcal{C}_{1}$ of radius $R_{1}$ and an inner circle $\mathcal{C}_{2}$ of radius $R_{2}$ as in Fig. 6.5. We integrate $f(z)$ along a contour $\mathcal{C}_{12}$ within the annulus that encircles the point $z$ in a counterclockwise fashion by following $\mathcal{C}_{1}$ counterclockwise and $\mathcal{C}_{2}$ clockwise and a line joining them in both directions. By Cauchy's integral formula (6.40), this contour integral yields $f(z)$

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \oint_{\mathcal{C}_{12}} \frac{f\left(z^{\prime}\right)}{z^{\prime}-z} d z^{\prime} . \tag{6.80}
\end{equation*}
$$

The integrations in opposite directions along the line joining $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ cancel, and we are left with a counterclockwise integral around the outer circle $\mathcal{C}_{1}$


Figure 6.5 A contour consisting of two concentric circles with center at $z_{0}$ encircles the point $z$ in a counterclockwise sense. The asterisks are poles or other singularities of the function $f(z)$.
and a clockwise one around $\mathcal{C}_{2}$ or minus a counterclockwise integral around $\mathcal{C}_{2}$

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \oint_{\mathcal{C}_{1}} \frac{f\left(z^{\prime}\right)}{z^{\prime}-z} d z^{\prime}-\frac{1}{2 \pi i} \oint_{\mathcal{C}_{2}} \frac{f\left(z^{\prime \prime}\right)}{z^{\prime \prime}-z} d z^{\prime \prime} \tag{6.81}
\end{equation*}
$$

Now the figure (6.5) shows that the center $z_{0}$ of the two concentric circles is closer to the points $z^{\prime \prime}$ on the inner circle $\mathcal{C}_{2}$ than it is to $z$; it also shows that $z_{0}$ is closer to $z$ than to the points $z^{\prime}$ on $\mathcal{C}_{1}$

$$
\begin{equation*}
\left|\frac{z^{\prime \prime}-z_{0}}{z-z_{0}}\right|<1 \quad \text { and } \quad\left|\frac{z-z_{0}}{z^{\prime}-z_{0}}\right|<1 . \tag{6.82}
\end{equation*}
$$

We add and subtract $z_{0}$ from each of the denominators in (6.81) and absorb the minus sign before the second integral into its denominator

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \oint_{\mathcal{C}_{1}} \frac{f\left(z^{\prime}\right)}{z^{\prime}-z_{0}-\left(z-z_{0}\right)} d z^{\prime}+\frac{1}{2 \pi i} \oint_{\mathcal{C}_{2}} \frac{f\left(z^{\prime \prime}\right)}{z-z_{0}-\left(z^{\prime \prime}-z_{0}\right)} d z^{\prime \prime} \tag{6.83}
\end{equation*}
$$

After factoring the two denominators

$$
\begin{align*}
f(z)= & \frac{1}{2 \pi i} \oint_{\mathcal{C}_{1}} \frac{f\left(z^{\prime}\right)}{\left(z^{\prime}-z_{0}\right)\left[1-\left(z-z_{0}\right) /\left(z^{\prime}-z_{0}\right)\right]} d z^{\prime} \\
& +\frac{1}{2 \pi i} \oint_{\mathcal{C}_{2}} \frac{f\left(z^{\prime \prime}\right)}{\left(z-z_{0}\right)\left[1-\left(z^{\prime \prime}-z_{0}\right) /\left(z-z_{0}\right)\right]} d z^{\prime \prime} \tag{6.84}
\end{align*}
$$

we expand them in power series (6.68) that converge absolutely and uniformly on the two contours

$$
\begin{align*}
f(z)= & \sum_{n=0}^{\infty}\left(z-z_{0}\right)^{n} \frac{1}{2 \pi i} \oint_{\mathcal{C}_{1}} \frac{f\left(z^{\prime}\right)}{\left(z^{\prime}-z_{0}\right)^{n+1}} d z^{\prime} \\
& +\sum_{m=0}^{\infty} \frac{1}{\left(z-z_{0}\right)^{m+1}} \frac{1}{2 \pi i} \oint_{\mathcal{C}_{2}}\left(z^{\prime \prime}-z_{0}\right)^{m} f\left(z^{\prime \prime}\right) d z^{\prime \prime} . \tag{6.85}
\end{align*}
$$

Since the functions being integrated are analytic between the two circles, we may move the contours, without changing the values of the integrals, to a common counterclockwise contour $\mathcal{C}$ about any circle of radius $R_{2} \leq R \leq R_{1}$ between the two circles $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$. We then set $m=-n-1$ so as to combine the two sums into one sum on $n$ from $-\infty$ to $\infty$

$$
\begin{equation*}
f(z)=\sum_{n=-\infty}^{\infty}\left(z-z_{0}\right)^{n} \frac{1}{2 \pi i} \oint_{\mathcal{C}} \frac{f\left(z^{\prime}\right)}{\left(z^{\prime}-z_{0}\right)^{n+1}} d z^{\prime} \tag{6.86}
\end{equation*}
$$

This Laurent series often is written as

$$
\begin{equation*}
f(z)=\sum_{n=-\infty}^{\infty} a_{n}\left(z_{0}\right)\left(z-z_{0}\right)^{n} \tag{6.87}
\end{equation*}
$$

with

$$
\begin{equation*}
a_{n}\left(z_{0}\right)=\frac{1}{2 \pi i} \oint_{\mathcal{C}} \frac{f(z)}{\left(z-z_{0}\right)^{n+1}} d z \tag{6.88}
\end{equation*}
$$

(Pierre Laurent, 1813-1854).
The coefficient $a_{-1}\left(z_{0}\right)$ is called the residue of the function $f(z)$ at $z_{0}$. Its significance will be discussed in section 6.13. Useful functions typically have Laurent series that start at some least integer $-\ell$

$$
\begin{equation*}
f(z)=\sum_{n=-\ell}^{\infty} a_{n}\left(z_{0}\right)\left(z-z_{0}\right)^{n} \tag{6.89}
\end{equation*}
$$

rather than at $-\infty$. For such functions, we can find the coefficients $a_{n}$ one by one without doing the integrals (6.88). The first one $a_{-\ell}$ is the limit

$$
\begin{equation*}
a_{-\ell}\left(z_{0}\right)=\lim _{z \rightarrow z_{0}}\left(z-z_{0}\right)^{\ell} f(z) \tag{6.90}
\end{equation*}
$$

The second Laurent coefficient $a_{-\ell+1}\left(z_{0}\right)$ is given by the recipe

$$
\begin{equation*}
a_{-\ell+1}\left(z_{0}\right)=\lim _{z \rightarrow z_{0}}\left(z-z_{0}\right)^{\ell-1}\left[f(z)-\left(z-z_{0}\right)^{-\ell} a_{-\ell}\left(z_{0}\right)\right] . \tag{6.91}
\end{equation*}
$$

The third coefficient requires a recipe with two subtractions, and so forth.

Example 6.12 (Laurent series for $f(z)=1 /(\exp (z)-1))$ The Matlab commands syms $z$; series $(1 /(\exp (z)-1))$ give

$$
\begin{equation*}
\frac{z}{12}+\frac{1}{z}-\frac{z^{3}}{720}-\frac{1}{2} \tag{6.92}
\end{equation*}
$$

while syms $z$; series $(1 /(\exp (z)-1), z$,'ExpansionPoint', 0 , 'Order', 10$)$ gives

$$
\begin{equation*}
\frac{z}{12}+\frac{1}{z}-\frac{z^{3}}{720}+\frac{z^{5}}{30240}-\frac{z^{7}}{1209600}+\frac{z^{9}}{47900160}-\frac{1}{2} \tag{6.93}
\end{equation*}
$$

The Mathematica command Series $[1 /(\operatorname{Exp}[z]-1), z, 0,9]$ gives

$$
\begin{equation*}
\frac{1}{z}-\frac{1}{2}+\frac{z}{12}-\frac{z^{3}}{720}+\frac{z^{5}}{30240}-\frac{z^{7}}{1209600}+\frac{z^{9}}{47900160}+O\left(z^{10}\right) \tag{6.94}
\end{equation*}
$$

Newton invented series with fractional exponents. The Matlab command syms z; series $\left(1 /\left(\exp \left(z^{1 / 3}\right)-1\right), z\right.$,'Order', 6 ) gives the Puiseux series $1 / z^{1 / 3}-z / 720+z^{1 / 3} / 12+z^{5 / 3} / 30240-1 / 2$. (Isaac Newton, 1643-1727)

### 6.11 Singularities

A function $f(z)$ that is analytic for all $z$ is entire. Entire functions have no singularities but, unless they are constants, they diverge as the real or imaginary part of $z$ goes to infinity. Some call $|z|=\infty$ the point at infinity.

A function $f(z)$ has an isolated singularity at $z_{0}$ if it is analytic in a small disk about $z_{0}$ but not analytic at that point.

A function $f(z)$ has a pole of order $n>0$ at a point $z_{0}$ if $\left(z-z_{0}\right)^{n} f(z)$ is analytic at $z_{0}$ but $\left(z-z_{0}\right)^{n-1} f(z)$ has an isolated singularity at $z_{0}$. A pole of order $n=1$ is called a simple pole. Poles are isolated singularities. A function is meromorphic if it is analytic for all $z$ except for poles.

Example 6.13 (Poles) The function

$$
\begin{equation*}
f(z)=\prod_{j=1}^{n} \frac{1}{(z-j)^{j}} \tag{6.95}
\end{equation*}
$$

has a pole of order $j$ at $z=j$ for $j=1,2, \ldots, n$. It is meromorphic.

An essential singularity is a pole of infinite order. If a function $f(z)$ has an essential singularity at $z_{0}$, then its Laurent series (6.86) really runs from $n=-\infty$ and not from $n=-\ell$ as in (6.89). Essential singularities are spooky: if a function $f(z)$ has an essential singularity at $w$, then inside every disk around $w, f(z)$ takes on every complex number, with at most one exception, an infinite number of times (Émile Picard, 1856-1941).

Example 6.14 (Essential singularity) The function $f(z)=\exp (1 / z)$ has an essential singularity at $z=0$ because its Laurent series (6.86)

$$
\begin{equation*}
f(z)=e^{1 / z}=\sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{z^{m}}=\sum_{n=-\infty}^{0} \frac{1}{|n|!} z^{n} \tag{6.96}
\end{equation*}
$$

runs from $n=-\infty$. Near $z=0, f(z)=\exp (1 / z)$ takes on every complex number except 0 an infinite number of times.

Example 6.15 (Meromorphic function with two poles) The function $f(z)=$ $1 / z(z+1)$ has simple poles at $z=0$ and at $z=-1$ but otherwise is analytic; it is meromorphic. We may expand it in a Laurent series (6.87-6.88)

$$
\begin{equation*}
f(z)=\frac{1}{z(z+1)}=\sum_{n=-\infty}^{\infty} a_{n} z^{n} \tag{6.97}
\end{equation*}
$$

about $z=0$ for $|z|<1$. The coefficient $a_{n}$ is the integral

$$
\begin{equation*}
a_{n}=\frac{1}{2 \pi i} \oint_{\mathcal{C}} \frac{d z}{z^{n+2}(z+1)} \tag{6.98}
\end{equation*}
$$

in which the contour $\mathcal{C}$ is a counterclockwise circle of radius $r<1$. Since $|z|<1$, we may expand $1 /(1+z)$ as the power series

$$
\begin{equation*}
\frac{1}{1+z}=\sum_{m=0}^{\infty}(-z)^{m} \tag{6.99}
\end{equation*}
$$

Doing the integrals, we find

$$
\begin{equation*}
a_{n}=\sum_{m=0}^{\infty} \frac{1}{2 \pi i} \oint_{\mathcal{C}}(-z)^{m} \frac{d z}{z^{n+2}}=\sum_{m=0}^{\infty}(-1)^{m} r^{m-n-1} \delta_{m, n+1} \tag{6.100}
\end{equation*}
$$

for $n \geq-1$ and zero otherwise. Thus the Laurent series about $z=0$ for $f(z)$ is

$$
\begin{equation*}
f(z)=\frac{1}{z(z+1)}=\sum_{n=-1}^{\infty}(-1)^{n+1} z^{n} . \tag{6.101}
\end{equation*}
$$

It starts at $n=-1$, not at $n=-\infty$, because $f(z)$ is meromorphic with only a simple pole at $z=0$.

Example 6.16 (Argument principle) Consider the counterclockwise integral

$$
\begin{equation*}
\frac{1}{2 \pi i} \oint_{\mathcal{C}} f(z) \frac{g^{\prime}(z)}{g(z)} d z \tag{6.102}
\end{equation*}
$$

along a closed contour $\mathcal{C}$ that lies inside a simply connected region $R$ in which $f(z)$ is analytic and $g(z)$ meromorphic. If the function $g(z)$ has a zero or a pole of order $n$ at $z=w \in R$

$$
\begin{equation*}
g(z)=a_{n}(w)(z-w)^{n}, \tag{6.103}
\end{equation*}
$$

then the ratio $g^{\prime} / g$ is

$$
\begin{equation*}
\frac{g^{\prime}(z)}{g(z)}=\frac{n(z-w)^{n-1}}{(z-w)^{n}}=\frac{n}{z-w} \tag{6.104}
\end{equation*}
$$

and the integral is

$$
\begin{equation*}
\frac{1}{2 \pi i} \oint_{\mathcal{C}} f(z) \frac{g^{\prime}(z)}{g(z)} d z=\frac{1}{2 \pi i} \oint_{\mathcal{C}} f(z) \frac{n}{z-w} d z=n f(w) \tag{6.105}
\end{equation*}
$$

Any function $g(z)$ meromorphic in $R$ will have a Laurent series

$$
\begin{equation*}
g(z)=\sum_{k=n}^{\infty} a_{k}(w)(z-w)^{k} \tag{6.106}
\end{equation*}
$$

about each point $w \in R$. One may show (exercise 6.19) that as $z \rightarrow w$ the ratio $g^{\prime} / g$ again approaches (6.104). It follows that the integral (6.102) is a sum of $n_{\ell} f\left(w_{\ell}\right)$ at the zeros and poles of $g(z)$ that lie within the contour $\mathcal{C}$

$$
\begin{equation*}
\frac{1}{2 \pi i} \oint_{\mathcal{C}} f(z) \frac{g^{\prime}(z)}{g(z)} d z=\sum_{\ell} \frac{1}{2 \pi i} \oint_{\mathcal{C}} f(z) \frac{n_{\ell}}{z-w_{\ell}}=\sum_{\ell} n_{\ell} f\left(w_{\ell}\right) \tag{6.107}
\end{equation*}
$$

in which $\left|n_{\ell}\right|$ is the multiplicity of the $\ell$ th zero or pole.

### 6.12 Analytic Continuation

We saw in Sec. 6.6 that a function $f(z)$ that is analytic within a circle of radius $R$ about a point $z_{0}$ possesses a Taylor series (6.70)

$$
\begin{equation*}
f(z)=\sum_{n=0}^{\infty} \frac{\left(z-z_{0}\right)^{n}}{n!} f^{(n)}\left(z_{0}\right) \tag{6.108}
\end{equation*}
$$

that converges for all $z$ inside the disk $\left|z-z_{0}\right|<R$. Suppose $z^{\prime}$ is the singularity of $f(z)$ that is closest to $z_{0}$. Pick a point $z_{1}$ in the disk $\left|z-z_{0}\right|<R$ that is not on the line from $z_{0}$ to the nearest singularity $z^{\prime}$. The function $f(z)$ is analytic at $z_{1}$ because $z_{1}$ is within the circle of radius $R$ about the point $z_{0}$, and so $f(z)$ has a Taylor series expansion like (6.108) but about the point $z_{1}$. Often the circle of convergence of this power series about $z_{1}$ will extend beyond the original disk $\left|z-z_{0}\right|<R$. If so, the two power series, one about $z_{0}$ and the other about $z_{1}$, define the function $f(z)$ and extend its domain of analyticity beyond the original disk $\left|z-z_{0}\right|<R$. Such an extension of the range of an analytic function is an analytic continuation.

Example 6.17 (Geometric series) The power series

$$
\begin{equation*}
f(z)=\sum_{n=0}^{\infty} z^{n} \tag{6.109}
\end{equation*}
$$

converges and defines an analytic function for $|z|<1$. But for such $z$, we may sum the series to

$$
\begin{equation*}
f(z)=\frac{1}{1-z} \tag{6.110}
\end{equation*}
$$

By summing the series (6.109), we have analytically continued the function $f(z)$ to the whole complex plane apart from its simple pole at $z=1$.

Example 6.18 (Gamma function) Euler's form of the gamma function is the integral

$$
\begin{equation*}
\Gamma(z)=\int_{0}^{\infty} e^{-t} t^{z-1} d t=(z-1)! \tag{6.111}
\end{equation*}
$$

which makes $\Gamma(z)$ analytic in the right half-plane $\operatorname{Re} z>0$. But by successively using the relation $\Gamma(z+1)=z \Gamma(z)$, we may extend $\Gamma(z)$ into the left half-plane

$$
\begin{equation*}
\Gamma(z)=\frac{1}{z} \Gamma(z+1)=\frac{1}{z} \frac{1}{z+1} \Gamma(z+2)=\frac{1}{z} \frac{1}{z+1} \frac{1}{z+2} \Gamma(z+3) \tag{6.112}
\end{equation*}
$$

The last expression defines $\Gamma(z)$ as a function that is analytic for $\operatorname{Re} z>-3$ apart from simple poles at $z=0,-1$, and -2 . Proceeding in this way, we may analytically continue the gamma function to the whole complex plane apart from the negative integers and zero. The analytically continued gamma function is represented by Weierstrass's formula

$$
\begin{equation*}
\Gamma(z)=\frac{1}{z} e^{-\gamma z}\left[\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right) e^{-z / n}\right]^{-1} \tag{6.113}
\end{equation*}
$$

Example 6.19 (Riemann's zeta function) Ser found an analytic continuation

$$
\begin{equation*}
\zeta(z)=\sum_{n=1}^{\infty} \frac{1}{n^{z}}=\frac{1}{z-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n}\binom{n}{k} \frac{(-1)^{k}}{(k+1)^{z-1}} \tag{6.114}
\end{equation*}
$$

of Riemann's zeta function (5.107) to the whole complex plane except for the point $z=1$ (Joseph Ser, 1875-1954).

Example 6.20 (Dimensional regularization) The loop diagrams of quantum field theory involve badly divergent integrals like

$$
\begin{equation*}
I(4)=\int \frac{d^{4} q}{(2 \pi)^{4}} \frac{\left(q^{2}\right)^{a}}{\left(q^{2}+\alpha^{2}\right)^{b}} \tag{6.115}
\end{equation*}
$$

where often $a=0$ and $b=2$ and $\alpha^{2}>0$. Gerardus 't Hooft (1946-) and Martinus J. G. Veltman (1931-2021) promoted the number of spacetime dimensions from 4 to a complex number $d$. The resulting integral has the value (Srednicki, 2007, p. 102)

$$
\begin{equation*}
I(d)=\int \frac{d^{d} q}{(2 \pi)^{d}} \frac{\left(q^{2}\right)^{a}}{\left(q^{2}+\alpha^{2}\right)^{b}}=\frac{\Gamma(b-a-d / 2) \Gamma(a+d / 2)}{(4 \pi)^{d / 2} \Gamma(b) \Gamma(d / 2)} \frac{1}{\left(\alpha^{2}\right)^{b-a-d / 2}} \tag{6.116}
\end{equation*}
$$

and so defines a function of the complex variable $d$ that is analytic everywhere except for simple poles at $d=2(n-a+b)$ where $n=0,1,2, \ldots, \infty$. At these poles, the formula

$$
\begin{equation*}
\Gamma(-n+z)=\frac{(-1)^{n}}{n!}\left(\frac{1}{z}-\gamma+\sum_{k=1}^{n} \frac{1}{k}+O(z)\right) \tag{6.117}
\end{equation*}
$$

where $\gamma=0.5772 \ldots$ is the Euler-Mascheroni constant (5.8) can be useful.

