Massless particles are not observed to have any continuous degree of freedom like θ; to avoid such a continuum of states, we must require that physical states (now called $\Psi_{k,\sigma}$) are eigenvectors of A and B with $a = b = 0$:

$$A\Psi_{k,\sigma} = B\Psi_{k,\sigma} = 0 \,. \tag{2.5.38}$$

These states are then distinguished by the eigenvalue of the remaining generator

$$J_3\Psi_{k,\sigma} = \sigma\Psi_{k,\sigma} \,. \tag{2.5.39}$$

Since the momentum **k** is in the three-direction, σ gives the component of angular momentum in the direction of motion, or *helicity*.

We are now in a position to calculate the Lorentz transformation properties of general massless particle states. First note that by use of the general arguments of Section 2.2, Eq. (2.5.32) generalizes for finite α and β to

$$U(S(\alpha,\beta)) = \exp(i\alpha A + i\beta B) \tag{2.5.40}$$

and for finite θ to

$$U(R(\theta)) = \exp(iJ_3\theta) \,. \tag{2.5.41}$$

An arbitrary element W of the little group can be put in the form (2.5.28), so that

$$U(W)\Psi_{k,\sigma} = \exp(i\alpha A + i\beta B)\exp(i\theta J_3)\Psi_{k,\sigma} = \exp(i\theta\sigma)\Psi_{k,\sigma}$$

and therefore Eq. (2.5.8) gives

$$D_{\sigma'\sigma}(W) = \exp(i\theta\sigma)\delta_{\sigma'\sigma} \,,$$

where θ is the angle defined by expressing W as in Eq. (2.5.28). The Lorentz transformation rule for a massless particle of arbitrary helicity is now given by Eqs. (2.5.11) and (2.5.18) as

$$U(\Lambda)\Psi_{p,\sigma} = \sqrt{\frac{(\Lambda p)^0}{p^0}} \exp\left(i\sigma\theta(\Lambda,p)\right)\Psi_{\Lambda p,\sigma} \tag{2.5.42}$$

with $\theta(\Lambda,p)$ defined by

$$W(\Lambda,p) \equiv L^{-1}(\Lambda p)\Lambda L(p) \equiv S\left(\alpha(\Lambda,p),\beta(\Lambda,p)\right)R\left(\theta(\Lambda,p)\right) \,. \tag{2.5.43}$$

We shall see in Section 5.9 that electromagnetic gauge invariance arises from the part of the little group parameterized by α and β.

At this point we have not yet encountered any reason that would forbid the helicity σ of a massless particle from being an arbitrary real number. As we shall see in Section 2.7, there are topological considerations that restrict the allowed values of σ to integers and half-integers, just as for massive particles.

To calculate the little-group element (2.5.43) for a given Λ and p, (and also to enable us to calculate the effect of space or time inversion on these states in the next section) we need to fix a convention for the standard Lorentz transformation that takes us from $k^\mu = (0, 0, \kappa, \kappa)$ to p^μ. This may conveniently be chosen to have the form

$$L(p) = R(\hat{\mathbf{p}})B(|\mathbf{p}|/\kappa) \tag{2.5.44}$$

where $B(u)$ is a pure boost along the three-direction:

$$B(u) \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (u^2+1)/2u & (u^2-1)/2u \\ 0 & 0 & (u^2-1)/2u & (u^2+1)/2u \end{bmatrix} \tag{2.5.45}$$

and $R(\hat{\mathbf{p}})$ is a pure rotation that carries the three-axis into the direction of the unit vector $\hat{\mathbf{p}}$. For instance, suppose we take $\hat{\mathbf{p}}$ to have polar and azimuthal angles θ and ϕ:

$$\hat{\mathbf{p}} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)\,. \tag{2.5.46}$$

