obeys the **Klein-Gordon equation**

$$(\nabla^2 - \partial_0^2 - m^2)\,\phi(x) \equiv (\Box - m^2)\,\phi(x) = 0. \tag{1.56}$$

1.8 Zero-point energy

The hamiltonian of a neutral scalar field is

$$H = \tfrac{1}{2}\int d^3x\, \dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2. \tag{1.57}$$

Substituting the Fourier expansion (1.55) of the field into this equation, we find

$$\begin{aligned} H = \tfrac{1}{2}\int d^3x & \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\frac{d^3k}{\sqrt{(2\pi)^3 2k^0}} \\ & \times \Big\{ \Big[-i\omega_p a(p)\, e^{ip\cdot x} + i\omega_p a^\dagger(p)\, e^{-ip\cdot x}\Big] \Big[-i\omega_k a(k)\, e^{ik\cdot x} + i\omega_k a^\dagger(k)\, e^{-ik\cdot x}\Big] \\ & + \Big[i\boldsymbol{p}\, a(p)\, e^{ip\cdot x} - i\boldsymbol{p}\, a^\dagger(p)\, e^{-ip\cdot x}\Big] \cdot \Big[i\boldsymbol{k}\, a(k)\, e^{ik\cdot x} - i\boldsymbol{k}\, a^\dagger(k)\, e^{-ik\cdot x}\Big] \\ & + m^2 \Big[a(p)\, e^{ip\cdot x} + a^\dagger(p)\, e^{-ip\cdot x}\Big] \Big[a(k)\, e^{ik\cdot x} + a^\dagger(k)\, e^{-ik\cdot x}\Big] \Big\} \end{aligned} \tag{1.58}$$

Terms proportional to $\delta(\boldsymbol{p}+\boldsymbol{k})$ are multiplied by $\boldsymbol{p}^2 + m^2 - \omega_p^2$ and vanish. We are left with

$$H = \int d^3p \left(\omega_p\, a^\dagger(\boldsymbol{p}) a(\boldsymbol{p}) + \frac{1}{2}\omega_p \left[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p})\right]\right). \tag{1.59}$$

which is often abbrevviated as

$$\begin{aligned} H &= \int d^3p\, p^0 \,\left(a^\dagger(p) a(p) + \tfrac{1}{2}\delta^3(\boldsymbol{0})\right) \\ &= \int d^3p\, p^0 \,\left(a^\dagger(p) a(p) + \frac{1}{2}\frac{V}{(2\pi)^3}\right). \end{aligned} \tag{1.60}$$

The first term may be taken to be the "renormalized" hamiltonian

$$H = \int d^3p\, p^0\, a^\dagger(p) a(p). \tag{1.61}$$

The second term is the zero-point energy density

$$\rho = \frac{E_0}{V} = \frac{1}{16\pi^3}\int d^3p\ p^0 = \frac{1}{4\pi^2}\int_0^\infty dp\ p^2\,\sqrt{p^2+m^2}. \tag{1.62}$$

Substituting the Fourier-series expansion

$$\phi(x) = \sum_{\boldsymbol{k}} \frac{1}{\sqrt{2\omega_k V}}\left[a_{\boldsymbol{k}}\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_k t)} + a_{\boldsymbol{k}}^\dagger\, e^{-i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_k t)}\right] \tag{1.63}$$

in which (4.189 of PM)

$$a_{\boldsymbol{k}} \equiv \sqrt{\frac{(2\pi)^3}{V}}\; a(\boldsymbol{k}) \tag{1.64}$$

into the hamiltonian (1.57), we find

$$\begin{aligned} H &= \frac{1}{2}\int d^3x \sum_{\boldsymbol{k},\boldsymbol{p}} \frac{\sqrt{\omega_k\omega_p}}{V}\left(a_k a_p^\dagger e^{ix\cdot(k-p)} + a_p^\dagger a_k e^{-ix\cdot(k-p)}\right)\\ &= \frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{p}} \sqrt{\omega_k\omega_p}\left(a_k a_p^\dagger + a_p^\dagger a_k\right)\delta_{\boldsymbol{k},\boldsymbol{p}}\\ &= \sum_{\boldsymbol{k},\boldsymbol{p}} \sqrt{\omega_k\omega_p}\left(a_k^\dagger a_p + \tfrac{1}{2}\delta_{\boldsymbol{k},\boldsymbol{p}}\right)\delta_{\boldsymbol{k},\boldsymbol{p}}\\ &= \sum_{\boldsymbol{k}} \omega_k\left(a_k^\dagger a_k + \tfrac{1}{2}\delta_{\boldsymbol{k},\boldsymbol{k}}\right). \end{aligned} \tag{1.65}$$