### 6.13 Calculus of residues

A contour integral of an analytic function $f(z)$ does not change unless the end points move or the contour crosses a singularity or leaves the region of analyticity (section 6.3). Let us consider the integral of a function $f(z)$ along a counterclockwise contour $\mathcal{C}$ that encircles $n$ poles at $z_{k}$ for $k=1, \ldots, n$ in a simply connected region $\mathcal{R}$ in which $f(z)$ is meromorphic. We may shrink the area within the contour $\mathcal{C}$ without changing the value of the integral until
the area is infinitesimal and the contour is a sum of $n$ tiny counterclockwise circles $\mathcal{C}_{k}$ around the $n$ poles

$$
\begin{equation*}
\oint_{\mathcal{C}} f(z) d z=\sum_{k=1}^{n} \oint_{\mathcal{C}_{k}} f(z) d z \tag{6.118}
\end{equation*}
$$

These tiny counterclockwise integrals around the poles at $z_{i}$ are $2 \pi i$ times the residues $a_{-1}\left(z_{i}\right)$ defined by Laurent's formula (6.88) with $n=-1$. So the whole counterclockwise integral is $2 \pi i$ times the sum of the residues of the enclosed poles of the function $f(z)$

$$
\begin{equation*}
\oint_{\mathcal{C}} f(z) d z=2 \pi i \sum_{k=1}^{n} a_{-1}\left(z_{k}\right)=2 \pi i \sum_{k=1}^{n} \operatorname{Res}\left(f, z_{k}\right) \tag{6.119}
\end{equation*}
$$

a result that is known as the residue theorem.
Example 6.21 (Pole of order $n$ ) Setting $z=w+\epsilon e^{i \theta}$, we do a counterclockwise integral around a circle $\mathcal{C}_{w}$ of radius $\epsilon$ with center $w$

$$
\begin{align*}
& \oint_{\mathcal{C}_{w}} a_{n}(w)(z-w)^{n} d z=a_{n}(w) \int_{0}^{2 \pi}\left(\epsilon e^{i \theta}\right)^{n} i \epsilon e^{i \theta} d \theta \\
& =i a_{n}(w) \epsilon^{n+1} \int_{0}^{2 \pi} e^{i(n+1) \theta} d \theta=2 \pi i a_{-1}(w) \delta_{n,-1} \tag{6.120}
\end{align*}
$$

This is why only the $n=-1$ term $a_{-1}(w)$ of the Laurent series (6.86-6.88) for $f(z)$ contributes to the integral

$$
\begin{equation*}
\oint_{\mathcal{C}_{w}} f(z) d z=\oint_{\mathcal{C}_{w}} \sum_{n=-\infty}^{\infty} a_{n}(w)(z-w)^{n} d z=2 \pi i a_{-1}(w) \tag{6.121}
\end{equation*}
$$

In general, one must do each tiny counterclockwise integral about each pole $z_{i}$, but simple poles are an important special case. If $w$ is a simple pole of the function $f(z)$, then near it $f(z)$ is given by its Laurent series (6.87) as

$$
\begin{equation*}
f(z)=\frac{a_{-1}(w)}{z-w}+\sum_{n=0}^{\infty} a_{n}(w)(z-w)^{n} \tag{6.122}
\end{equation*}
$$

In this case, its residue is by (6.90) with $-\ell=-1$

$$
\begin{equation*}
a_{-1}(w)=\lim _{z \rightarrow w}(z-w) f(z) \tag{6.123}
\end{equation*}
$$

which usually is easier to do than the integral (6.88)

$$
\begin{equation*}
a_{-1}(w)=\frac{1}{2 \pi i} \oint_{\mathcal{C}} f(z) d z \tag{6.124}
\end{equation*}
$$

Example 6.22 (A function with simple poles) The integral of the function

$$
\begin{equation*}
f(z)=\sum_{n=1}^{\infty} \frac{z}{z-n^{-s}} \tag{6.125}
\end{equation*}
$$

along a circle of radius 2 with center at $z=0$ is just the sum of its residues

$$
\begin{equation*}
\frac{1}{2 \pi i} \oint f(z) d z=\sum_{n=1}^{\infty} \lim _{z \rightarrow n^{-s}}\left(z-n^{-s}\right) f(z)=\sum_{n=1}^{\infty} \frac{1}{n^{s}}=\zeta(s) \tag{6.126}
\end{equation*}
$$

which is the zeta function (5.107).
Example 6.23 (Cauchy's Integral Formula) Suppose the function $f(z)$ is analytic within a region $\mathcal{R}$ and that $\mathcal{C}$ is a counterclockwise contour that encircles a point $w \in \mathcal{R}$. Then the counterclockwise contour $\mathcal{C}$ encircles the simple pole at $w$ of the function $f(z) /(z-w)$, which is its only singularity in $\mathcal{R}$. By applying the residue theorem and formula (6.123) for the residue $a_{-1}(w)$ of the function $f(z) /(z-w)$, we find

$$
\begin{equation*}
\oint_{\mathcal{C}} \frac{f(z)}{z-w} d z=2 \pi i a_{-1}(w)=2 \pi i \lim _{z \rightarrow w}(z-w) \frac{f(z)}{z-w}=2 \pi i f(w) \tag{6.127}
\end{equation*}
$$

So Cauchy's integral formula (6.40) is an example of the calculus of residues.

Example 6.24 (A meromorphic function) By the residue theorem (6.119), the integral of the function

$$
\begin{equation*}
f(z)=\frac{1}{z-1} \frac{1}{(z-2)^{2}} \tag{6.128}
\end{equation*}
$$

along the circle $\mathcal{C}=4 e^{i \theta}$ for $0 \leq \theta \leq 2 \pi$ is the sum of the residues at $z=1$ and $z=2$

$$
\begin{equation*}
\oint_{\mathcal{C}} f(z) d z=2 \pi i\left[a_{-1}(1)+a_{-1}(2)\right] \tag{6.129}
\end{equation*}
$$

The function $f(z)$ has a simple pole at $z=1$, and so we may use the formula (6.123) to evaluate the residue $a_{-1}(1)$ as

$$
\begin{equation*}
a_{-1}(1)=\lim _{z \rightarrow 1}(z-1) f(z)=\lim _{z \rightarrow 1} \frac{1}{(z-2)^{2}}=1 \tag{6.130}
\end{equation*}
$$

instead of using Cauchy's integral formula (6.40) to do the integral of $f(z)$ along a tiny circle about $z=1$, which gives the same result

$$
\begin{equation*}
a_{-1}(1)=\frac{1}{2 \pi i} \oint \frac{d z}{z-1} \frac{1}{(z-2)^{2}}=\frac{1}{(1-2)^{2}}=1 \tag{6.131}
\end{equation*}
$$

The residue $a_{-1}(2)$ is the integral of $f(z)$ along a tiny circle about $z=2$, which we do by using Cauchy's integral formula (6.42)

$$
\begin{equation*}
a_{-1}(2)=\frac{1}{2 \pi i} \oint \frac{d z}{(z-2)^{2}} \frac{1}{z-1}=\left.\frac{d}{d z} \frac{1}{z-1}\right|_{z=2}=-\frac{1}{(2-1)^{2}}=-1 \tag{6.132}
\end{equation*}
$$

getting the same answer as if we had used the recipe (6.90) for $a_{-2}$

$$
\begin{equation*}
a_{-2}(2)=\lim _{z \rightarrow 2}(z-2)^{2} \frac{1}{(z-1)(z-2)^{2}}=1 \tag{6.133}
\end{equation*}
$$

and (6.91) for $a_{-1}$

$$
\begin{equation*}
a_{-1}(2)=\lim _{z \rightarrow 2}(z-2)\left[\frac{1}{(z-1)(z-2)^{2}}-\frac{a_{-2}(2)}{(z-2)^{2}}\right]=-1 \tag{6.134}
\end{equation*}
$$

The sum of the residues $a_{-1}(1)$ and $a_{-1}(2)$ is zero, and so the integral (6.129) vanishes. Another way of evaluating this integral is to deform it, not into two tiny circles about the two poles, but rather into a huge circle $z=R e^{i \theta}$ and to notice that as $R \rightarrow \infty$ the modulus of this integral vanishes

$$
\begin{equation*}
|\oint f(z) d z| \approx \frac{2 \pi}{R^{2}} \rightarrow 0 \tag{6.135}
\end{equation*}
$$

This contour is an example of a ghost contour.

### 6.14 Ghost contours

Often one needs to do an integral that is not a closed counterclockwise contour. Integrals along the real axis occur frequently. One sometimes can convert a line integral into a closed contour by adding a contour along which the integral vanishes, a ghost contour. We have just seen an example (6.135) of a ghost contour, and we shall see more of them in what follows.

Example 6.25 (Using ghost contours) Consider the integral

$$
\begin{equation*}
I=\int_{-\infty}^{\infty} \frac{1}{(x-i)(x-2 i)(x-3 i)} d x \tag{6.136}
\end{equation*}
$$

We could do the integral by adding a contour $R e^{i \theta}$ from $\theta=0$ to $\theta=\pi$. In the limit $R \rightarrow \infty$, the integral of $1 /[(z-i)(z-2 i)(z-3 i)]$ along this contour vanishes; it is a ghost contour. The original integral $I$ and the ghost contour encircle the three poles, and so we could compute $I$ by evaluating the residues at those poles. But we also could add a ghost contour around
the lower half plane. This contour and the real line encircle no poles. So we get $I=0$ without doing any work at all.

Example 6.26 (Fourier transform of a gaussian) During our computation of the Fourier transform of a gaussian (4.17-4.20), we promised to justify the shift in the variable of integration from $x$ to $x+i k / 2 m^{2}$ in this chapter. So let us consider the contour integral of the entire function $f(z)=\exp \left(-m^{2} z^{2}\right)$ over a rectangular closed contour along the real axis from $-R$ to $R$ and then from $z=R$ to $z=R+i c$ and then from there to $z=-R+i c$ and then to $z=-R$. Since $f(z)$ is analytic within the contour, the integral is zero

$$
\oint d z e^{-m^{2} z^{2}}=\int_{-R}^{R} d z e^{-m^{2} z^{2}}+\int_{R}^{R+i c} d z e^{-m^{2} z^{2}}+\int_{R+i c}^{-R+i c} d z e^{-m^{2} z^{2}}+\int_{-R+i c}^{-R} d z e^{-m^{2} z^{2}}=0
$$

for all finite positive values of $R$ and so also in the limit $R \rightarrow \infty$. The two contours in the imaginary direction are of length $c$ and are damped by the factor $\exp \left(-m^{2} R^{2}\right)$, and so they vanish in the limit $R \rightarrow \infty$. They are ghost contours. It follows then from this last equation in the limit $R \rightarrow \infty$ that

$$
\begin{equation*}
\int_{-\infty}^{\infty} d x e^{-m^{2}(x+i c)^{2}}=\int_{-\infty}^{\infty} d x e^{-m^{2} x^{2}}=\frac{\sqrt{\pi}}{m} \tag{6.137}
\end{equation*}
$$

which is the promised result (4.19). Setting $c=k /\left(2 m^{2}\right)$ and dividing both sides of (6.137) by $\sqrt{2 \pi} e^{m^{2} c^{2}}$, we see that the Fourier transform of a gaussian is a gaussian (4.20)

$$
\begin{equation*}
\tilde{f}(k)=\int_{-\infty}^{\infty} \frac{d x}{\sqrt{2 \pi}} e^{-i k x} e^{-m^{2} x^{2}}=\frac{1}{\sqrt{2} m} e^{-k^{2} / 4 m^{2}} \tag{6.138}
\end{equation*}
$$

Dividing both sides of this formula by $\sqrt{2 \pi}$ and setting $x=p / \hbar, k=-\epsilon \dot{q}$, and $m^{2}=\epsilon \hbar^{2} /(2 m)$, we get

$$
\begin{equation*}
\int_{-\infty}^{\infty} \exp \left(-\epsilon \frac{p^{2}}{2 m}+i \epsilon \frac{\dot{q} p}{\hbar}\right) \frac{d p}{2 \pi \hbar}=\sqrt{\frac{m}{2 \pi \epsilon \hbar^{2}}} \exp \left(-\epsilon \frac{m \dot{q}^{2}}{2 \hbar^{2}}\right) \tag{6.139}
\end{equation*}
$$

a formula we'll use in section 20.5 to derive path integrals for partition functions.

The earlier relation (6.137) implies (exercise 6.22) that

$$
\begin{equation*}
\int_{-\infty}^{\infty} d x e^{-m^{2}(x+z)^{2}}=\int_{-\infty}^{\infty} d x e^{-m^{2} x^{2}}=\frac{\sqrt{\pi}}{m} \tag{6.140}
\end{equation*}
$$

for $m>0$ and arbitrary complex $z$.
Example 6.27 (A cosine integral) To compute the integral

$$
\begin{equation*}
I_{c}=\int_{0}^{\infty} \frac{\cos x}{q^{2}+x^{2}} d x, \quad q>0, \tag{6.141}
\end{equation*}
$$

we use the evenness of the integrand to extend the integration

$$
\begin{equation*}
I_{c}=\frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos x}{q^{2}+x^{2}} d x \tag{6.142}
\end{equation*}
$$

write the cosine as $[\exp (i x)+\exp (-i x)] / 2$, and factor the denominators

$$
\begin{equation*}
I_{c}=\frac{1}{4} \int_{-\infty}^{\infty} \frac{e^{i x}}{(x-i q)(x+i q)} d x+\frac{1}{4} \int_{-\infty}^{\infty} \frac{e^{-i x}}{(x-i q)(x+i q)} d x \tag{6.143}
\end{equation*}
$$

We promote $x$ to a complex variable $z$ and add the contour $z=R e^{i \theta}$ to the first integral and $z=R e^{-i \theta}$ to the second integral both over $\theta$ from 0 to $\pi$. The term $\exp (i z) d z /\left(q^{2}+z^{2}\right)=\exp (i R \cos \theta-R \sin \theta) i R e^{i \theta} d \theta /\left(q^{2}+R^{2} e^{2 i \theta}\right)$ vanishes in the limit $R \rightarrow \infty$, so the first contour is a counterclockwise ghost contour. A similar argument applies to the second (clockwise) contour, and we have

$$
\begin{equation*}
I_{c}=\frac{1}{4} \oint \frac{e^{i z}}{(z-i q)(z+i q)} d z+\frac{1}{4} \oint \frac{e^{-i z}}{(z-i q)(z+i q)} d z \tag{6.144}
\end{equation*}
$$

The first integral picks up the pole at $i q$ and the second the pole at $-i q$, so by Cauchy's integral formula (6.40)

$$
\begin{equation*}
I_{c}=\frac{i \pi}{2}\left(\frac{e^{-q}}{2 i q}+\frac{e^{-q}}{2 i q}\right)=\frac{\pi e^{-q}}{2 q} . \tag{6.145}
\end{equation*}
$$

Example 6.28 (Third-harmonic microscopy) An ultra-short laser pulse intensely focused in a medium generates a third-harmonic electric field $E_{3}$ in the forward direction proportional to the integral (Boyd, 2000)

$$
\begin{equation*}
E_{3} \propto \chi^{(3)} E_{0}^{3} \int_{-\infty}^{\infty} e^{i \Delta k z} \frac{d z}{(1+2 i z / b)^{2}} \tag{6.146}
\end{equation*}
$$

along the axis of the beam as in Fig. 6.6. Here $b=2 \pi t_{0}^{2} n / \lambda=k t_{0}^{2}$ in which $n=n(\omega)$ is the index of refraction of the medium, $\lambda$ is the wavelength of the laser light in the medium, and $t_{0}$ is the transverse or waist radius of the gaussian beam, defined by $E(r)=E \exp \left(-r^{2} / t_{0}^{2}\right)$.

When the dispersion is normal, that is when $d n(\omega) / d \omega>0$, the shift in the wave vector $\Delta k=3 \omega[n(\omega)-n(3 \omega)] / c$ is negative. Since $\Delta k<0$, the exponential is damped when $z=x+i y$ is in the lower half plane (LHP)

$$
\begin{equation*}
e^{i \Delta k z}=e^{i \Delta k(x+i y)}=e^{i \Delta k x} e^{-\Delta k y} . \tag{6.147}
\end{equation*}
$$

So as we did in example 6.27, we will add a contour around the lower half plane ( $z=R e^{i \theta}, \pi \leq \theta \leq 2 \pi$, and $\left.d z=i R e^{i \theta} d \theta\right)$ because in the limit $R \rightarrow \infty$, the integral along it vanishes; it is a ghost contour.

## Third-harmonic microscopy



Figure 6.6 In the limit in which the distance $L$ is much larger than the wavelength $\lambda$, the integral (6.146) is non-zero when an edge (solid line) lies where the beam is focused but not when a feature (...) lies where the beam is not focused. Only features within the focused region are visible.

The function $f(z)=\exp (i \Delta k z) /(1+2 i z / b)^{2}$ has a double pole at $z=i b / 2$ which is in the UHP since the length $b>0$, but no singularity in the LHP $y<0$. So the integral of $f(z)$ along the closed contour from $z=-R$ to $z=R$ and then along the ghost contour vanishes. But since the integral along the ghost contour vanishes, so does the integral from $-R$ to $R$. Thus when the dispersion is normal, the third-harmonic signal vanishes, $E_{3}=0$, as long as the medium with constant $\chi^{(3)}(z)$ effectively extends from $-\infty$ to $\infty$ so that its edges are in the unfocused region like the dotted lines of Fig. 6.6. But an edge with $\Delta k>0$ in the focused region like the solid line of the figure does make a third-harmonic signal $E_{3}$. Third-harmonic microscopy lets us see features instead of background.