Then we can take $R(\hat{\mathbf{p}})$ as a rotation by angle θ around the two-axis, which takes $(0, 0, 1)$ into $(\sin\theta,\ 0, \cos\theta)$, followed by a rotation by angle ϕ around the three-axis:

$$U(R(\hat{\mathbf{p}})) = \exp(i\phi J_3)\,\exp(i\theta J_2)\,, \tag{2.5.47}$$

where $0 \leq \theta \leq \pi,\quad 0 \leq \phi < 2\pi$. (We give $U(R(\hat{\mathbf{p}}))$ rather than $R(\hat{\mathbf{p}})$, together with a specification of the range of ϕ and θ, because shifting θ or ϕ by 2π would give the same rotation $R(\hat{\mathbf{p}})$, but a different sign for $U(R(\hat{\mathbf{p}}))$ when acting on half-integer spin states.) Since (2.5.47) is a rotation, and does take the three-axis into the direction (2.5.46), any other choice of such an $R(\hat{\mathbf{p}})$ would differ from this one by at most an initial rotation around the three-axis, corresponding to a mere redefinition of the phase of the one-particle states.

Note that the helicity is Lorentz-invariant; a massless particle of a given helicity σ looks the same (aside from its momentum) in all inertial frames. Indeed, we would be justified in thinking of massless particles of each different helicity as different species of particles. However, as we shall see in the next section, particles of opposite helicity are related by the symmetry of space inversion. Thus, because electromagnetic and gravitational forces obey space inversion symmetry, the massless particles of helicity ± 1 associated with electromagnetic phenomena are both called *photons*, and the massless particles of helicity ± 2 that are believed to be associated with gravitation are both called *gravitons*. On the other hand, the supposedly massless particles of helicity $\pm 1/2$ that are emitted in nuclear beta decay have no interactions (apart from gravitation) that respect the symmetry of space inversion, so these particles are given different names: *neutrinos* for helicity $+1/2$, and *antineutrinos* for helicity $-1/2$.

Even though the helicity of a massless particle is Lorentz-invariant, the state itself is not. In particular, because of the helicity-dependent phase factor $\exp(i\sigma\theta)$ in Eq. (2.5.42), a state formed as a linear superposition of one-particle states with opposite helicities will be changed by a Lorentz transformation into a different superposition. For instance, a general one-photon state of four-momenta may be written

$$\Psi_{p;\alpha} = \alpha_+ \Psi_{p,+1} + \alpha_- \Psi_{p,-1} ,$$

where

$$|\alpha_+|^2 + |\alpha_-|^2 = 1 .$$

The generic case is one of *elliptic polarization*, with $|\alpha_\pm|$ both non-zero and unequal. *Circular polarization* is the limiting case where either α_+ or α_- vanishes, and *linear polarization* is the opposite extreme, with $|\alpha_+| = |\alpha_-|$. The overall phase of α_+ and α_- has no physical significance, and for linear polarization may be adjusted so that $\alpha_- = \alpha_+^*$, but the relative phase is still important. Indeed, for linear polarizations with $\alpha_- = \alpha_+^*$, the phase of α_+ may be identified as the angle between the plane of polarization and some fixed reference direction perpendicular to $\mathbf{p}$. Eq. (2.5.42) shows that under a Lorentz transformation $\Lambda^\mu{}_\nu$, this angle rotates by an amount $\theta(\Lambda, p)$. Plane polarized gravitons can be defined in a similar way, and here Eq. (2.5.42) has the consequence that a Lorentz transformation Λ rotates the plane of polarization by an angle $2\theta(\Lambda, p)$.

2.6 Space Inversion and Time-Reversal

We saw in Section 2.3 that any homogeneous Lorentz transformation is either proper and orthochronous (i.e., $\mathrm{Det}\Lambda = +1$ and $\Lambda^0{}_0 \geq +1$) or else equal to a proper orthochronous transformation times either $\mathscr{P}$ or $\mathscr{T}$ or $\mathscr{P}\mathscr{T}$, where $\mathscr{P}$ and $\mathscr{T}$ are the space inversion and time-reversal transformations

$$\mathscr{P}^\mu{}_\nu = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \quad \mathscr{T}^\mu{}_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} .$$

It used to be thought self-evident that the fundamental multiplication rule of the Poincaré group

$$U(\bar{\Lambda}, \bar{a})\, U(\Lambda, a) = U(\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a})$$