Once again, the first term may be taken to be the "renormalized" hamiltonian

$$H = \sum_{\boldsymbol{k}} \omega_k\, a_k^\dagger a_k. \tag{1.66}$$

To evaluate the second term, we write the Kronecker delta as

$$\delta_{\boldsymbol{k},\boldsymbol{k}} = \int e^{i\boldsymbol{x}\cdot(\boldsymbol{k}-\boldsymbol{k})}\frac{d^3x}{V} = \int \frac{d^3x}{V} = 1. \tag{1.67}$$

So the zero-point energy is

$$E_0 = \sum_{\boldsymbol{k}} \tfrac{1}{2}\omega_k, \tag{1.68}$$

and the zero-point energy density is

$$\rho = \frac{E_0}{V} = \frac{1}{V}\sum_{\boldsymbol{k}} \tfrac{1}{2}\,\omega_k. \tag{1.69}$$

The relation (1.64) between the operators of box and continuum quantization reveals that the two forms (1.61) and (1.66) of the renormalized hamiltonian are more similar than they might seem at first glance

$$H = \int d^3k\; \omega_k\, a^\dagger(k)\, a(k) = \frac{V}{(2\pi)^3}\int d^3k\; \omega_k\, a_k^\dagger\, a_k \sim \sum_{\boldsymbol{k}} \omega_k\, a_k^\dagger a_k \tag{1.70}$$

which is an instance of the relation

$$\int d^3k \sim \frac{(2\pi)^3}{V}\sum_{\boldsymbol{k}} \tag{1.71}$$

This relation (1.71) between integration and summation also reveals that the two forms (1.62) and (1.69) of the zero-point energy density are more similar than they might seem at first glance

$$\rho = \frac{1}{16\pi^3}\int d^3k\ \omega_k \sim \frac{1}{2(2\pi)^3}\frac{(2\pi)^3}{V}\sum_{\boldsymbol{k}}\ \omega_k = \frac{1}{V}\sum_{\boldsymbol{k}}\tfrac{1}{2}\omega_k. \tag{1.72}$$

1.9 Conserved charges

If the field ϕ adds and deletes charged particles, an interaction $\mathcal{H}(x)$ that is a polynomial in ϕ will not commute with the charge operator Q because ϕ^+ will lower the charge and ϕ^- will raise it. The standard way to solve this problem is to start with two hermitian fields ϕ_1 and ϕ_2 of the same mass. One defines a complex scalar field as a complex linear combination of the two fields

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right)\\ &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\left[\frac{1}{\sqrt{2}}\left(a_1(p) + ia_2(p)\right)e^{ip\cdot x} + \frac{1}{\sqrt{2}}\left(a_1^\dagger(p) + ia_2^\dagger(p)\right)e^{-ip\cdot x}\right].\end{aligned} \tag{1.73}$$

Setting

$$a(p) = \frac{1}{\sqrt{2}}\left(a_1(p) + ia_2(p)\right) \quad \text{and} \quad b^\dagger(p) = \frac{1}{\sqrt{2}}\left(a_1^\dagger(p) + ia_2^\dagger(p)\right) \tag{1.74}$$

so that

$$b(p) = \frac{1}{\sqrt{2}}\left(a_1(p) - ia_2(p)\right) \quad \text{and} \quad a^\dagger(p) = \frac{1}{\sqrt{2}}\left(a_1^\dagger(p) - ia_2^\dagger(p)\right) \tag{1.75}$$

we have

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\left[a(p)\,e^{ip\cdot x} + b^\dagger(p)\,e^{-ip\cdot x}\right] \tag{1.76}$$

and

$$\phi^\dagger(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\left[b(p)\,e^{ip\cdot x} + a^\dagger(p)\,e^{-ip\cdot x}\right]. \tag{1.77}$$

Since the commutation relations of the real creation and annihilation operators are for $i, j = 1, 2$

$$[a_i(p), a_j^\dagger(p')] = \delta_{ij}\,\delta^3(\boldsymbol{p} - \boldsymbol{p}') \quad \text{and} \quad [a_i(p), a_j(p')] = 0 = [a_i^\dagger(p), a_j^\dagger(p')] \tag{1.78}$$