Example 6.29 (Green and Bessel) Let us evaluate the Fourier transform

$$
\begin{equation*}
I(x)=\int_{-\infty}^{\infty} d k \frac{e^{i k x}}{k^{2}+m^{2}} \tag{6.148}
\end{equation*}
$$

of the function $1 /\left(k^{2}+m^{2}\right)$. If $x>0$, then the exponential deceases with $\operatorname{Im} k$ in the upper half plane. So as in example 6.27, the semicircular contour $k=R e^{i \theta}$ for $0 \leq \theta \leq \pi$ on which $d k=i R e^{i \theta} d \theta$ is a ghost contour. So if $x>0$, then we can add this contour to the integral $I(x)$ without changing it. Thus $I(x)$ is equal to the closed contour integral along the real axis and
the semicircular ghost contour

$$
\begin{equation*}
I(x)=\oint d k \frac{e^{i k x}}{k^{2}+m^{2}}=\oint d k \frac{e^{i k x}}{(k+i m)(k-i m)} \tag{6.149}
\end{equation*}
$$

This closed contour encircles the simple pole at $k=i m$ (and no other singularity) once counterclockwise, and Cauchy's integral formula (6.40) gives the integral (for $m>0$ ) as

$$
\begin{equation*}
I(x)=2 \pi i \frac{e^{i(i m) x}}{2 i m}=\frac{\pi}{m} e^{-m x} \quad \text { for } \quad x>0 \tag{6.150}
\end{equation*}
$$

Similarly if $x<0$, we can add the semicircular ghost contour $k=R e^{i \theta}$, $\pi \leq \theta \leq 2 \pi, d k=i R e^{i \theta} d \theta$ with $k$ running around the perimeter of the lower half plane and so once clockwise about the pole at $k=-i m$. So for $x<0$, Cauchy's integral formula (6.40) gives the integral (for $m>0$ ) as

$$
\begin{equation*}
I(x)=-2 \pi i \frac{e^{i(-i m) x}}{-2 i m}=\frac{\pi}{m} e^{m x} \quad \text { for } \quad x<0 \tag{6.151}
\end{equation*}
$$

We combine the two cases (6.150) and (6.151) into the result

$$
\begin{equation*}
\int_{-\infty}^{\infty} d k \frac{e^{i k x}}{k^{2}+m^{2}}=\frac{\pi}{m} e^{-m|x|} \tag{6.152}
\end{equation*}
$$

We can use this formula to develop an expression for the Green's function of the laplacian in cylindical coordinates. Setting $\boldsymbol{x}^{\prime}=0$ and $r=|\boldsymbol{x}|=$ $\sqrt{\rho^{2}+z^{2}}$ in the Coulomb Green's function (4.121), we have

$$
\begin{equation*}
G(r)=\frac{1}{4 \pi r}=\frac{1}{4 \pi \sqrt{\rho^{2}+z^{2}}}=\int \frac{d^{3} \boldsymbol{k}}{(2 \pi)^{3}} \frac{1}{\boldsymbol{k}^{2}} e^{i \boldsymbol{k} \cdot \boldsymbol{x}} \tag{6.153}
\end{equation*}
$$

The integral over the $z$-component of $\boldsymbol{k}$ is (6.152) with $m^{2}=k_{x}^{2}+k_{y}^{2} \equiv k^{2}$

$$
\begin{equation*}
\int_{-\infty}^{\infty} d k_{z} \frac{e^{i k_{z} z}}{k_{z}^{2}+k^{2}}=\frac{\pi}{k} e^{-k|z|} \tag{6.154}
\end{equation*}
$$

So with $k_{x} x+k_{y} y \equiv k \rho \cos \phi$, the Green's function is

$$
\begin{equation*}
\frac{1}{4 \pi \sqrt{\rho^{2}+z^{2}}}=\int_{0}^{\infty} \frac{\pi d k}{(2 \pi)^{3}} \int_{0}^{2 \pi} d \phi e^{i k \rho \cos \phi} e^{-k|z|} \tag{6.155}
\end{equation*}
$$

The $\phi$ integral is a representation $(6.54 \& 10.7)$ of the Bessel function $J_{0}(k \rho)$

$$
\begin{equation*}
J_{0}(k \rho)=\int_{0}^{2 \pi} \frac{d \phi}{2 \pi} e^{i k \rho \cos \phi} \tag{6.156}
\end{equation*}
$$

Thus we arrive at the Coulomb Green's function

$$
\begin{equation*}
\frac{1}{4 \pi \sqrt{\rho^{2}+z^{2}}}=\int_{0}^{\infty} \frac{d k}{4 \pi} J_{0}(k \rho) e^{-k|z|} \tag{6.157}
\end{equation*}
$$

in cylindical coordinates (Schwinger et al., 1998a, p. 166).
Example 6.30 (Yukawa and Green) We saw in example 4.16 that the Green's function for Yukawa's differential operator (4.134) is

$$
\begin{equation*}
G_{Y}(\boldsymbol{x})=\int \frac{d^{3} k}{(2 \pi)^{3}} \frac{e^{i \boldsymbol{k} \cdot \boldsymbol{x}}}{\boldsymbol{k}^{2}+m^{2}} \tag{6.158}
\end{equation*}
$$

Letting $\boldsymbol{k} \cdot \boldsymbol{x}=k r \cos \theta$ in which $r=|\boldsymbol{x}|$, we find

$$
\begin{aligned}
G_{Y}(r) & =\int_{0}^{\infty} \frac{k^{2} d k}{(2 \pi)^{2}} \int_{-1}^{1} \frac{e^{i k r \cos \theta}}{k^{2}+m^{2}} d \cos \theta=\frac{1}{i r} \int_{0}^{\infty} \frac{d k}{(2 \pi)^{2}} \frac{k}{k^{2}+m^{2}}\left(e^{i k r}-e^{-i k r}\right) \\
& =\frac{1}{i r} \int_{-\infty}^{\infty} \frac{d k}{(2 \pi)^{2}} \frac{k}{k^{2}+m^{2}} e^{i k r}=\frac{1}{i r} \int_{-\infty}^{\infty} \frac{d k}{(2 \pi)^{2}} \frac{k}{(k-i m)(k+i m)} e^{i k r}
\end{aligned}
$$

We add a ghost contour that loops over the upper-half plane and get

$$
\begin{equation*}
G_{Y}(r)=\frac{2 \pi i}{(2 \pi)^{2} i r} \frac{i m}{2 i m} e^{-m r}=\frac{e^{-m r}}{4 \pi r} \tag{6.159}
\end{equation*}
$$

which Yukawa proposed as the potential between two hadrons due to the exchange of a particle of mass $m$, the pion. Because the mass of the pion is 140 MeV , the range of the Yukawa potential is $\hbar / m c=1.4 \times 10^{-15} \mathrm{~m}$.

Example 6.31 (Green's function for the laplacian in $n$ dimensions) The Green's function for the laplacian $-\triangle G(x)=\delta^{(n)}(x)$ is

$$
\begin{equation*}
G(x)=\int \frac{1}{k^{2}} e^{i k \cdot x} \frac{d^{n} k}{(2 \pi)^{n}} \tag{6.160}
\end{equation*}
$$

in $n$ dimensions. We use the formula

$$
\begin{equation*}
\frac{1}{k^{2}}=\int_{0}^{\infty} e^{-\lambda k^{2}} d \lambda \tag{6.161}
\end{equation*}
$$

to write it as a gaussian integral

$$
\begin{equation*}
G(x)=\int e^{-\lambda k^{2}+i k \cdot x} d \lambda \frac{d^{n} k}{(2 \pi)^{n}} \tag{6.162}
\end{equation*}
$$

We now complete the square in the exponent

$$
\begin{equation*}
-\lambda k^{2}+i k \cdot x=-\lambda(k-i x / 2 \lambda)^{2}-x^{2} / 4 \lambda \tag{6.163}
\end{equation*}
$$

use our gaussian formula (6.137), and with $\alpha=x^{2} / 4 \lambda$ write the Green's function as

$$
\begin{align*}
G(x) & =\int_{0}^{\infty} d \lambda \int \frac{d^{n} k}{(2 \pi)^{n}} e^{-x^{2} / 4 \lambda} e^{-\lambda(k-i x / 2 \lambda)^{2}}=\int_{0}^{\infty} d \lambda \int \frac{d^{n} k}{(2 \pi)^{n}} e^{-x^{2} / 4 \lambda} e^{-\lambda k^{2}} \\
& =\int_{0}^{\infty} e^{-x^{2} / 4 \lambda} \frac{d \lambda}{(4 \pi \lambda)^{n / 2}}=\frac{\left(x^{2}\right)^{1-n / 2}}{4 \pi^{n / 2}} \int_{0}^{\infty} e^{-\alpha} \alpha^{n / 2-2} d \alpha \\
& =\frac{\Gamma(n / 2-1)}{4 \pi^{n / 2}\left(x^{2}\right)^{(n / 2-1)}} . \tag{6.164}
\end{align*}
$$

Our formula (5.65) for $\Gamma\left(n+\frac{1}{2}\right)$ says that $\Gamma(1 / 2)=\sqrt{\pi}$, and so this formula (6.164) for $n=3$ gives $G(x)=1 / 4 \pi|x|$ which is (4.121); since $\Gamma(1)=1$, it also gives for $n=4$

$$
\begin{equation*}
G(x)=\frac{1}{4 \pi^{2} x^{2}} . \tag{6.165}
\end{equation*}
$$

Example 6.32 (The Yukawa Green's function in $n$ dimensions) The Yukawa Green's function which satisfies $\left(-\triangle+m^{2}\right) G(x)=\delta^{(n)}(x)$ in $n$ dimensions is the integral (6.160) with $k^{2}$ replaced by $k^{2}+m^{2}$

$$
\begin{equation*}
G(x)=\int \frac{1}{k^{2}+m^{2}} e^{i k \cdot x} \frac{d^{n} k}{(2 \pi)^{n}} . \tag{6.166}
\end{equation*}
$$

Using the integral formula (6.161), we write it as a gaussian integral

$$
\begin{equation*}
G(x)=\int e^{-\lambda\left(k^{2}+m^{2}\right)+i k \cdot x} \frac{d \lambda d^{n} k}{(2 \pi)^{n}} . \tag{6.167}
\end{equation*}
$$

Completing the square as in (6.163), we have

$$
\begin{align*}
G(x) & =\int e^{-x^{2} / 4 \lambda} e^{-\lambda(k-i x / 2 \lambda)^{2}-\lambda m^{2}} \frac{d \lambda d^{n} k}{(2 \pi)^{n}}=\int e^{-x^{2} / 4 \lambda} e^{-\lambda\left(k^{2}+m^{2}\right)} \frac{d \lambda d^{n} k}{(2 \pi)^{n}} \\
& =\int_{0}^{\infty} e^{-x^{2} / 4 \lambda-\lambda m^{2}} \frac{d \lambda}{(4 \pi \lambda)^{n / 2}} . \tag{6.168}
\end{align*}
$$

We can relate this to a Bessel function by setting $\lambda=(|x| / 2 m) \exp (-y)$

$$
\begin{align*}
G(x) & =\frac{1}{(4 \pi)^{n / 2}}\left(\frac{2 m}{x}\right)^{(n / 2-1)} \int_{-\infty}^{\infty} e^{-m x \cosh y+(n / 2-1) y} d y \\
& =\frac{2}{(4 \pi)^{n / 2}}\left(\frac{2 m}{x}\right)^{(n / 2-1)} \int_{0}^{\infty} e^{-m x \cosh y} \cosh (n / 2-1) y d y \\
& =\frac{2}{(4 \pi)^{n / 2}}\left(\frac{2 m}{x}\right)^{(n / 2-1)} K_{n / 2-1}(m x) \tag{6.169}
\end{align*}
$$

where $x=|x|=\sqrt{x^{2}}$ and $K$ is a modified Bessel function of the second kind (10.77). If $n=3$, this is (exercise 6.30) the Yukawa potential (6.159).

Example 6.33 (A Fourier transform) To the integral

$$
\begin{equation*}
J(x)=\int_{-\infty}^{\infty} \frac{e^{i k x}}{\left(k^{2}+m^{2}\right)^{2}} d k \tag{6.170}
\end{equation*}
$$

we may add ghost contours as in the preceding example, but now the integrand has double poles at $k= \pm i m$, and so we must use Cauchy's integral formula (6.44) for the case of $n=1$, which is Eq.(6.42). For $x>0$, we add a ghost contour in the UHP and find

$$
\begin{align*}
J(x) & =\oint \frac{e^{i k x}}{(k+i m)^{2}(k-i m)^{2}} d k=\left.2 \pi i \frac{d}{d k} \frac{e^{i k x}}{(k+i m)^{2}}\right|_{k=i m} \\
& =\frac{\pi}{2 m^{2}}\left(x+\frac{1}{m}\right) e^{-m x} . \tag{6.171}
\end{align*}
$$

If $x<0$, then we add a ghost contour in the LHP and find

$$
\begin{align*}
J(x) & =\oint \frac{e^{i k x}}{(k+i m)^{2}(k-i m)^{2}} d k=-\left.2 \pi i \frac{d}{d k} \frac{e^{i k x}}{(k-i m)^{2}}\right|_{k=-i m} \\
& =\frac{\pi}{2 m^{2}}\left(-x+\frac{1}{m}\right) e^{m x} \tag{6.172}
\end{align*}
$$

Putting the two together, we get

$$
\begin{equation*}
J(x)=\int_{-\infty}^{\infty} \frac{e^{i k x}}{\left(k^{2}+m^{2}\right)^{2}} d k=\frac{\pi}{2 m^{2}}\left(|x|+\frac{1}{m}\right) e^{-m|x|} . \tag{6.173}
\end{equation*}
$$

as the Fourier transform of $1 /\left(k^{2}+m^{2}\right)^{2}$.
Example 6.34 (Integral of a complex gaussian) We can use a ghost contour to do the integral

$$
\begin{equation*}
I=\int_{-\infty}^{\infty} e^{w x^{2}} d x \tag{6.174}
\end{equation*}
$$

in which the real part of the nonzero complex number $w=u+i v=\rho e^{i \phi}$ is negative or zero

$$
\begin{equation*}
u \leq 0 \quad \Longleftrightarrow \quad \frac{\pi}{2} \leq \phi \leq \frac{3 \pi}{2} \tag{6.175}
\end{equation*}
$$

We first write the integral $I$ as twice the same integral along half the $x$-axis

$$
\begin{equation*}
I=2 \int_{0}^{\infty} e^{w x^{2}} d x \tag{6.176}
\end{equation*}
$$

If we promote $x$ to a complex variable $z=r e^{i \theta}$, then $w z^{2}$ will be negative if $\phi+2 \theta=\pi$, that is, if $\theta=(\pi-\phi) / 2$ where in view of (6.175) $\theta$ lies in the interval $-\pi / 4 \leq \theta \leq \pi / 4$.

The closed pie-shaped contour of Fig. 6.7 (down the real axis from $z=0$ to $z=R$, along the arc $z=R \exp \left(i \theta^{\prime}\right)$ as $\theta^{\prime}$ goes from 0 to $\theta$, and then down the line $z=r \exp (i \theta)$ from $z=R \exp (i \theta)$ to $z=0$ ) encloses no singularities of the function $f(z)=\exp \left(w z^{2}\right)$. Hence the integral of $\exp \left(w z^{2}\right)$ along that contour vanishes.

To show that the arc is a ghost contour, we bound it by

$$
\begin{align*}
\left|\int_{0}^{\theta} e^{(u+i v) R^{2} e^{2 i \theta^{\prime}}} R d \theta^{\prime}\right| & \leq \int_{0}^{\theta} \exp \left[u R^{2} \cos 2 \theta^{\prime}-v R^{2} \sin 2 \theta^{\prime}\right] R d \theta^{\prime} \\
& \leq \int_{0}^{\theta} e^{-v R^{2} \sin 2 \theta^{\prime}} R d \theta^{\prime} \tag{6.177}
\end{align*}
$$

Here $v \sin 2 \theta^{\prime} \geq 0$, and so if $v$ is positive, then so is $\theta^{\prime}$. Then $0 \leq \theta^{\prime} \leq \pi / 4$, and so $\sin \left(2 \theta^{\prime}\right) \geq 4 \theta^{\prime} / \pi$. Thus since $u<0$, we have the upper bound

$$
\begin{equation*}
\left|\int_{0}^{\theta} e^{(u+i v) R^{2} e^{2 i \theta^{\prime}}} R d \theta^{\prime}\right| \leq \int_{0}^{\theta} e^{-4 v R^{2} \theta^{\prime} / \pi} R d \theta^{\prime}=\frac{\pi\left(e^{-4 v R^{2} \theta^{\prime} / \pi}-1\right)}{4 v R} \tag{6.178}
\end{equation*}
$$

which vanishes in the limit $R \rightarrow \infty$. (If $v$ is negative, then so is $\theta^{\prime}$, the pie-shaped contour is in the fourth quadrant, $\sin \left(2 \theta^{\prime}\right) \leq 4 \theta^{\prime} / \pi$, and the inequality (6.178) holds with absolute-value signs around the second integral.)

Since by Cauchy's integral theorem (6.21) the integral along the pieshaped contour of Fig. 6.7 vanishes, it follows that

$$
\begin{equation*}
\frac{1}{2} I+\int_{R e^{i \theta}}^{0} e^{w z^{2}} d z=0 \tag{6.179}
\end{equation*}
$$

But the choice $\theta=(\pi-\phi) / 2$ implies that on the line $z=r \exp (i \theta)$ the quantity $w z^{2}$ is negative, $w z^{2}=-\rho r^{2}$. Thus with $d z=\exp (i \theta) d r$, we have

$$
\begin{equation*}
I=2 \int_{0}^{R e^{i \theta}} e^{w z^{2}} d z=2 e^{i \theta} \int_{0}^{R} e^{-\rho r^{2}} d r \tag{6.180}
\end{equation*}
$$

so that as $R \rightarrow \infty$

$$
\begin{equation*}
I=2 e^{i \theta} \int_{0}^{\infty} e^{-\rho r^{2}} d r=e^{i \theta} \sqrt{\frac{\pi}{\rho}}=\sqrt{\frac{\pi}{\rho e^{-2 i \theta}}} \tag{6.181}
\end{equation*}
$$

Finally from $\theta=(\pi-\phi) / 2$ and $w=\rho \exp (i \phi)$, we find that for $\operatorname{Re} w \leq 0$

$$
\begin{equation*}
\int_{-\infty}^{\infty} e^{w x^{2}} d x=\sqrt{\frac{\pi}{-w}} \tag{6.182}
\end{equation*}
$$

## Pie-shaped contour



Figure 6.7 The integral of the entire function $\exp \left(w z^{2}\right)$ along the pieshaped closed contour vanishes by Cauchy's theorem.
as long as $w \neq 0$. Shifting $x$ by a complex number $b$, we still have

$$
\begin{equation*}
\int_{-\infty}^{\infty} e^{w(x-b)^{2}} d x=\sqrt{\frac{\pi}{-w}} \tag{6.183}
\end{equation*}
$$

as long as $\operatorname{Re} w<0$. If $w=i a \neq 0$ and $a$ and $b$ are real, then

$$
\begin{equation*}
\int_{-\infty}^{\infty} e^{i a(x-b)^{2}} d x=\sqrt{\frac{i \pi}{a}} \quad \text { or } \quad \int_{-\infty}^{\infty} e^{i a x^{2}-2 i a b x} d x=\sqrt{\frac{i \pi}{a}} e^{-i a b^{2}} \tag{6.184}
\end{equation*}
$$

Setting $x=p, a=-\epsilon /(2 m \hbar)$, and $b=m \dot{q}$ in the last equation, and dividing both sides by $2 \pi \hbar$, we find

$$
\begin{equation*}
\int_{-\infty}^{\infty} \exp \left(-i \epsilon \frac{p^{2}}{2 m \hbar}+i \epsilon \frac{\dot{q} p}{\hbar}\right) \frac{d p}{2 \pi \hbar}=\sqrt{\frac{m}{2 \pi i \epsilon \hbar}} \exp \left(i \epsilon \frac{m \dot{q}^{2}}{2 \hbar}\right) \tag{6.185}
\end{equation*}
$$

a formula we'll use in section 20.3 to derive path integrals for probability amplitudes.

Let us try to express the line integral of a not necessarily analytic function $f(x, y)=u(x, y)+i v(x, y)$ along a closed counterclockwise contour $\mathcal{C}$ as an integral over the surface enclosed by the contour. The contour integral is

$$
\begin{equation*}
\oint_{\mathcal{C}}(u+i v)(d x+i d y)=\oint_{\mathcal{C}}(u d x-v d y)+i \oint_{\mathcal{C}}(v d x+u d y) \tag{6.186}
\end{equation*}
$$

Now since the contour $\mathcal{C}$ is counterclockwise, the differential $d x$ is negative at the top of the curve with coordinates $\left(x, y_{+}(x)\right)$ and positive at the bottom $\left(x, y_{-}(x)\right)$. So the first line integral is the surface integral

$$
\begin{align*}
\oint_{\mathcal{C}} u d x & =\int\left[u\left(x, y_{-}(x)\right)-u\left(x, y_{+}(x)\right)\right] d x \\
& =-\int\left[\int_{y_{-}(x)}^{y_{+}(x)} u_{y}(x, y) d y\right] d x \\
& =-\int u_{y}|d x d y|=-\int u_{y} d a \tag{6.187}
\end{align*}
$$

in which $d a=|d x d y|$ is a positive element of area. Similarly, we find

$$
\begin{equation*}
i \oint_{\mathcal{C}} v d x=-i \int v_{y}|d x d y|=-i \int v_{y} d a \tag{6.188}
\end{equation*}
$$

The $d y$ integrals are then:

$$
\begin{align*}
-\oint_{\mathcal{C}} v d y & =-\int v_{x}|d x d y|=-\int v_{x} d a  \tag{6.189}\\
i \oint_{\mathcal{C}} u d y & =i \int u_{x}|d x d y|=i \int u_{x} d a \tag{6.190}
\end{align*}
$$

Combining (6.186-6.190), we find

$$
\begin{equation*}
\oint_{\mathcal{C}}(u+i v)(d x+i d y)=-\int\left(u_{y}+v_{x}\right) d a+i \int\left(-v_{y}+u_{x}\right) d a \tag{6.191}
\end{equation*}
$$

This formula holds whether or not the function $f(x, y)$ is analytic. But if $f(x, y)$ is analytic on and within the contour $\mathcal{C}$, then it satisfies the CauchyRiemann conditions (6.10) within the contour, and so both surface integrals vanish. The contour integral then is zero, which is Cauchy's integral theorem (6.32).

The contour integral of the function $f(x, y)=u(x, y)+i v(x, y)$ differs from zero (its value if $f(x, y)$ is analytic in $z=x+i y)$ by the surface integrals of $u_{y}+v_{x}$ and $u_{x}-v_{y}$
$\left|\oint_{\mathcal{C}} f(z) d z\right|^{2}=\left|\oint_{\mathcal{C}}(u+i v)(d x+i d y)\right|^{2}=\left|\int\left(u_{y}+v_{x}\right) d a\right|^{2}+\left|\int\left(u_{x}-v_{y}\right) d a\right|^{2}$
which vanish when $f=u+i v$ satisfies the Cauchy-Riemann conditions (6.10).

Example 6.35 (The integral of a nonanalytic function) The integral formula (6.191) can help us evaluate contour integrals of functions that are not
analytic. The function

$$
\begin{equation*}
f(x, y)=\frac{1}{x+i y+i \epsilon} \frac{1}{1+x^{2}+y^{2}} \tag{6.193}
\end{equation*}
$$

is the product of an analytic function $1 /(z+i \epsilon)$, where $\epsilon$ is tiny and positive, and a nonanalytic real one $r(x, y)=1 /\left(1+z^{*} z\right)$. The $i \epsilon$ pushes the pole in $u+i v=1 /(z+i \epsilon)$ into the lower half plane. The real and imaginary parts of $f$ are

$$
\begin{equation*}
U(x, y)=u(x, y) r(x, y)=\frac{x}{x^{2}+(y+\epsilon)^{2}} \frac{1}{1+x^{2}+y^{2}} \tag{6.194}
\end{equation*}
$$

and

$$
\begin{equation*}
V(x, y)=v(x, y) r(x, y)=\frac{-y-\epsilon}{x^{2}+(y+\epsilon)^{2}} \frac{1}{1+x^{2}+y^{2}} . \tag{6.195}
\end{equation*}
$$

We will use (6.191) to compute the contour integral $I$ of $f$ along the real axis from $-\infty$ to $\infty$ and then along the ghost contour $z=x+i y=R e^{i \theta}$ for $0 \leq \theta \leq \pi$ and $R \rightarrow \infty$ around the upper half plane

$$
\begin{equation*}
I=\oint f(x, y) d z=\int_{-\infty}^{\infty} d x \int_{0}^{\infty} d y\left[-U_{y}-V_{x}+i\left(-V_{y}+U_{x}\right)\right] . \tag{6.196}
\end{equation*}
$$

Since $u$ and $v$ satisfy the Cauchy-Riemann conditions (6.10), the terms in the area integral simplify to $-U_{y}-V_{x}=-u r_{y}-v r_{x}$ and $-V_{y}+U_{x}=-v r_{y}+u r_{x}$. So the integral $I$ is

$$
\begin{equation*}
I=\int_{-\infty}^{\infty} d x \int_{0}^{\infty} d y\left[-u r_{y}-v r_{x}+i\left(-v r_{y}+u r_{x}\right)\right] \tag{6.197}
\end{equation*}
$$

or explicitly

$$
\begin{equation*}
I=\int_{-\infty}^{\infty} d x \int_{0}^{\infty} d y \frac{-2 \epsilon x-2 i\left(x^{2}+y^{2}+\epsilon y\right)}{\left[x^{2}+(y+\epsilon)^{2}\right]\left(1+x^{2}+y^{2}\right)^{2}} . \tag{6.198}
\end{equation*}
$$

We let $\epsilon \rightarrow 0$ and find

$$
\begin{equation*}
I=-2 i \int_{-\infty}^{\infty} d x \int_{0}^{\infty} d y \frac{1}{\left(1+x^{2}+y^{2}\right)^{2}} . \tag{6.199}
\end{equation*}
$$

Changing variables to $\rho^{2}=x^{2}+y^{2}$, we have

$$
\begin{equation*}
I=-4 \pi i \int_{0}^{\infty} d \rho \frac{\rho}{\left(1+\rho^{2}\right)^{2}}=2 \pi i \int_{0}^{\infty} d \rho \frac{d}{d \rho} \frac{1}{1+\rho^{2}}=-2 \pi i \tag{6.200}
\end{equation*}
$$

which is simpler than evaluating the integral (6.196) directly.

Example 6.36 (Inverse Laplace transform of $1 /(s-k)$ ) To find the inverse Laplace transform (4.151) of $f(s)=1 /(s-k)(4.152)$, we add a ghost contour that goes around the upper half-plane encircling the pole at $u=i(s-k)$

$$
\begin{align*}
F(t) & =e^{s t} \int_{-\infty}^{\infty} \frac{d u}{2 \pi} e^{i u t} f(s+i u)=\frac{e^{s t}}{2 \pi} \int_{-\infty}^{\infty} d u \frac{e^{i u t}}{s+i u-k} \\
& =\frac{e^{s t}}{2 \pi i} \oint d u \frac{e^{i u t}}{u-i(s-k)}=e^{s t} e^{i^{2} t(s-k)}=e^{k t} \tag{6.201}
\end{align*}
$$

Example 6.37 (Inverse Laplace transform of $\left.f(s)=1 /(s-i a)^{3}\right)$ We add to the inverse Laplace transform (4.151) a contour that goes over the upper half-plane encircling the pole at $u=a-i s$ and use Cauchy's integral formula (6.44)

$$
\begin{align*}
F(t) & =\frac{e^{s t}}{2 \pi} \oint d u \frac{e^{i u t}}{(s+i u-i a)^{3}}=\frac{e^{s t}}{2 \pi i^{3}} \oint d u \frac{e^{i u t}}{(u-a-i s)^{3}} \\
& =\frac{e^{s t}}{2 \pi i^{3}} \oint d u \frac{e^{i u t}}{(u-a-i s)^{3}}=-\left.\frac{e^{s t}}{2} \frac{d^{2} e^{i u t}}{d u^{2}}\right|_{u=a+i s}=\frac{t^{2}}{2} e^{i a t} \tag{6.202}
\end{align*}
$$

as given by Mathematica's "InverseLaplaceTransform $\left[1 /(s-i a)^{3}, s, t\right]$ " and by Matlab's "syms s a t; $F=1 /(s-i * a)^{3} ; f=$ ilaplace $(F)$."

### 6.15 Logarithms and cuts

By definition a function $f$ is single valued: it maps every number $z$ in its domain into a unique number $f(z)$. A function that maps only one number $z$ in its domain into each number $f(z)$ in its range is said to be one to one. A one-to-one function $f$ has a well-defined inverse function $f^{-1}$ which maps $f(z)$ back to $z$.

The exponential function is one to one when restricted to the real numbers. It maps every real number $x$ into a positive number $\exp (x)$. It has an inverse function $\log (x)$ that maps every positive number $\exp (x)$ back into $x$. But the exponential function is not one to one on the complex numbers because $\exp (z+2 \pi n i)=\exp (z)$ for every integer $n$. Because it is many to one, the exponential function has no inverse function on the complex numbers. Its would-be inverse function $\log$ maps $z$ to $\log (\exp (z))$ or $z+2 \pi n i$ which is not unique. It has in it an arbitrary integer $n$.

In other words if $z=r \exp (i \theta)$, then suitable logarithms of $z$ are $\log (z)=$ $\log (r)+i \theta+i 2 \pi n$ because for every integer $n$

$$
\begin{equation*}
\exp (\log (r)+i \theta+i 2 \pi n)=r e^{i \theta+i 2 \pi n}=r e^{i \theta}=z \tag{6.203}
\end{equation*}
$$

People usually want one of the correct values of a logarithm, rather than all of them. Two conventions for choosing $n$ are common. In both conventions, $n=0$ when $z$ is in the upper half-plane.

In the first convention, the angle $\theta$ is zero for $z$ along the positive real axis $z>0$, and increases continuously as the point $z$ moves counterclockwise around the origin, until at points just below the positive real axis, $\theta$ is slightly less than $2 \pi$. In this convention, the value of $\theta$ drops by $2 \pi$ as $z$ crosses the positive real axis moving counterclockwise. This discontinuity on the positive real axis is called a cut.

The second common convention puts the cut on the negative real axis. Here the value of $\theta$ is the same as in the first convention when $z$ is in the upper half-plane. But in the lower half-plane, as $z$ moves clockwise from the positive real axis to just below the negative real axis $\theta$ decreases from 0 to slighty more than $-\pi$. As $z$ continues to move clockwise and crosses the cut on the negative real axis, $\theta$ jumps by $2 \pi$.

The two conventions agree in the upper half-plane but differ by $2 \pi$ in the lower half-plane.

Sometimes it is convenient to place the cut on the positive or negative imaginary axis - or along a line that makes an arbitrary angle with the real axis. In any particular calculation, we are at liberty to define the polar angle $\theta$ by placing the cut anywhere we like, but we must not change from one convention to another in the same computation.

### 6.16 Powers and roots

The logarithm is used to define many functions to which it passes its arbitrariness. For instance, $z=r \exp (i \theta)$ raised to any power $a$ is

$$
\begin{equation*}
z^{a}=\exp (a \log z)=\exp [a(\log r+i \theta+i 2 \pi n)]=r^{a} e^{i a \theta} e^{i 2 \pi n a} . \tag{6.204}
\end{equation*}
$$

So $z^{a}$ is not unique unless $a$ is an integer. The square root, for example, has a sign ambiguity

$$
\begin{equation*}
\sqrt{z}=\exp \left[\frac{1}{2}(\log r+i \theta+i 2 \pi n)\right]=\sqrt{r} e^{i \theta / 2} e^{i n \pi}=(-1)^{n} \sqrt{r} e^{i \theta / 2} \tag{6.205}
\end{equation*}
$$

It changes sign when $z$ crosses a cut. The $m$ th root

$$
\begin{equation*}
\sqrt[m]{z}=z^{1 / m}=\exp [(\log r+i \theta+i 2 \pi n) / m]=r^{1 / m} e^{i \theta / m} e^{i 2 \pi n / m} \tag{6.206}
\end{equation*}
$$

changes by $\exp ( \pm 2 \pi i / m)$ when $z$ crosses a cut. And when $a=u+i v$ is a complex number, $z^{a}$ is

$$
\begin{equation*}
z^{a}=e^{a \log z}=e^{(u+i v)(\log r+i \theta+i 2 \pi n)}=r^{u+i v} e^{(-v+i u)(\theta+2 \pi n)} \tag{6.207}
\end{equation*}
$$

which changes by $\exp [2 \pi(-v+i u)]$ when $z$ crosses a cut.
Example $6.38\left(i^{i}\right) \quad$ The number $i=\exp (i \pi / 2+i 2 \pi n)$ for any integer $n$. So the general value of $i^{i}$ is $i^{i}=\exp [i(i \pi / 2+i 2 \pi n)]=\exp (-\pi / 2-2 \pi n)$.

One can define a sequence of $m$ th-root functions

$$
\begin{equation*}
\left(z^{1 / m}\right)_{n}=\exp \left(\frac{\log r+i(\theta+2 \pi n)}{m}\right) \tag{6.208}
\end{equation*}
$$

one for each integer $n$. These functions are the branches of the $m$ th-root function. One can merge all the branches into one multivalued $m$ th-root function. Using a convention for $\theta$, one would extend the $n=0$ branch to the $n=1$ branch by winding counterclockwise around the point $z=0$. One would encounter no discontinuity as one passed from one branch to another. The point $z=0$, where any cut starts, is called a branch point because by winding around it, one passes smoothly from one branch to another. Such branches, introduced by Riemann, can be associated with any multivalued analytic function not just with the $m$ th root.

Example 6.39 (Explicit square roots) If the cut in the square root $\sqrt{z}$ is on the negative real axis, then an explicit formula for the square root of $x+i y$ is

$$
\begin{equation*}
\sqrt{x+i y}=\sqrt{\frac{\sqrt{x^{2}+y^{2}}+x}{2}}+i \operatorname{sign}(y) \sqrt{\frac{\sqrt{x^{2}+y^{2}}-x}{2}} \tag{6.209}
\end{equation*}
$$

in which $\operatorname{sign}(y)=\operatorname{sgn}(y)=y /|y|$. On the other hand, if the cut in the square root $\sqrt{z}$ is on the positive real axis, then an explicit formula for the square root of $x+i y$ is

$$
\begin{equation*}
\sqrt{x+i y}=\operatorname{sign}(y) \sqrt{\frac{\sqrt{x^{2}+y^{2}}+x}{2}}+i \sqrt{\frac{\sqrt{x^{2}+y^{2}}-x}{2}} \tag{6.210}
\end{equation*}
$$

(exercise 6.31).
Example 6.40 (Cuts) Cuts are discontinuities, so people place them where they do the least harm. For the function

$$
\begin{equation*}
f(z)=\sqrt{z^{2}-1}=\sqrt{(z-1)(z+1)} \tag{6.211}
\end{equation*}
$$

the two common conventions work well. If we put the cut in the definition of the angle $\theta$ along the positive or negative real axis, then the sign discontinuity (a factor of -1 ) from $\sqrt{z-1}$ cancels the one from $\sqrt{z+1}$ except for $-1 \leq z \leq 1$. So the function $f(z)$ then has a discontinuity or a cut only for $-1 \leq z \leq 1$.

But if we used one of the common conventions to define the function

$$
\begin{equation*}
f(z)=\sqrt{z^{2}+1}=\sqrt{(z-i)(z+i)}, \tag{6.212}
\end{equation*}
$$

then we'd have two semi-infinite cuts. If instead we put the $\theta$-cut on the positive or negative imaginary axis, then the function $f(z)$ would have a single cut running along the imaginary axis from $-i$ to $i$.

Example 6.41 (Square-root cut) To evaluate the integral

$$
\begin{equation*}
I_{s}=\int_{0}^{\infty} \frac{d x}{(x+a)^{2} \sqrt{x}}, \quad a>0 \tag{6.213}
\end{equation*}
$$

we put the cut on the positive real axis. The integral backwards along and just below the positive real axis

$$
\begin{equation*}
I_{-}=\int_{\infty}^{0} \frac{d x}{(x+a)^{2} \sqrt{x-i \epsilon}}=-\int_{\infty}^{0} \frac{d x}{(x+a)^{2} \sqrt{x+i \epsilon}}=I_{s} \tag{6.214}
\end{equation*}
$$

is the same as $I_{s}$ since a minus sign from the square root cancels the minus sign due to the backwards direction.

Since

$$
\begin{equation*}
\lim _{|z| \rightarrow \infty} \frac{|z|}{|z+a|^{2}|\sqrt{z}|}=0 \tag{6.215}
\end{equation*}
$$

the integrals of $f(z)=1 /\left[(z+a)^{2} \sqrt{z}\right]$ along the contours $z=R \exp (i \theta)$ for $0<\theta<\pi$ and for $\pi<\theta<2 \pi$ vanish as $R \rightarrow \infty$. So these contours are ghost contours. We then add a pair of cancelling integrals along the negative real axis up to the pole at $z=-a$ and then add a clockwise loop $\mathcal{C}$ around it. As in Fig. 6.8, the integral along this collection of contours encloses no singularity and therefore vanishes

$$
\begin{equation*}
0=I_{s}+I_{-}+I_{\mathcal{G}_{+}}+I_{\mathcal{G}_{-}}+I_{\mathcal{C}} . \tag{6.216}
\end{equation*}
$$

Thus $2 I_{s}=-I_{\mathcal{C}}$, and so from Cauchy's integral formula (6.44) for $n=1$, we have

$$
\begin{equation*}
I_{s}=-\frac{1}{2} I_{\mathcal{C}}=-\frac{1}{2} \oint_{\mathcal{C}} \frac{1}{(z+a)^{2} \sqrt{z}} d z=-\left.i \pi \frac{d}{d z} z^{-1 / 2}\right|_{z=-a}=\frac{\pi}{2 a^{3 / 2}} \tag{6.217}
\end{equation*}
$$

which one may check with the Mathematica command Assuming $[a>0$, Integrate $\left[1 /\left((x+a)^{2} * \operatorname{Sqrt}[x]\right),\{x, 0\right.$, Infinity $\left.\left.\}\right]\right]$.

Example 6.42 (Contour integral with a cut) Let's compute the integral

$$
\begin{equation*}
I=\int_{0}^{\infty} \frac{x^{a}}{(x+1)^{2}} d x \tag{6.218}
\end{equation*}
$$



Figure 6.8 The integrals of $f(z)=1 /\left[(x+a)^{2} \sqrt{z}\right]$ as well as that of $f(z)=z^{a} /(z+1)^{2}$ along the ghost contours $\mathcal{G}_{+}$and $\mathcal{G}_{-}$and the contours $\mathcal{C}, \mathcal{I}_{-}$, and $\mathcal{I}_{+}$vanish because the combined contour encircles no poles of either $f(z)$. The cut (solid line) runs from the origin to infinity along the positive real axis.
for $-1<a<1$. We promote $x$ to a complex variable $z$ and put the cut on the positive real axis. Since

$$
\begin{equation*}
\lim _{|z| \rightarrow \infty} \frac{|z|^{a+1}}{|z+1|^{2}}=0 \tag{6.219}
\end{equation*}
$$

the integrand vanishes faster than $1 /|z|$, and we may add two ghost contours, $\mathcal{G}_{+}$counterclockwise around the upper half-plane and $\mathcal{G}_{-}$counterclockwise around the lower half-plane, as shown in Fig. 6.8.

We add a contour $\mathcal{C}$ that runs from $-\infty$ to the double pole at $z=-1$, loops around that pole, and then runs back to $-\infty$; the two long contours along the negative real axis cancel because the cut in $\theta$ lies on the positive real axis. So the contour integral along $\mathcal{C}$ is just the clockwise integral around the double pole which by Cauchy's integral formula (6.42) is

$$
\begin{equation*}
\oint_{\mathcal{C}} \frac{z^{a}}{(z-(-1))^{2}} d z=-\left.2 \pi i \frac{d z^{a}}{d z}\right|_{z=-1}=2 \pi i a e^{\pi a i} \tag{6.220}
\end{equation*}
$$

We also add the integral $I_{-}$from $\infty$ to 0 just below the real axis

$$
\begin{equation*}
I_{-}=\int_{\infty}^{0} \frac{(x-i \epsilon)^{a}}{(x-i \epsilon+1)^{2}} d x=\int_{\infty}^{0} \frac{\exp (a(\log (x)+2 \pi i))}{(x+1)^{2}} d x \tag{6.221}
\end{equation*}
$$

which is

$$
\begin{equation*}
I_{-}=-e^{2 \pi a i} \int_{0}^{\infty} \frac{x^{a}}{(x+1)^{2}} d x=-e^{2 \pi a i} I . \tag{6.222}
\end{equation*}
$$

Now the sum of all these contour integrals is zero because it is a closed contour that encloses no singularity. So we have

$$
\begin{equation*}
0=\left(1-e^{2 \pi a i}\right) I+2 \pi i a e^{\pi a i} \tag{6.223}
\end{equation*}
$$

or

$$
\begin{equation*}
I=\int_{0}^{\infty} \frac{x^{a}}{(x+1)^{2}} d x=\frac{\pi a}{\sin (\pi a)} \tag{6.224}
\end{equation*}
$$

as the value of the integral (6.218).
Example 6.43 (Euler's reflection formula) The beta function (5.77) for $x=z$ and $y=1-z$ is the integral

$$
\begin{equation*}
\mathrm{B}(z, 1-z)=\Gamma(z) \Gamma(1-z)=\int_{0}^{1} t^{z-1}(1-t)^{-z} d t \tag{6.225}
\end{equation*}
$$

Setting $t=u /(1+u)$, so that $u=t /(1-t)$ and $d t=1 /(1+u)^{2}$, we have

$$
\begin{equation*}
\mathrm{B}(z, 1-z)=\int_{0}^{\infty} \frac{u^{z-1}}{1+u} d u \tag{6.226}
\end{equation*}
$$

We integrate $f(u)=u^{z-1} /(1+u)$ along the contour of the preceding example (6.42) which includes the ghost contour $\mathcal{G}=\mathcal{G}_{+} \cup \mathcal{G}_{-}$and runs down both sides of the cut along the positive real axis. Since $f(u)$ is analytic inside the contour, the integral vanishes

$$
\begin{equation*}
0=\int_{\mathcal{I}_{+}} f(u) d u+\int_{\mathcal{G}} f(u) d u+\int_{\mathcal{C}} f(u) d u+\int_{\mathcal{I}_{-}} f(u) d u . \tag{6.227}
\end{equation*}
$$

The clockwise contour $\mathcal{C}$ is

$$
\begin{equation*}
\int_{\mathcal{C}} \frac{u^{z-1}}{1+u} d u=-2 \pi i(-1)^{z-1}=-2 \pi i e^{i \pi(z-1)}=2 \pi i e^{i \pi z} \tag{6.228}
\end{equation*}
$$

The contour $\mathcal{I}_{+}$runs just above the positive real axis, and the integral of $f(u)$ along it is the desired integral $\mathrm{B}(z, 1-z)$. The contour $\mathcal{I}_{-}$runs backwards and just below the cut where $u=|u|-i \epsilon$

$$
\begin{equation*}
\int_{\mathcal{I}_{-}} f(u) d u=-\int_{0}^{\infty} \frac{\left(|u| e^{2 \pi i-\epsilon}\right)^{z-1}}{1+u} d u=-e^{2 \pi i z} \int_{0}^{\infty} \frac{u^{z-1}}{1+u} d u \tag{6.229}
\end{equation*}
$$

Thus the vanishing (6.227) of the contour integral

$$
\begin{equation*}
0=\mathrm{B}(z, 1-z)+2 \pi i e^{i \pi z}-e^{2 \pi i z} \mathrm{~B}(z, 1-z) \tag{6.230}
\end{equation*}
$$

gives us Euler's reflection formula

$$
\begin{equation*}
\mathrm{B}(z, 1-z)=\Gamma(z) \Gamma(1-z)=\frac{\pi}{\sin \pi z} . \tag{6.231}
\end{equation*}
$$

Example 6.44 (A Matthews and Walker integral) To do the integral

$$
\begin{equation*}
I=\int_{0}^{\infty} \frac{d x}{1+x^{3}} \tag{6.232}
\end{equation*}
$$

we promote $x$ to a complex variable $z$ and consider the function $f(z)=$ $\log z /\left(1+z^{3}\right)$. If we put the cut in the logarithm on the positive real axis, then $f(z)$ is analytic everywhere except for $z \geq 0$ and at $z^{3}=-1$. The integral of $f(z)$ along the ghost contour $z=R \exp (i \theta)$ from $\theta=\epsilon$ to $\theta=$ $2 \pi-\epsilon$ and along both sides of the real axis from $z=i \epsilon$ to $z=R+i \epsilon$ and from $z=R-i \epsilon$ to $z=-i \epsilon$ is by the residue theorem (6.119)

$$
\begin{equation*}
\oint f(z) d z=\oint \frac{\log (z)}{\left(z-e^{i \pi / 3}\right)\left(z-e^{3 i \pi / 3}\right)\left(z-e^{5 i \pi / 3}\right)} d z=-\frac{4 \pi^{2} i}{3 \sqrt{3}} \tag{6.233}
\end{equation*}
$$

Since $\log (x+i \epsilon)=\log (x)$ and $\log (x-i \epsilon)=\log (x)+2 \pi i$, while $|\epsilon \log (\epsilon)| \rightarrow 0$ as $\epsilon \rightarrow 0$, that same integral approaches $-2 \pi i I$ as $R \rightarrow \infty$. Thus the integral (6.232) is $I=2 \pi /(3 \sqrt{3})$.

### 6.17 Conformal mapping

An analytic function $f(z)$ maps curves in the $z$ plane into curves in the $f(z)$ plane. In general, this mapping preserves angles. To see why, consider the angle $d \theta$ between two tiny complex lines $d z=\epsilon \exp (i \theta)$ and $d z^{\prime}=\epsilon \exp \left(i \theta^{\prime}\right)$ that radiate from the same point $z$. The angle between $d z$ and $d z^{\prime}$ is the phase of the ratio

$$
\begin{equation*}
\frac{d z^{\prime}}{d z}=\frac{\epsilon e^{i \theta^{\prime}}}{\epsilon e^{i \theta}}=e^{i\left(\theta^{\prime}-\theta\right)} \tag{6.234}
\end{equation*}
$$

Let's use $w=\rho e^{i \phi}$ for $f(z)$. Then the analytic function $f(z)$ maps $d z$ into

$$
\begin{equation*}
d w=f(z+d z)-f(z) \approx f^{\prime}(z) d z=f^{\prime}(z) \epsilon e^{i \theta} \tag{6.235}
\end{equation*}
$$

and $d z^{\prime}$ into

$$
\begin{equation*}
d w^{\prime}=f\left(z+d z^{\prime}\right)-f(z) \approx f^{\prime}(z) d z^{\prime}=f^{\prime}(z) \epsilon e^{i \theta^{\prime}} \tag{6.236}
\end{equation*}
$$

The angle $d \phi=\phi^{\prime}-\phi$ between $d w$ and $d w^{\prime}$ is the phase of the ratio

$$
\begin{equation*}
\frac{d w^{\prime}}{d w}=\frac{e^{i \phi^{\prime}}}{e^{i \phi}}=\frac{f^{\prime}(z) d z^{\prime}}{f^{\prime}(z) d z}=\frac{d z^{\prime}}{d z}=\frac{e^{i \theta^{\prime}}}{e^{i \theta}}=e^{i\left(\theta^{\prime}-\theta\right)} \tag{6.237}
\end{equation*}
$$

So as long as the derivative $f^{\prime}(z)$ does not vanish, the angle in the $w$-plane is the same as the angle in the $z$-plane

$$
\begin{equation*}
d \phi=d \theta . \tag{6.238}
\end{equation*}
$$

Analytic functions preserve angles. They are conformal maps.
What if $f^{\prime}(z)=0$ ? In this case, $d w \approx f^{\prime \prime}(z) d z^{2} / 2$ and $d w^{\prime} \approx f^{\prime \prime}(z) d z^{\prime 2} / 2$, and so the angle $d \phi=d \phi^{\prime}-d \phi$ between $d w$ and $d w^{\prime}$ is the phase of the ratio

$$
\begin{equation*}
\frac{d w^{\prime}}{d w}=\frac{e^{i \phi^{\prime}}}{e^{i \phi}}=\frac{f^{\prime \prime}(z) d z^{\prime 2}}{f^{\prime \prime}(z) d z^{2}}=\frac{d z^{\prime 2}}{d z^{2}}=e^{2 i\left(\theta^{\prime}-\theta\right)} \tag{6.239}
\end{equation*}
$$

So angles are doubled, $d \phi=2 d \theta$.
In general, if the first nonzero derivative is $f^{(n)}(z)$, then

$$
\begin{equation*}
\frac{d w^{\prime}}{d w}=\frac{e^{i \phi^{\prime}}}{e^{i \phi}}=\frac{f^{(n)}(z) d z^{\prime n}}{f^{(n)}(z) d z^{n}}=\frac{d z^{\prime n}}{d z^{n}}=e^{n i\left(\theta^{\prime}-\theta\right)} \tag{6.240}
\end{equation*}
$$

and so $d \phi=n d \theta$. Angles increase by a factor of $n$.
Example $6.45\left(z^{n}\right)$ The function $f(z)=z^{n}$ has only one nonzero derivative $f^{(k)}(0)=n!\delta_{n k}$ at the origin $z=0$. So at $z=0$ the map $z \rightarrow z^{n}$ scales angles by $n, d \phi=n d \theta$, but at $z \neq 0$ the first derivative $f^{(1)}(z)=n z^{n-1}$ is not equal to zero. So $z^{n}$ is conformal except at the origin.

Example 6.46 (Möbius transformation) The function

$$
\begin{equation*}
f(z)=\frac{a z+b}{c z+d} \tag{6.241}
\end{equation*}
$$

maps (straight) lines into lines and circles and maps circles into circles and lines, unless $a d=b c$ in which case it is the constant $b / d$.

### 6.18 Cauchy's principal value

Suppose that $f(x)$ is differentiable or analytic at and near the point $x=0$, and that we wish to evaluate the integral

$$
\begin{equation*}
K=\lim _{\epsilon \rightarrow 0} \int_{-a}^{b} d x \frac{f(x)}{x-i \epsilon} \tag{6.242}
\end{equation*}
$$

for $a>0$ and $b>0$. First we regularize the pole at $x=0$ by using a method devised by Cauchy

$$
\begin{equation*}
K=\lim _{\delta \rightarrow 0}\left[\lim _{\epsilon \rightarrow 0}\left(\int_{-a}^{-\delta} d x \frac{f(x)}{x-i \epsilon}+\int_{-\delta}^{\delta} d x \frac{f(x)}{x-i \epsilon}+\int_{\delta}^{b} d x \frac{f(x)}{x-i \epsilon}\right)\right] . \tag{6.243}
\end{equation*}
$$

In the first and third integrals, since $|x| \geq \delta>\epsilon$, we may set $\epsilon=0$

$$
\begin{equation*}
K=\lim _{\delta \rightarrow 0}\left(\int_{-a}^{-\delta} d x \frac{f(x)}{x}+\int_{\delta}^{b} d x \frac{f(x)}{x}\right)+\lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \int_{-\delta}^{\delta} d x \frac{f(x)}{x-i \epsilon} \tag{6.244}
\end{equation*}
$$

We'll discuss the first two integrals before analyzing the last one.
The limit of the first two integrals is Cauchy's principal value

$$
\begin{equation*}
P \int_{-a}^{b} d x \frac{f(x)}{x} \equiv \lim _{\delta \rightarrow 0}\left(\int_{-a}^{-\delta} d x \frac{f(x)}{x}+\int_{\delta}^{b} d x \frac{f(x)}{x}\right) . \tag{6.245}
\end{equation*}
$$

If the function $f(x)$ is nearly constant near $x=0$, then the large negative values of $1 / x$ for $x$ slightly less than zero cancel the large positive values of $1 / x$ for $x$ slightly greater than zero. The point $x=0$ is not special; Cauchy's principal value about $x=y$ is defined by the limit

$$
\begin{equation*}
P \int_{-a}^{b} d x \frac{f(x)}{x-y} \equiv \lim _{\delta \rightarrow 0}\left(\int_{-a}^{y-\delta} d x \frac{f(x)}{x-y}+\int_{y+\delta}^{b} d x \frac{f(x)}{x-y}\right) . \tag{6.246}
\end{equation*}
$$

Using Cauchy's principal value, we may write $K$ as

$$
\begin{equation*}
K=P \int_{-a}^{b} d x \frac{f(x)}{x}+\lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \int_{-\delta}^{\delta} d x \frac{f(x)}{x-i \epsilon} . \tag{6.247}
\end{equation*}
$$

To evaluate the second integral, we use differentiability of $f(x)$ near $x=0$
to write $f(x)=f(0)+x f^{\prime}(0)$ and then extract the constants $f(0)$ and $f^{\prime}(0)$

$$
\begin{align*}
\lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \int_{-\delta}^{\delta} d x \frac{f(x)}{x-i \epsilon} & =\lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \int_{-\delta}^{\delta} d x \frac{f(0)+x f^{\prime}(0)}{x-i \epsilon} \\
& =f(0) \lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \int_{-\delta}^{\delta} \frac{d x}{x-i \epsilon}+f^{\prime}(0) \lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \int_{-\delta}^{\delta} \frac{x d x}{x-i \epsilon} \\
& =f(0) \lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \int_{-\delta}^{\delta} \frac{d x}{x-i \epsilon}+f^{\prime}(0) \lim _{\delta \rightarrow 0} 2 \delta \\
& =f(0) \lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \int_{-\delta}^{\delta} \frac{d x}{x-i \epsilon} \tag{6.248}
\end{align*}
$$

Since $1 /(z-i \epsilon)$ is analytic in the lower half-plane, we may deform the straight contour from $x=-\delta$ to $x=\delta$ into a tiny semicircle that avoids the point $x=0$ by setting $z=\delta e^{i \theta}$ and letting $\theta$ run from $\pi$ to $2 \pi$

$$
\begin{equation*}
K=P \int_{-a}^{b} d x \frac{f(x)}{x}+f(0) \lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \int_{-\delta}^{\delta} d z \frac{1}{z-i \epsilon} \tag{6.249}
\end{equation*}
$$

We now can set $\epsilon=0$ and so write $K$ as

$$
\begin{align*}
K & =P \int_{-a}^{b} d x \frac{f(x)}{x}+f(0) \lim _{\delta \rightarrow 0} \int_{\pi}^{2 \pi} i \delta e^{i \theta} d \theta \frac{1}{\delta e^{i \theta}} \\
& =P \int_{-a}^{b} d x \frac{f(x)}{x}+i \pi f(0) . \tag{6.250}
\end{align*}
$$

Recalling the definition (6.242) of $K$, we have

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0} \int_{-a}^{b} d x \frac{f(x)}{x-i \epsilon}=P \int_{-a}^{b} d x \frac{f(x)}{x}+i \pi f(0) \tag{6.251}
\end{equation*}
$$

for any function $f(x)$ that is differentiable at $x=0$. This is often written as

$$
\begin{equation*}
\frac{1}{x-i \epsilon}=P \frac{1}{x}+i \pi \delta(x) \quad \text { and } \quad \frac{1}{x+i \epsilon}=P \frac{1}{x}-i \pi \delta(x) \tag{6.252}
\end{equation*}
$$

and as

$$
\begin{equation*}
\frac{1}{x-y \pm i \epsilon}=P \frac{1}{x-y} \mp i \pi \delta(x-y) \tag{6.253}
\end{equation*}
$$

Example 6.47 (An application of Cauchy's trick) We use (6.252) to evaluate the integral

$$
\begin{equation*}
I=\int_{-\infty}^{\infty} d x \frac{1}{x+i \epsilon} \frac{1}{1+x^{2}} \tag{6.254}
\end{equation*}
$$

as

$$
\begin{equation*}
I=P \int_{-\infty}^{\infty} d x \frac{1}{x} \frac{1}{1+x^{2}}-i \pi \int_{-\infty}^{\infty} d x \frac{\delta(x)}{1+x^{2}} \tag{6.255}
\end{equation*}
$$

Because the function $1 / x\left(1+x^{2}\right)$ is odd, the principal part is zero. The integral over the delta function gives unity, so we have $I=-i \pi$.

Example 6.48 (Cubic form of Cauchy's principal value) Cauchy's principal value of the integral

$$
\begin{equation*}
P \int_{-a}^{b} \frac{f(x)}{x^{3}} d x \tag{6.256}
\end{equation*}
$$

is finite as long as $f(z)$ is analytic at $z=0$ with a vanishing first derivative there, $f^{\prime}(0)=0$. In this case Cauchy's integral formula (6.43) says that

$$
\begin{align*}
\int_{-a}^{b} d x \frac{f(x)}{(x-i \epsilon)^{3}} & =P \int_{-a}^{b} d x \frac{f(x)}{x^{3}}+\lim _{\delta \rightarrow 0} \int_{\pi}^{2 \pi} i \delta e^{i \theta} d \theta \frac{f\left(\delta e^{i \theta}\right)}{\left(\delta e^{i \theta}\right)^{3}} \\
& =P \int_{-a}^{b} d x \frac{f(x)}{x^{3}}+i \frac{\pi}{2} f^{\prime \prime}(0) . \tag{6.257}
\end{align*}
$$

Example 6.49 (Cauchy's principal value) By explicit use of the formula

$$
\begin{equation*}
\int \frac{d x}{x^{2}-a^{2}}=-\frac{1}{2 a} \log \frac{x+a}{x-a} \tag{6.258}
\end{equation*}
$$

one may show (exercise 6.33) that

$$
\begin{equation*}
P \int_{0}^{\infty} \frac{d x}{x^{2}-a^{2}}=\int_{0}^{a-\delta} \frac{d x}{x^{2}-a^{2}}+\int_{a+\delta}^{\infty} \frac{d x}{x^{2}-a^{2}}=0 \tag{6.259}
\end{equation*}
$$

a result we'll use in section 6.21.
Example $6.50(\sin k / k)$ To compute the integral

$$
\begin{equation*}
I_{s}=\int_{0}^{\infty} \frac{d k}{k} \sin k \tag{6.260}
\end{equation*}
$$

which we used to derive the formula (4.121) for the Green's function of the laplacian in three dimensions, we first express $I_{s}$ as an integral along the whole real axis

$$
\begin{equation*}
I_{s}=\int_{0}^{\infty} \frac{d k}{2 i k}\left(e^{i k}-e^{-i k}\right)=\int_{-\infty}^{\infty} \frac{d k}{2 i k} e^{i k} \tag{6.261}
\end{equation*}
$$

by which we actually mean the Cauchy principal part

$$
\begin{equation*}
I_{s}=\lim _{\delta \rightarrow 0}\left(\int_{-\infty}^{-\delta} d k \frac{e^{i k}}{2 i k}+\int_{\delta}^{\infty} d k \frac{e^{i k}}{2 i k}\right)=P \int_{-\infty}^{\infty} d k \frac{e^{i k}}{2 i k} \tag{6.262}
\end{equation*}
$$

Using Cauchy's trick (6.252), we have

$$
\begin{equation*}
I_{s}=P \int_{-\infty}^{\infty} d k \frac{e^{i k}}{2 i k}=\int_{-\infty}^{\infty} d k \frac{e^{i k}}{2 i(k+i \epsilon)}+\int_{-\infty}^{\infty} d k i \pi \delta(k) \frac{e^{i k}}{2 i} . \tag{6.263}
\end{equation*}
$$

To the first integral, we add a ghost contour around the upper half-plane. For the contour from $k=L$ to $k=L+i H$ and then to $k=-L+i H$ and then down to $k=-L$, one may show (exercise 6.36) that the integral of $\exp (i k) / k$ vanishes in the double limit $L \rightarrow \infty$ and $H \rightarrow \infty$. With this ghost contour, the first integral therefore vanishes because the pole at $k=-i \epsilon$ is in the lower half plane. The delta function in the second integral then gives $\pi / 2$, so that

$$
\begin{equation*}
I_{s}=\oint d k \frac{e^{i k}}{2 i(k+i \epsilon)}+\frac{\pi}{2}=\frac{\pi}{2} \tag{6.264}
\end{equation*}
$$

as stated in (4.120).

Example 6.51 (The Feynman propagator) Adding $\pm i \epsilon$ to the denominator of a pole term of an integral formula for a function $f(x)$ can slightly shift the pole into the upper or lower half plane, causing the pole to contribute if a ghost contour goes around the upper half-plane or the lower half-plane. Such an $i \epsilon$ can impose a boundary condition on a Green's function.

The Feynman propagator $\Delta_{F}(x)$ is a Green's function for the KleinGordon differential operator (Weinberg, 1995, pp. 274-280)

$$
\begin{equation*}
\left(m^{2}-\square\right) \Delta_{F}(x)=\delta^{4}(x) \tag{6.265}
\end{equation*}
$$

in which $x=\left(x^{0}, \boldsymbol{x}\right)$ and

$$
\begin{equation*}
\square=\triangle-\frac{\partial^{2}}{\partial t^{2}}=\triangle-\frac{\partial^{2}}{\partial\left(x^{0}\right)^{2}} \tag{6.266}
\end{equation*}
$$

is the four-dimensional version of the laplacian $\triangle \equiv \nabla \cdot \nabla$. Here $\delta^{4}(x)$ is the four-dimensional Dirac delta function (4.39)

$$
\begin{equation*}
\delta^{4}(x)=\int \frac{d^{4} q}{(2 \pi)^{4}} \exp \left[i\left(\boldsymbol{q} \cdot \boldsymbol{x}-q^{0} x^{0}\right)\right]=\int \frac{d^{4} q}{(2 \pi)^{4}} e^{i q x} \tag{6.267}
\end{equation*}
$$

in which $q x=\boldsymbol{q} \cdot \boldsymbol{x}-q^{0} x^{0}$ is the Lorentz-invariant inner product of the 4vectors $q$ and $x$. There are many Green's functions that satisfy Eq.(6.265). Feynman's propagator $\Delta_{F}(x)$

$$
\begin{equation*}
\Delta_{F}(x)=\int \frac{d^{4} q}{(2 \pi)^{4}} \frac{\exp (i q x)}{q^{2}+m^{2}-i \epsilon}=\int \frac{d^{3} q}{(2 \pi)^{3}} \int_{-\infty}^{\infty} \frac{d q^{0}}{2 \pi} \frac{e^{i \boldsymbol{q} \cdot \boldsymbol{x}-i q^{0} x^{0}}}{q^{2}+m^{2}-i \epsilon} \tag{6.268}
\end{equation*}
$$

is the one that satisfies boundary conditions that will become evident when we analyze the effect of its $i \epsilon$. The quantity $E_{\boldsymbol{q}}=\sqrt{\boldsymbol{q}^{2}+m^{2}}$ is the energy of a particle of mass $m$ and momentum $\boldsymbol{q}$ in natural units with the speed of light $c=1$. Using this abbreviation and setting $\epsilon^{\prime}=\epsilon / 2 E_{q}$, we may write the denominator as
$q^{2}+m^{2}-i \epsilon=\boldsymbol{q} \cdot \boldsymbol{q}-\left(q^{0}\right)^{2}+m^{2}-i \epsilon=\left(E_{q}-i \epsilon^{\prime}-q^{0}\right)\left(E_{q}-i \epsilon^{\prime}+q^{0}\right)+\epsilon^{\prime 2}$
in which $\epsilon^{\prime 2}$ is negligible. Dropping the prime on $\epsilon$, we do the $q^{0}$ integral

$$
\begin{equation*}
I(\boldsymbol{q})=-\int_{-\infty}^{\infty} \frac{d q^{0}}{2 \pi} e^{-i q^{0} x^{0}} \frac{1}{\left[q^{0}-\left(E_{\boldsymbol{q}}-i \epsilon\right)\right]\left[q^{0}-\left(-E_{\boldsymbol{q}}+i \epsilon\right)\right]} . \tag{6.270}
\end{equation*}
$$

As shown in Fig. 6.9, the integrand

$$
\begin{equation*}
e^{-i q^{0} x^{0}} \frac{1}{\left[q^{0}-\left(E_{\boldsymbol{q}}-i \epsilon\right)\right]\left[q^{0}-\left(-E_{\boldsymbol{q}}+i \epsilon\right)\right]} \tag{6.271}
\end{equation*}
$$

has poles at $E_{\boldsymbol{q}}-i \epsilon$ and at $-E_{\boldsymbol{q}}+i \epsilon$. When $x^{0}>0$, we can add a ghost contour that goes clockwise around the lower half-plane and get

$$
\begin{equation*}
I(\boldsymbol{q})=i e^{-i E_{\boldsymbol{q}} x^{0}} \frac{1}{2 E_{\boldsymbol{q}}} \text { for } x^{0}>0 \tag{6.272}
\end{equation*}
$$

When $x^{0}<0$, our ghost contour goes counterclockwise around the upper half-plane, and we get

$$
\begin{equation*}
I(\boldsymbol{q})=i e^{i E_{\boldsymbol{q}} x^{0}} \frac{1}{2 E_{\boldsymbol{q}}} \text { for } x^{0}<0 \tag{6.273}
\end{equation*}
$$

Using the step function $\theta(x)=(x+|x|) / 2$, we combine (6.272) and (6.273)

$$
\begin{equation*}
-i I(\boldsymbol{q})=\frac{1}{2 E_{\boldsymbol{q}}}\left[\theta\left(x^{0}\right) e^{-i E_{\boldsymbol{q}} x^{0}}+\theta\left(-x^{0}\right) e^{i E_{\boldsymbol{q}} x^{0}}\right] . \tag{6.274}
\end{equation*}
$$

In terms of the Lorentz-invariant function

$$
\begin{equation*}
\Delta_{+}(x)=\frac{1}{(2 \pi)^{3}} \int \frac{d^{3} q}{2 E_{\boldsymbol{q}}} \exp \left[i\left(\boldsymbol{q} \cdot \boldsymbol{x}-E_{\boldsymbol{q}} x^{0}\right)\right] \tag{6.275}
\end{equation*}
$$

and with a factor of $-i$, Feynman's propagator (25.161) is

$$
\begin{equation*}
-i \Delta_{F}(x)=\theta\left(x^{0}\right) \Delta_{+}(x)+\theta\left(-x^{0}\right) \Delta_{+}\left(\boldsymbol{x},-x^{0}\right) . \tag{6.276}
\end{equation*}
$$

## Ghost contours and the Feynman propagator



Figure 6.9 In equation (6.271), the function $f\left(q^{0}\right)$ has poles at $\pm\left(E_{\boldsymbol{q}}-i \epsilon\right)$, and the function $\exp \left(-i q^{0} x^{0}\right)$ is exponentially suppressed in the lower half plane if $x^{0}>0$ and in the upper half plane if $x^{0}<0$. So we can add a ghost contour (...) in the LHP if $x^{0}>0$ and in the UHP if $x^{0}<0$.

The integral (6.275) defining $\Delta_{+}(x)$ is insensitive to the sign of $\boldsymbol{q}$, and so

$$
\begin{align*}
\Delta_{+}(-x) & =\frac{1}{(2 \pi)^{3}} \int \frac{d^{3} q}{2 E_{\boldsymbol{q}}} \exp \left[i\left(-\boldsymbol{q} \cdot \boldsymbol{x}+E_{\boldsymbol{q}} x^{0}\right)\right]  \tag{6.277}\\
& =\frac{1}{(2 \pi)^{3}} \int \frac{d^{3} q}{2 E_{\boldsymbol{q}}} \exp \left[i\left(\boldsymbol{q} \cdot \boldsymbol{x}+E_{\boldsymbol{q}} x^{0}\right)\right]=\Delta_{+}\left(\boldsymbol{x},-x^{0}\right)
\end{align*}
$$

Thus we arrive at the standard form of the Feynman propagator

$$
\begin{equation*}
-i \Delta_{F}(x)=\theta\left(x^{0}\right) \Delta_{+}(x)+\theta\left(-x^{0}\right) \Delta_{+}(-x) . \tag{6.278}
\end{equation*}
$$

The annihilation operators $a(\boldsymbol{q})$ and the creation operators $a^{\dagger}(\boldsymbol{p})$ of a scalar field $\phi(x)$ satisfy the commutation relations

$$
\begin{equation*}
\left[a(\boldsymbol{q}), a^{\dagger}(\boldsymbol{p})\right]=\delta^{3}(\boldsymbol{q}-\boldsymbol{p}) \quad \text { and } \quad[a(\boldsymbol{q}), a(\boldsymbol{p})]=\left[a^{\dagger}(\boldsymbol{q}), a^{\dagger}(\boldsymbol{p})\right]=0 . \tag{6.279}
\end{equation*}
$$

Thus the commutator of the positive-frequency part

$$
\begin{equation*}
\phi^{+}(x)=\int \frac{d^{3} p}{\sqrt{(2 \pi)^{3} 2 p^{0}}} \exp \left[i\left(\boldsymbol{p} \cdot \boldsymbol{x}-p^{0} x^{0}\right)\right] a(\boldsymbol{p}) \tag{6.280}
\end{equation*}
$$

of a scalar field $\phi=\phi^{+}+\phi^{-}$with its negative-frequency part

$$
\begin{equation*}
\phi^{-}(y)=\int \frac{d^{3} q}{\sqrt{(2 \pi)^{3} 2 q^{0}}} \exp \left[-i\left(\boldsymbol{q} \cdot \boldsymbol{y}-q^{0} y^{0}\right)\right] a^{\dagger}(\boldsymbol{q}) \tag{6.281}
\end{equation*}
$$

is the Lorentz-invariant function $\Delta_{+}(x-y)$

$$
\begin{align*}
{\left[\phi^{+}(x), \phi^{-}(y)\right] } & =\int \frac{d^{3} p d^{3} q}{(2 \pi)^{3} 2 \sqrt{q^{0} p^{0}}} e^{i p x-i q y}\left[a(\boldsymbol{p}), a^{\dagger}(\boldsymbol{q})\right] \\
& =\int \frac{d^{3} p}{(2 \pi)^{3} 2 p^{0}} e^{i p(x-y)}=\Delta_{+}(x-y) \tag{6.282}
\end{align*}
$$

in which $p(x-y)=\boldsymbol{p} \cdot(\boldsymbol{x}-\boldsymbol{y})-p^{0}\left(x^{0}-y^{0}\right)$.
The Lorentz-invariant function $\Delta_{+}(x)$ depends only upon $x^{2}=\boldsymbol{x}^{2}-\left(x^{0}\right)^{2}$, and for spacelike separations, $x^{2} \equiv r^{2}>0$, has the value (Weinberg, 1995, p. 202)

$$
\begin{equation*}
\Delta_{+}(x)=\frac{m}{4 \pi^{2} r} \int_{0}^{\infty} d u \frac{u \sin (m r u)}{\sqrt{u^{2}+1}}=\frac{m}{4 \pi^{2} r} K_{1}(m r) \tag{6.283}
\end{equation*}
$$

in which $r=\sqrt{x^{2}}, u=p / m$, and $K_{1}(x)$ is a Hankel function (section 10.6). For $|x| \ll 1, K_{1}(x)$ is approximately

$$
\begin{equation*}
K_{1}(x)=\frac{1}{x}+\frac{x}{2}\left[\log \left(\frac{x}{2}\right)+\gamma-\frac{1}{2}\right]+\frac{x^{3}}{16}\left[\log \left(\frac{x}{2}\right)+\gamma-\frac{5}{4}\right]+\cdots . \tag{6.284}
\end{equation*}
$$

But at timelike separations $x^{2} \equiv-t^{2}<0$, the Lorentz-invariant function $\Delta_{+}(x)$

$$
\begin{equation*}
\Delta_{+}(x)=\frac{m^{2}}{4 \pi^{2}} \int_{0}^{\infty} e^{-i m u \sqrt{-x^{2}}} \frac{u^{2} d u}{\sqrt{u^{2}+1}} \tag{6.285}
\end{equation*}
$$

is a singular distribution.
The Feynman propagator arises most simply as the mean value in the vacuum of the time-ordered product of the fields $\phi(x)$ and $\phi(y)$

$$
\begin{equation*}
\mathcal{T}\{\phi(x) \phi(y)\} \equiv \theta\left(x^{0}-y^{0}\right) \phi(x) \phi(y)+\theta\left(y^{0}-x^{0}\right) \phi(y) \phi(x) . \tag{6.286}
\end{equation*}
$$

The operators $a(\boldsymbol{p})$ and $a^{\dagger}(\boldsymbol{p})$ respectively annihilate the vacuum ket $a(\boldsymbol{p})|0\rangle=$ 0 and $\operatorname{bra}\langle 0| a^{\dagger}(\boldsymbol{p})=0$, and so by $(6.280 \& 6.281)$ do the positive- and negative-frequency parts of the field $\phi^{+}(z)|0\rangle=0$ and $\langle 0| \phi^{-}(z)=0$. Thus the mean value in the vacuum of the time-ordered product is

$$
\begin{align*}
\langle 0| \mathcal{T}\{\phi(x) \phi(y)\}|0\rangle= & \langle 0| \theta\left(x^{0}-y^{0}\right) \phi(x) \phi(y)+\theta\left(y^{0}-x^{0}\right) \phi(y) \phi(x)|0\rangle \\
= & \langle 0| \theta\left(x^{0}-y^{0}\right) \phi^{+}(x) \phi^{-}(y)+\theta\left(y^{0}-x^{0}\right) \phi^{+}(y) \phi^{-}(x)|0\rangle \\
= & \langle 0| \theta\left(x^{0}-y^{0}\right)\left[\phi^{+}(x), \phi^{-}(y)\right] \\
& \quad+\theta\left(y^{0}-x^{0}\right)\left[\phi^{+}(y), \phi^{-}(x)\right]|0\rangle . \tag{6.287}
\end{align*}
$$

But by (25.158), these commutators are $\Delta_{+}(x-y)$ and $\Delta_{+}(y-x)$. Thus the mean value in the vacuum of the time-ordered product

$$
\begin{align*}
\langle 0| \mathcal{T}\{\phi(x) \phi(y)\}|0\rangle & =\theta\left(x^{0}-y^{0}\right) \Delta_{+}(x-y)+\theta\left(y^{0}-x^{0}\right) \Delta_{+}(y-x) \\
& =-i \Delta_{F}(x-y) \tag{6.288}
\end{align*}
$$

is the Feynman propagator (6.276) multiplied by $-i$.

### 6.19 Dispersion relations

In many physical contexts, functions occur that are analytic in the upper half-plane. Suppose for instance that $\hat{f}(t)$ is a transfer function that determines an effect $e(t)$ due to a cause $c(t)$

$$
\begin{equation*}
e(t)=\int_{-\infty}^{\infty} d t^{\prime} \hat{f}\left(t-t^{\prime}\right) c\left(t^{\prime}\right) . \tag{6.289}
\end{equation*}
$$

If the system is causal, then the transfer function $\hat{f}\left(t-t^{\prime}\right)$ is zero for $t-t^{\prime}<0$, and so its Fourier transform

$$
\begin{equation*}
f(z)=\int_{-\infty}^{\infty} \frac{d t}{\sqrt{2 \pi}} \hat{f}(t) e^{i z t}=\int_{0}^{\infty} \frac{d t}{\sqrt{2 \pi}} \hat{f}(t) e^{i z t} \tag{6.290}
\end{equation*}
$$

will be analytic in the upper half-plane and will shrink as the imaginary part of $z=x+i y$ increases.

So let us assume that a function $f(z)$ is analytic in the upper half-plane and on the real axis and further that

$$
\begin{equation*}
\lim _{r \rightarrow \infty}\left|f\left(r e^{i \theta}\right)\right|=0 \quad \text { for } \quad 0 \leq \theta \leq \pi \tag{6.291}
\end{equation*}
$$

By Cauchy's integral formula (6.40), if $z_{0}$ lies in the upper half-plane, then $f\left(z_{0}\right)$ is given by the closed counterclockwise contour integral

$$
\begin{equation*}
f\left(z_{0}\right)=\frac{1}{2 \pi i} \oint \frac{f(z)}{z-z_{0}} d z \tag{6.292}
\end{equation*}
$$

in which the contour runs along the real axis and then loops over the semicircle

$$
\begin{equation*}
\lim _{r \rightarrow \infty} r e^{i \theta} \quad \text { for } \quad 0 \leq \theta \leq \pi \tag{6.293}
\end{equation*}
$$

Our assumption (6.291) about the behavior of $f(z)$ in the upper half plane
implies that this contour (6.293) is a ghost contour because its modulus is bounded by

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \frac{1}{2 \pi} \int \frac{\left|f\left(r e^{i \theta}\right)\right| r}{r} d \theta=\lim _{r \rightarrow \infty}\left|f\left(r e^{i \theta}\right)\right|=0 \tag{6.294}
\end{equation*}
$$

So we may drop the ghost contour and write $f\left(z_{0}\right)$ as

$$
\begin{equation*}
f\left(z_{0}\right)=\frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x-z_{0}} d x \tag{6.295}
\end{equation*}
$$

Letting the imaginary part $y_{0}$ of $z_{0}=x_{0}+i y_{0}$ shrink to $\epsilon$

$$
\begin{equation*}
f\left(x_{0}\right)=\frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x-x_{0}-i \epsilon} d x \tag{6.296}
\end{equation*}
$$

and using Cauchy's trick (6.253), we get

$$
\begin{equation*}
f\left(x_{0}\right)=\frac{1}{2 \pi i} P \int_{-\infty}^{\infty} \frac{f(x)}{x-x_{0}} d x+\frac{i \pi}{2 \pi i} \int_{-\infty}^{\infty} f(x) \delta\left(x-x_{0}\right) d x \tag{6.297}
\end{equation*}
$$

or

$$
\begin{equation*}
f\left(x_{0}\right)=\frac{1}{2 \pi i} P \int_{-\infty}^{\infty} \frac{f(x)}{x-x_{0}} d x+\frac{1}{2} f\left(x_{0}\right) \tag{6.298}
\end{equation*}
$$

which is the dispersion relation

$$
\begin{equation*}
f\left(x_{0}\right)=\frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{f(x)}{x-x_{0}} d x \tag{6.299}
\end{equation*}
$$

If we separate $f(z)=u(z)+i v(z)$ into its real $u(z)$ and imaginary $v(z)$ parts, then this dispersion relation (6.299)

$$
\begin{align*}
u\left(x_{0}\right)+i v\left(x_{0}\right) & =\frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{u(x)+i v(x)}{x-x_{0}} d x  \tag{6.300}\\
& =\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(x)}{x-x_{0}} d x-\frac{i}{\pi} P \int_{-\infty}^{\infty} \frac{u(x)}{x-x_{0}} d x
\end{align*}
$$

breaks into its real and imaginary parts

$$
\begin{equation*}
u\left(x_{0}\right)=\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(x)}{x-x_{0}} d x \quad \text { and } \quad v\left(x_{0}\right)=-\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(x)}{x-x_{0}} d x \tag{6.301}
\end{equation*}
$$

which express $u$ and $v$ as Hilbert transforms of each other.
In applications of dispersion relations, the function $f(x)$ for $x<0$ sometimes is either physically meaningless or experimentally inaccessible. In such cases, there may be a symmetry that relates $f(-x)$ to $f(x)$. For instance, if
$f(x)$ is the Fourier transform of a real function $\hat{f}(k)$, then by Eq.(4.28) it obeys the symmetry relation

$$
\begin{equation*}
f^{*}(x)=u(x)-i v(x)=f(-x)=u(-x)+i v(-x) \tag{6.302}
\end{equation*}
$$

which says that $u$ is even, $u(-x)=u(x)$, and $v$ odd, $v(-x)=-v(x)$. Using these symmetries, one may show (exercise 6.39) that the Hilbert transformations (6.301) become

$$
\begin{equation*}
u\left(x_{0}\right)=\frac{2}{\pi} P \int_{0}^{\infty} \frac{x v(x)}{x^{2}-x_{0}^{2}} d x \quad \text { and } \quad v\left(x_{0}\right)=-\frac{2 x_{0}}{\pi} P \int_{0}^{\infty} \frac{u(x)}{x^{2}-x_{0}^{2}} d x \tag{6.303}
\end{equation*}
$$

which do not require input at negative values of $x$.

### 6.20 Kramers-Kronig relations

If we use $\sigma \boldsymbol{E}$ for the current density $\boldsymbol{J}$ and $\boldsymbol{E}(t)=e^{-i \omega t} \boldsymbol{E}$ for the electric field, then Maxwell's equation $\nabla \times \boldsymbol{B}=\mu \boldsymbol{J}+\epsilon \mu \dot{\boldsymbol{E}}$ becomes

$$
\begin{equation*}
\boldsymbol{\nabla} \times \boldsymbol{B}=-i \omega \epsilon \mu\left(1+i \frac{\sigma}{\epsilon \omega}\right) \boldsymbol{E} \equiv-i \omega n^{2} \epsilon_{0} \mu_{0} \boldsymbol{E} \tag{6.304}
\end{equation*}
$$

in which the squared index of refraction is

$$
\begin{equation*}
n^{2}(\omega)=\frac{\epsilon \mu}{\epsilon_{0} \mu_{0}}\left(1+i \frac{\sigma}{\epsilon \omega}\right) \tag{6.305}
\end{equation*}
$$

The imaginary part of $n^{2}$ represents the scattering of light mainly by electrons. At high frequencies in nonmagnetic materials $n^{2}(\omega) \rightarrow 1$, and so Kramers and Kronig applied the Hilbert-transform relations (6.303) to the function $n^{2}(\omega)-1$ in order to satisfy condition (6.291). Their relations are

$$
\begin{equation*}
\operatorname{Re}\left(n^{2}\left(\omega_{0}\right)\right)=1+\frac{2}{\pi} P \int_{0}^{\infty} \frac{\omega \operatorname{Im}\left(n^{2}(\omega)\right)}{\omega^{2}-\omega_{0}^{2}} d \omega \tag{6.306}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Im}\left(n^{2}\left(\omega_{0}\right)\right)=-\frac{2 \omega_{0}}{\pi} P \int_{0}^{\infty} \frac{\operatorname{Re}\left(n^{2}(\omega)\right)-1}{\omega^{2}-\omega_{0}^{2}} d \omega \tag{6.307}
\end{equation*}
$$

What Kramers and Kronig actually wrote was slightly different from these dispersion relations ( $6.306 \& 6.307$ ). H. A. Lorentz had shown that the index of refraction $n(\omega)$ is related to the forward scattering amplitude $f(\omega)$ for the scattering of light by a density $N$ of scatterers (Sakurai, 1982)

$$
\begin{equation*}
n(\omega)=1+\frac{2 \pi c^{2}}{\omega^{2}} N f(\omega) \tag{6.308}
\end{equation*}
$$

They used this formula to infer that the real part of the index of refraction approached unity in the limit of infinite frequency and applied the Hilbert transform (6.303)

$$
\begin{equation*}
\operatorname{Re}[n(\omega)]=1+\frac{2}{\pi} P \int_{0}^{\infty} \frac{\omega^{\prime} \operatorname{Im}\left[n\left(\omega^{\prime}\right)\right]}{\omega^{\prime 2}-\omega^{2}} d \omega^{\prime} \tag{6.309}
\end{equation*}
$$

The Lorentz relation (6.308) expresses the imaginary part $\operatorname{Im}[n(\omega)]$ of the index of refraction in terms of the imaginary part of the forward scattering amplitude $f(\omega)$

$$
\begin{equation*}
\operatorname{Im}[n(\omega)]=2 \pi(c / \omega)^{2} N \operatorname{Im}[f(\omega)] . \tag{6.310}
\end{equation*}
$$

And the optical theorem relates $\operatorname{Im}[f(\omega)]$ to the total cross-section

$$
\begin{equation*}
\sigma_{\text {tot }}=\frac{4 \pi}{|\boldsymbol{k}|} \operatorname{Im}[f(\omega)]=\frac{4 \pi c}{\omega} \operatorname{Im}[f(\omega)] . \tag{6.311}
\end{equation*}
$$

Thus we have $\operatorname{Im}[n(\omega)]=c N \sigma_{\text {tot }} /(2 \omega)$, and by the Lorentz relation (6.308) $\operatorname{Re}[n(\omega)]=1+2 \pi(c / \omega)^{2} N \operatorname{Re}[f(\omega)]$. Insertion of these formulas into the Kramers-Kronig integral (6.309) gives a dispersion relation for the real part of the forward scattering amplitude $f(\omega)$ in terms of the total cross-section

$$
\begin{equation*}
\operatorname{Re}[f(\omega)]=\frac{\omega^{2}}{2 \pi^{2} c} P \int_{0}^{\infty} \frac{\sigma_{\mathrm{tot}}\left(\omega^{\prime}\right)}{\omega^{\prime 2}-\omega^{2}} d \omega^{\prime} \tag{6.312}
\end{equation*}
$$

### 6.21 Phase and group velocities

Suppose $A(\boldsymbol{x}, t)$ is the amplitude

$$
\begin{equation*}
A(\boldsymbol{x}, t)=\int e^{i(\boldsymbol{p} \cdot \boldsymbol{x}-E t) / \hbar} A(\boldsymbol{p}) d^{3} p=\int e^{i(\boldsymbol{k} \cdot \boldsymbol{x}-\omega t)} B(\boldsymbol{k}) d^{3} k \tag{6.313}
\end{equation*}
$$

where $B(\boldsymbol{k})=\hbar^{3} A(\hbar \boldsymbol{k})$ varies slowly compared to the phase $\exp [i(\boldsymbol{k} \cdot \boldsymbol{x}-\omega t)]$. The phase velocity $\boldsymbol{v}_{\boldsymbol{p}}$ is the linear relation $\boldsymbol{x}=\boldsymbol{v}_{\boldsymbol{p}} t$ between $\boldsymbol{x}$ and $t$ that keeps the phase $\phi=\boldsymbol{p} \cdot \boldsymbol{x}-E t$ constant as a function of the time

$$
\begin{equation*}
0=\boldsymbol{p} \cdot d \boldsymbol{x}-E d t=\left(\boldsymbol{p} \cdot \boldsymbol{v}_{\boldsymbol{p}}-E\right) d t \quad \Longleftrightarrow \quad \boldsymbol{v}_{\boldsymbol{p}}=\frac{E}{p} \hat{\boldsymbol{p}}=\frac{\omega}{k} \hat{\boldsymbol{k}} \tag{6.314}
\end{equation*}
$$

in which $p=|\boldsymbol{p}|$, and $k=|\boldsymbol{k}|$. For light in the vacuum, $\boldsymbol{v}_{\boldsymbol{p}}=c=(\omega / k) \hat{\boldsymbol{k}}$. For a particle of mass $m>0$, the phase velocity exceeds the speed of light, $v_{\boldsymbol{p}}=\sqrt{c^{2} p^{2}+m^{2} c^{4}} / p \geq c$.

The more physical group velocity $\boldsymbol{v}_{g}$ is the linear relation $\boldsymbol{x}=\boldsymbol{v}_{g} t$
between $\boldsymbol{x}$ and $t$ that maximizes the amplitude $A(\boldsymbol{x}, t)$ by keeping the phase $\phi=\boldsymbol{p} \cdot \boldsymbol{x}-E t$ constant as a function of the momentum $\boldsymbol{p}$

$$
\begin{equation*}
\boldsymbol{\nabla}_{\boldsymbol{p}}(\boldsymbol{p} \cdot \boldsymbol{x}-E t)=\boldsymbol{x}-\boldsymbol{\nabla}_{\boldsymbol{p}} E(\boldsymbol{p}) t=0 . \tag{6.315}
\end{equation*}
$$

This condition of stationary phase gives the group velocity as

$$
\begin{equation*}
\boldsymbol{v}_{g}=\boldsymbol{\nabla}_{\boldsymbol{p}} E(\boldsymbol{p})=\boldsymbol{\nabla}_{\boldsymbol{k}} \omega(\boldsymbol{k}) . \tag{6.316}
\end{equation*}
$$

If $E=\boldsymbol{p}^{2} /(2 m)$, then $\boldsymbol{v}_{g}=\boldsymbol{p} / m$. For a relativistic particle with $E=$ $\sqrt{c^{2} p^{2}+m^{2} c^{4}}$, the group velocity is $\boldsymbol{v}_{g}=c^{2} \boldsymbol{p} / E \leq c$.

When light traverses a medium with a complex index of refraction $n(\boldsymbol{k})$, the wave vector $\boldsymbol{k}$ becomes complex, and its (positive) imaginary part represents the scattering of photons in the forward direction, mainly by electrons. For simplicity, we'll consider the propagation of light through a medium in one dimension, that of the forward direction of the beam. Then the (real) frequency $\omega(k)$ and the (complex) wave number $k$ are related by a complex index of refraction $n(k)=k c / \omega(k)$, and the phase velocity of the light is

$$
\begin{equation*}
v_{p}=\frac{\omega}{\operatorname{Re}(k)}=\frac{c}{\operatorname{Re}(n(k))} . \tag{6.317}
\end{equation*}
$$

If we regard the index of refraction as a function of the frequency $\omega$, instead of the wave number $k$, then by differentiating the real part of the relation $\omega n(\omega)=c k$ with respect to $\omega$, we find

$$
\begin{equation*}
n_{r}(\omega)+\omega \frac{d n_{r}(\omega)}{d \omega}=c \frac{d k_{r}}{d \omega} \tag{6.318}
\end{equation*}
$$

in which the subscript $r$ means real part. Thus the group velocity (6.316) of the light is

$$
\begin{equation*}
v_{g}=\frac{d \omega}{d k_{r}}=\frac{c}{n_{r}(\omega)+\omega d n_{r} / d \omega} . \tag{6.319}
\end{equation*}
$$

Optical physicists call the denominator the group index of refraction

$$
\begin{equation*}
n_{g}(\omega)=n_{r}(\omega)+\omega \frac{d n_{r}(\omega)}{d \omega} \tag{6.320}
\end{equation*}
$$

so that as in the expression (6.317) for the phase velocity $v_{p}=c / n_{r}(\omega)$, the group velocity is $v_{g}=c / n_{g}(\omega)$.

In some media, the derivative $d n_{r} / d \omega$ is large and positive, and the group velocity $v_{g}$ of light there can be much less than $c$ (Steinberg et al., 1993; Wang and Zhang, 1995) -as slow as $17 \mathrm{~m} / \mathrm{s}$ (Hau et al., 1999). This effect is called slow light. In certain other media, the derivative $d n / d \omega$ is so negative that the group index of refraction $n_{g}(\omega)$ is less than unity, and in them the group velocity $v_{g}$ exceeds $c$ ! This effect is called fast light. In some media,
the derivative $d n_{r} / d \omega$ is so negative that $d n_{r} / d \omega<-n_{r}(\omega) / \omega$, and then $n_{g}(\omega)$ is not only less than unity but also less than zero. In such a medium, the group velocity $v_{g}$ of light is negative! This effect is called backwards light.

Sommerfeld and Brillouin (Brillouin, 1960, ch. II \& III) anticipated fast light and concluded that it would not violate special relativity as long as the signal velocity - defined as the speed of the front of a square pulseremained less than $c$. Fast light does not violate special relativity (Stenner et al., 2003; Brunner et al., 2004) (Léon Brillouin 1889-1969, Arnold Sommerfeld 1868-1951).

Slow, fast, and backwards light can occur when the frequency $\omega$ of the light is near a peak or resonance in the total cross-section $\sigma_{\text {tot }}$ for the scattering of light by the atoms of the medium. To see why, recall that the index of refraction $n(\omega)$ is related to the forward scattering amplitude $f(\omega)$ and the density $N$ of scatterers by the formula (6.308)

$$
\begin{equation*}
n(\omega)=1+\frac{2 \pi c^{2}}{\omega^{2}} N f(\omega) \tag{6.321}
\end{equation*}
$$

and that the real part of the forward scattering amplitude is given by the Kramers-Kronig integral (6.312) of the total cross-section

$$
\begin{equation*}
\operatorname{Re}(f(\omega))=\frac{\omega^{2}}{2 \pi^{2} c} P \int_{0}^{\infty} \frac{\sigma_{\mathrm{tot}}\left(\omega^{\prime}\right) d \omega^{\prime}}{\omega^{\prime 2}-\omega^{2}} \tag{6.322}
\end{equation*}
$$

So the real part of the index of refraction is

$$
\begin{equation*}
n_{r}(\omega)=1+\frac{c N}{\pi} P \int_{0}^{\infty} \frac{\sigma_{\mathrm{tot}}\left(\omega^{\prime}\right) d \omega^{\prime}}{\omega^{\prime 2}-\omega^{2}} \tag{6.323}
\end{equation*}
$$

If the amplitude for forward scattering is of the Breit-Wigner form

$$
\begin{equation*}
f(\omega)=f_{0} \frac{\Gamma / 2}{\omega_{0}-\omega-i \Gamma / 2} \tag{6.324}
\end{equation*}
$$

then by (6.321) the real part of the index of refraction is

$$
\begin{equation*}
n_{r}(\omega)=1+\frac{\pi c^{2} N f_{0} \Gamma\left(\omega_{0}-\omega\right)}{\omega^{2}\left[\left(\omega-\omega_{0}\right)^{2}+\Gamma^{2} / 4\right]} \tag{6.325}
\end{equation*}
$$

and by (6.319) the group velocity is

$$
\begin{equation*}
v_{g}=c\left[1+\frac{\pi c^{2} N f_{0} \Gamma \omega_{0}}{\omega^{2}} \frac{\left[\left(\omega-\omega_{0}\right)^{2}-\Gamma^{2} / 4\right]}{\left[\left(\omega-\omega_{0}\right)^{2}+\Gamma^{2} / 4\right]^{2}}\right]^{-1} \tag{6.326}
\end{equation*}
$$

This group velocity $v_{g}$ is less than $c$ whenever $\left(\omega-\omega_{0}\right)^{2}>\Gamma^{2} / 4$. But we get
fast light $v_{g}>c$, if $\left(\omega-\omega_{0}\right)^{2}<\Gamma^{2} / 4$, and even backwards light, $v_{g}<0$, if $\omega \approx \omega_{0}$ with $4 \pi c^{2} N f_{0} / \Gamma \omega_{0} \gg 1$. Robert W. Boyd's papers explain how to make slow and fast light (Bigelow et al., 2003) and backwards light (Gehring et al., 2006).

We can use the principal-part identity (6.259) to subtract

$$
\begin{equation*}
0=\frac{c N}{\pi} \sigma_{\mathrm{tot}}(\omega) P \int_{0}^{\infty} \frac{1}{\omega^{\prime 2}-\omega^{2}} d \omega^{\prime} \tag{6.327}
\end{equation*}
$$

from the Kramers-Kronig integral (6.323) so as to write the index of refraction in the regularized form

$$
\begin{equation*}
n_{r}(\omega)=1+\frac{c N}{\pi} P \int_{0}^{\infty} \frac{\sigma_{\mathrm{tot}}\left(\omega^{\prime}\right)-\sigma_{\mathrm{tot}}(\omega)}{\omega^{\prime 2}-\omega^{2}} d \omega^{\prime} \tag{6.328}
\end{equation*}
$$

which we can differentiate and use in the group-velocity formula (6.319)

$$
\begin{equation*}
v_{g}(\omega)=c\left[1+\frac{c N}{\pi} P \int_{0}^{\infty} \frac{\left[\sigma_{\mathrm{tot}}\left(\omega^{\prime}\right)-\sigma_{\mathrm{tot}}(\omega)\right]\left(\omega^{\prime 2}+\omega^{2}\right)}{\left(\omega^{\prime 2}-\omega^{2}\right)^{2}} d \omega^{\prime}\right]^{-1} \tag{6.329}
\end{equation*}
$$

### 6.22 Method of Steepest Descent

Integrals like

$$
\begin{equation*}
I(r)=\int_{a}^{b} d z h(z) \exp (r f(z)) \tag{6.330}
\end{equation*}
$$

often are dominated by the exponential. We'll first assume that the real part $u(z)$ of $f(z)$ has one rather than many saddle points (6.341) between $a$ and $b$. Then the value of the integral $I(r)$ is independent of the contour between the end points $a$ and but is sensitive to $r$ and to the real part $u(z)$ of $f(z)=u(z)+i v(z)$. But since $f(z)$ is analytic, its real and imaginary parts $u(z)$ and $v(z)$ are harmonic functions which have no minima or maxima, only saddle points (6.56).

For simplicity, we'll assume that the real part $u(z)$ of $f(z)$ has only one saddle point between the points $a$ and $b$. (If it has more than one, we must repeat the computation that follows.) If $w$ is the saddle point, then $u_{x}=$ $u_{y}=0$ at $z=w$ which by the Cauchy-Riemann equations (6.10) implies that $v_{x}=v_{y}=0$. Thus the derivative of the function $f$ also vanishes at the saddle point $f^{\prime}(w)=0$, and so near $w$ we may approximate $f(z)$ as

$$
\begin{equation*}
f(z) \approx f(w)+\frac{1}{2}(z-w)^{2} f^{\prime \prime}(w) \tag{6.331}
\end{equation*}
$$

Let's write the second derivative as $f^{\prime \prime}(w)=\rho e^{i \phi}$ and choose our contour
through the saddle point $w$ to be a straight line $z=w+s e^{i \theta}$ with $\theta$ fixed for $z$ near $w$. As we vary $s$ along this line, we want

$$
\begin{equation*}
(z-w)^{2} f^{\prime \prime}(w)=s^{2} \rho e^{2 i \theta} e^{i \phi}<0 \tag{6.332}
\end{equation*}
$$

so we keep $2 \theta+\phi=\pi$ which ensures that near $z=w$

$$
\begin{equation*}
f(z) \approx f(w)-\frac{1}{2} \rho s^{2} . \tag{6.333}
\end{equation*}
$$

Since $z=w+s e^{i \theta}$, its differential is $d z=e^{i \theta} d s$, and the integral $I(r)$ is

$$
\begin{align*}
I(r) & \approx \int_{-\infty}^{\infty} h(w) \exp \left\{r\left[f(w)+\frac{1}{2}(z-w)^{2} f^{\prime \prime}(w)\right]\right\} d z  \tag{6.334}\\
& =h(w) e^{i \theta} e^{r f(w)} \int_{-\infty}^{\infty} \exp \left(-\frac{1}{2} r \rho s^{2}\right) d s=h(w) e^{i \theta} e^{r f(w)} \sqrt{\frac{2 \pi}{r \rho}} .
\end{align*}
$$

Moving the phase $e^{i \theta}$ inside the square root

$$
\begin{equation*}
I(r) \approx h(w) e^{r f(w)} \sqrt{\frac{2 \pi}{r \rho e^{-2 i \theta}}} \tag{6.335}
\end{equation*}
$$

and using $f^{\prime \prime}(w)=\rho e^{i \phi}$ and $2 \theta+\phi=\pi$ to show that

$$
\begin{equation*}
\rho e^{-2 i \theta}=\rho e^{i \phi-i \pi}=-\rho e^{i \phi}=-f^{\prime \prime}(w), \tag{6.336}
\end{equation*}
$$

we get our formula for the saddle-point integral (6.330)

$$
\begin{equation*}
I(r) \approx\left(\frac{2 \pi}{-r f^{\prime \prime}(w)}\right)^{1 / 2} h(w) e^{r f(w)} \tag{6.337}
\end{equation*}
$$

Example 6.52 (Stirling's formula for $n$ !) An exact formula for $n$ !

$$
\begin{equation*}
n!=\left.(-1)^{n} \frac{d^{n} y^{-1}}{d y^{n}}\right|_{y=1} \tag{6.338}
\end{equation*}
$$

is the integral

$$
\begin{equation*}
n!=\left.(-1)^{n} \frac{d^{n}}{d y^{n}} \int_{0}^{\infty} e^{-y z} d z\right|_{y=1}=\int_{0}^{\infty} z^{n} e^{-z} d z=\int_{0}^{\infty} e^{n \log z-z} d z \tag{6.339}
\end{equation*}
$$

Comparing it to the integral (6.330) for $I(r)$, we set $f(z)=n \log z-z$ as well as $r=1$ and $h(z)=1$. The saddle point $w$ is where $f^{\prime}(w)=0$ which is at $w=n$. Since $f^{\prime \prime}(n)=-1 / n$, our steepest-descent approximation (6.337) to $n$ ! gives us Stirling's formula

$$
\begin{equation*}
n!\approx e^{n \log n-n} \int_{-\infty}^{\infty} e^{-s^{2} /(2 n)} d s=\sqrt{2 \pi n}\left(\frac{n}{e}\right)^{n} \tag{6.340}
\end{equation*}
$$

If there are $n$ saddle points $w_{j}$ for $j=1, \ldots, n$, then the steepest-descent approximation to the integral $I(r)$ is the sum

$$
\begin{equation*}
I(r) \approx \sum_{j=1}^{N}\left(\frac{2 \pi}{-r f^{\prime \prime}\left(w_{j}\right)}\right)^{1 / 2} h\left(w_{j}\right) e^{r f\left(w_{j}\right)} . \tag{6.341}
\end{equation*}
$$

### 6.23 Applications to string theory

This section is optional on a first reading.
String theory may or may not have anything to do with physics, but it does provide many amusing applications of complex-variable theory. The coordinates $\sigma$ and $\tau$ of the world sheet of a string form a complex variable $z=e^{2(\tau-i \sigma)}$. The product of two operators $U(z)$ and $V(w)$ often has poles in $z-w$ as $z \rightarrow w$ but is well defined if $z$ and $w$ are radially ordered

$$
\begin{equation*}
\mathcal{R}\{U(z) V(w)\} \equiv U(z) V(w) \theta(|z|-|w|)+V(w) U(z) \theta(|w|-|z|) \tag{6.342}
\end{equation*}
$$

in which $\theta(x)=(x+|x|) / 2|x|$ is the step function. Since the modulus of $z=e^{2(\tau-i \sigma)}$ depends only upon $\tau$, radial order is time order in $\tau_{z}$ and $\tau_{w}$.

The modes $L_{n}$ of the principal component of the energy-momentum tensor $T(z)$ are defined by its Laurent series

$$
\begin{equation*}
T(z)=\sum_{n=-\infty}^{\infty} \frac{L_{n}}{z^{n+2}} \tag{6.343}
\end{equation*}
$$

and the inverse relation

$$
\begin{equation*}
L_{n}=\frac{1}{2 \pi i} \oint z^{n+1} T(z) d z \tag{6.344}
\end{equation*}
$$

Thus the commutator of two modes involves two loop integrals

$$
\begin{equation*}
\left[L_{m}, L_{n}\right]=\left[\frac{1}{2 \pi i} \oint z^{m+1} T(z) d z, \frac{1}{2 \pi i} \oint w^{n+1} T(w) d w\right] \tag{6.345}
\end{equation*}
$$

which we may deform as long as we cross no poles. Let's hold $w$ fixed and deform the $z$ loop so as to keep the $T$ 's radially ordered when $z$ is near $w$ as in Fig. 6.10. The operator-product expansion of the radially ordered product $\mathcal{R}\{T(z) T(w)\}$ is

$$
\begin{equation*}
\mathcal{R}\{T(z) T(w)\}=\frac{c / 2}{(z-w)^{4}}+\frac{2}{(z-w)^{2}} T(w)+\frac{1}{z-w} T^{\prime}(w)+\ldots \tag{6.346}
\end{equation*}
$$

in which the prime means derivative, $c$ is a constant, and the dots denote


Figure 6.10 The two counterclockwise circles about the origin preserve radial order when $z$ is near $w$ by veering slightly to $|z|>|w|$ for the product $U(z) V(w)$ and to $|z|<|w|$ for the product $V(w) U(z)$.
terms that are analytic in $z$ and $w$. The commutator introduces a minus sign that cancels most of the two contour integrals and converts what remains into an integral along a tiny circle $C_{w}$ about the point $w$ as in Fig. 6.10

$$
\begin{equation*}
\left[L_{m}, L_{n}\right]=\oint \frac{d w}{2 \pi i} w^{n+1} \oint_{C_{w}} \frac{d z}{2 \pi i} z^{m+1}\left[\frac{c / 2}{(z-w)^{4}}+\frac{2 T(w)}{(z-w)^{2}}+\frac{T^{\prime}(w)}{z-w}\right] \tag{6.347}
\end{equation*}
$$

After doing the $z$-integral, which is left as a homework exercise (6.42), one may use the Laurent series (6.343) for $T(w)$ to do the $w$-integral, which one may choose to be along a tiny circle about $w=0$, and so find the commutator

$$
\begin{equation*}
\left[L_{m}, L_{n}\right]=(m-n) L_{m+n}+\frac{c}{12} m\left(m^{2}-1\right) \delta_{m+n, 0} \tag{6.348}
\end{equation*}
$$

of the Virasoro algebra.
Example 6.53 (Using ghost contours to sum series) Consider the integral

$$
I=\oint_{C} \frac{\csc \pi z}{(z-a)^{2}} d z
$$

along the counterclockwise rectangular contour $C$ from $z=N+1 / 2-i Y$
to $z=N+1 / 2+i Y$ to $z=-N-1 / 2+i Y$ to $z=-N-1 / 2-i Y$ and back to $z=N+1 / 2-i Y$ in which $N$ is a positive integer, and $a$ is not an integer. In the twin limits $N \rightarrow \infty$ and $Y \rightarrow \infty$, the integral vanishes because on the contour $1 /|z-a|^{2} \approx 1 / N^{2}$ or $1 / Y^{2}$ while $|\csc \pi z| \leq 1$. We now shrink the contour down to tiny circles about the poles of $\csc \pi z$ at all the integers, $z=n$, and about the nonintegral value, $z=a$. By Cauchy's integral formula (6.42), the tiny contour integral around $z=a$ is

$$
\oint_{a} \frac{\csc \pi z}{(z-a)^{2}} d z=\left.2 \pi i \frac{d \csc \pi z}{d z}\right|_{z=a}=-2 \pi^{2} i \frac{\cos \pi a}{\sin ^{2} \pi a}
$$

In the twin limits $N \rightarrow \infty$ and $Y \rightarrow \infty$, the tiny counterclockwise integrals around the poles of $1 / \sin \pi z$ at $z=n \pi$ are (exercise 6.45)

$$
\sum_{n=-\infty}^{\infty} \oint_{n} \frac{\csc \pi z}{(z-a)^{2}} d z=2 i \sum_{n=-\infty}^{\infty}(-1)^{n} \frac{1}{(n-a)^{2}}
$$

We thus have the sum rule

$$
\sum_{n=-\infty}^{\infty}(-1)^{n} \frac{1}{(n-a)^{2}}=\pi^{2} \cot \pi a \csc \pi a
$$

## Further reading

For examples of conformal mappings see (Lin, 2011, section 3.5.7).

## Exercises

6.1 Compute the two limits (6.6) and (6.7) of example 6.2 but for the function $f(x, y)=x^{2}-y^{2}+2 i x y$. Do the limits now agree? Explain.
6.2 Show that if $f(z)$ is analytic in a disk, then the integral of $f(z)$ around a tiny (isosceles) triangle of side $\epsilon \ll 1$ inside the disk is zero to order $\epsilon^{2}$.
6.3 Show that the product $f(z) g(z)$ of two functions is analytic at $z$ if both $f(z)$ and $g(z)$ are analytic at $z$.
6.4 Derive the two integral representations (6.54) for Bessel's functions $J_{n}(t)$ of the first kind from the integral formula (6.53). Hint: Think of the integral (6.53) as running from $-\pi$ to $\pi$.
6.5 Do the integral

$$
\oint_{C} \frac{d z}{z^{2}-1}
$$

in which the contour $C$ is counterclockwise about the circle $|z|=2$.
6.6 The function $f(z)=1 / z$ is analytic in the region $|z|>0$. Compute the integral of $f(z)$ counterclockwise along the unit circle $z=e^{i \theta}$ for $0 \leq \theta \leq 2 \pi$. The contour lies entirely within the domain of analyticity of the function $f(z)$. Did you get zero? Why? If not, why not?
6.7 Let $P(z)$ be the polynomial

$$
\begin{equation*}
P(z)=\left(z-a_{1}\right)\left(z-a_{2}\right)\left(z-a_{3}\right) \tag{6.349}
\end{equation*}
$$

with roots $a_{1}, a_{2}$, and $a_{3}$. Let $R$ be the maximum of the three moduli $\left|a_{k}\right|$. (a) If the three roots are all different, evaluate the integral

$$
\begin{equation*}
I=\oint_{C} \frac{d z}{P(z)} \tag{6.350}
\end{equation*}
$$

along the counterclockwise contour $z=2 R e^{i \theta}$ for $0 \leq \theta \leq 2 \pi$. (b) Same exercise, but for $a_{1}=a_{2} \neq a_{3}$.
6.8 Compute the integral of the function $f(z)=e^{a z} /\left(z^{2}-3 z+2\right)$ along the counterclockwise contour $\mathcal{C}_{\square}$ that follows the perimeter of a square of side 6 centered at the origin. That is, find

$$
\begin{equation*}
I=\oint_{\mathcal{C}_{\square}} \frac{e^{a z}}{z^{2}-3 z+2} d z \tag{6.351}
\end{equation*}
$$

6.9 Use Cauchy's integral formula (6.44) and Rodrigues's expression (6.45) for Legendre's polynomial $P_{n}(x)$ to derive Schlaefli's formula (6.46).
6.10 Use Schlaefli's formula (6.46) for the Legendre polynomials and Cauchy's integral formula (6.40) to compute the value of $P_{n}(-1)$.
6.11 Evaluate the counterclockwise integral around the unit circle $|z|=1$

$$
\begin{equation*}
\oint\left(3 \sinh ^{2} 2 z-4 \cosh ^{3} z\right) \frac{d z}{z} \tag{6.352}
\end{equation*}
$$

6.12 Evaluate the counterclockwise integral around the circle $|z|=2$

$$
\begin{equation*}
\oint \frac{z^{3}}{z^{4}-1} d z \tag{6.353}
\end{equation*}
$$

6.13 Evaluate the contour integral of the function $f(z)=\sin w z /(z-5)^{3}$ along the curve $z=6+4(\cos t+i \sin t)$ for $0 \leq t \leq 2 \pi$.
6.14 Evaluate the contour integral of the function $f(z)=\sin w z /(z-5)^{3}$ along the curve $z=-6+4(\cos t+i \sin t)$ for $0 \leq t \leq 2 \pi$.
6.15 Is the function $f(x, y)=x^{2}+i y^{2}$ analytic?
6.16 Is the function $f(x, y)=x^{3}-3 x y^{2}+3 i x^{2} y-i y^{3}$ analytic? Is the function $x^{3}-3 x y^{2}$ harmonic? Does it have a minimum or a maximum? If so, what are they?
6.17 Is the function $f(x, y)=x^{2}+y^{2}+i\left(x^{2}+y^{2}\right)$ analytic? Is $x^{2}+y^{2}$ a harmonic function? What is its minimum, if it has one?
6.18 Derive the first three nonzero terms of the Laurent series for $f(z)=$ $1 /\left(e^{z}-1\right)$ about $z=0$.
6.19 Assume that a function $g(z)$ is meromorphic in $R$ and has a Laurent series (6.106) about a point $w \in R$. Show that as $z \rightarrow w$, the ratio $g^{\prime}(z) / g(z)$ becomes (6.104).
6.20 Use a contour integral to evaluate the integral

$$
\begin{equation*}
I_{a}=\int_{0}^{\pi} \frac{d \theta}{a+\cos \theta}, \quad a>1 . \tag{6.354}
\end{equation*}
$$

6.21 Find the poles and residues of the functions $1 / \sin z$ and $1 / \cos z$.
6.22 Derive the integral formula (6.140) from (6.137).
6.23 Show that if $\operatorname{Re} w<0$, then for arbitrary complex $z$

$$
\begin{equation*}
\int_{-\infty}^{\infty} e^{w(x+z)^{2}} d x=\sqrt{\frac{\pi}{-w}} . \tag{6.355}
\end{equation*}
$$

6.24 Find the value of the integral

$$
\begin{equation*}
I=\int_{-\infty}^{\infty} \frac{1}{(x-i)(x-2 i)(x+3 i)} d x \tag{6.356}
\end{equation*}
$$

6.25 Use a ghost contour to evaluate the integral

$$
\int_{-\infty}^{\infty} \frac{x \sin x}{x^{2}+a^{2}} d x
$$

Show your work; do not just quote the result of a commercial math program.
6.26 For $a>0$ and $b^{2}-4 a c<0$, use a ghost contour to do the integral

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d x}{a x^{2}+b x+c} \tag{6.357}
\end{equation*}
$$

6.27 Show that

$$
\begin{equation*}
\int_{0}^{\infty} \cos a x e^{-x^{2}} d x=\frac{1}{2} \sqrt{\pi} e^{-a^{2} / 4} \tag{6.358}
\end{equation*}
$$

6.28 Show that

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d x}{1+x^{4}}=\frac{\pi}{\sqrt{2}} . \tag{6.359}
\end{equation*}
$$

6.29 Evaluate the integral

$$
\begin{equation*}
\int_{0}^{\infty} \frac{\cos x}{1+x^{4}} d x \tag{6.360}
\end{equation*}
$$

6.30 Show that the Yukawa Green's function (6.169) reproduces the Yukawa potential (6.159) when $n=3$. Use $K_{1 / 2}(x)=\sqrt{\pi / 2 x} e^{-x}(10.80)$.
6.31 Derive the two explicit formulas (6.209) and (6.210) for the square root of a complex number.
6.32 What is $(-i)^{i}$ ? What is the most general value of this expression?
6.33 Use the indefinite integral (6.258) to derive the principal-part formula (6.259).
6.34 The Bessel function $J_{n}(x)$ is given by the integral

$$
\begin{equation*}
J_{n}(x)=\frac{1}{2 \pi i} \oint_{C} e^{(x / 2)(z-1 / z)} \frac{d z}{z^{n+1}} \tag{6.361}
\end{equation*}
$$

along a counterclockwise contour about the origin. Find the generating function for these Bessel functions, that is, the function $G(x, z)$ whose Laurent series has the $J_{n}(x)$ 's as coefficients

$$
\begin{equation*}
G(x, z)=\sum_{n=-\infty}^{\infty} J_{n}(x) z^{n} . \tag{6.362}
\end{equation*}
$$

6.35 Show that the Heaviside step function $\theta(y)=(y+|y|) /(2|y|)$ is given by the integral

$$
\begin{equation*}
\theta(y)=\frac{1}{2 \pi i} \int_{-\infty}^{\infty} e^{i y x} \frac{d x}{x-i \epsilon} \tag{6.363}
\end{equation*}
$$

in which $\epsilon$ is an infinitesimal positive number.
6.36 Show that the integral of $\exp (i k) / k$ along the contour from $k=L$ to $k=L+i H$ and then to $k=-L+i H$ and then down to $k=-L$ vanishes in the double limit $L \rightarrow \infty$ and $H \rightarrow \infty$.
6.37 Use a ghost contour and a cut to evaluate the integral

$$
\begin{equation*}
I=\int_{-1}^{1} \frac{d x}{\left(x^{2}+1\right) \sqrt{1-x^{2}}} \tag{6.364}
\end{equation*}
$$

by imitating example 6.44. Be careful when picking up the poles at $z= \pm i$. If necessary, use the explicit square root formulas (6.209) and (6.210).
6.38 Redo the previous exercise (6.37) by defining the square roots so that the cuts run from $-\infty$ to -1 and from 1 to $\infty$. Take advantage of the evenness of the integrand and integrate on a contour that is slightly above the whole real axis. Then add a ghost contour around the upper half plane.
6.39 Show that if $u$ is even and $v$ is odd, then the Hilbert transforms (6.301) imply (6.303).
6.40 Show why the principal-part identity (6.259) lets one write the KramersKronig integral (6.323) for the index of refraction in the regularized form (6.328).
6.41 Use the formula (6.319) for the group velocity and the regularized expression (6.328) for the real part of the index of refraction $n_{r}(\omega)$ to derive formula (6.329) for the group velocity.
6.42 (a) Perform the $z$-integral in Eq.(6.347). (b) Use the result of part (a) to find the commutator [ $L_{m}, L_{n}$ ] of the Virasoro algebra. Hint: use the Laurent series (6.343).
6.43 Assume that $\epsilon(z)$ is analytic in a disk that contains a tiny circular contour $C_{w}$ about the point $w$ as in Fig. 6.10. Do the contour integral

$$
\begin{equation*}
\oint_{C_{w}} \epsilon(z)\left[\frac{c / 2}{(z-w)^{4}}+\frac{2 T(w)}{(z-w)^{2}}+\frac{T^{\prime}(w)}{z-w}\right] \frac{d z}{2 \pi i} \tag{6.365}
\end{equation*}
$$

and express your result in terms of $\epsilon(w), T(w)$, and their derivatives.
6.44 Show that if the coefficients $a_{k}$ of the equation $0=a_{0}+a_{1} z+\ldots+a_{n} z^{n}$ are real, then its $n$ roots $z_{k}$ are real or come in pairs that are complex conjugates, $z_{\ell}$ and $z_{\ell}^{*}$, of each other.
6.45 Show that if $a$ is not an integer, then the sum of the tiny counterclockwise integrals about the points $z=n$ of example 6.53 is

$$
\sum_{n=-\infty}^{\infty} \oint_{n} \frac{\csc \pi z}{(z-a)^{2}} d z=2 i \sum_{n=-\infty}^{\infty}(-1)^{n} \frac{1}{(n-a)^{2}}
$$

6.46 Use the trick of example 6.53 with $\csc \pi z \rightarrow \cot \pi z$ to show that

$$
\sum_{n=-\infty}^{\infty} \frac{1}{(n-a)^{2}}=\frac{\pi^{2}}{\sin ^{2} \pi a}
$$

as long as $a$ is not an integer.

