Fermi Fields without Tears

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Abstract

One can construct Majorana and Dirac fields from fields that are only slightly more complicated than scalar fields.

Introduction

Some problems in quantum field theory are intrinsically difficult or insoluble; We intend to shed no light on them. Others are solved and pose few difficulties for students. There is, however, a class of solved problems that consistently confuse students year after year. Among the more prickly and more important of these is the construction of fields of spin one-half.

Weinberg has written the clearest and most complete discussions of this subject in his papers on massive [Weinberg, 1964a] and massless [Weinberg, 1964b] fields of any spin and in his magnificent books [Weinberg, 1995, Weinberg, 1996, Weinberg, 2000] on quantum field theory. We follow his notation and show how to make Majorana and Dirac fields out of simple scalar-like fields.

A Scalar Field

First let's recall the usual formula for a spin-zero field $\varphi(x)$

$$\varphi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \left[a(\mathbf{p})e^{ipx} + a^\dagger(\mathbf{p})e^{-ipx}\right]. \tag{1}$$

The annihilation and creation operators $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$ satisfy the commutation relations

$$\begin{aligned} [a(\mathbf{p}), a^\dagger(\mathbf{p}')] &= \delta(\mathbf{p} - \mathbf{p}') \\ [a(\mathbf{p}), a(\mathbf{p}')] &= 0 \\ [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] &= 0. \end{aligned} \tag{2}$$

They destroy and create spin-zero particles of mass m, momentum $\mathbf{p}$, and energy $p^0 = \sqrt{m^2 + \mathbf{p}^2}$.

These particles are their own anti-particles.

In our units, $\hbar = c = 1$.

Gamma Matrices

Weinberg's [Weinberg, 1995] choice of γ-matrices is

$$\gamma^k = -i \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \quad k = 1, 2, 3, \quad \text{and} \quad \gamma^0 = -i \begin{pmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{pmatrix}. \tag{3}$$

They satisfy the anti-commutation relations

$$[\gamma^a, \gamma^b]_+ = 2\eta^{ab}, \tag{4}$$

in which the flat space-time metric is

$$\eta^{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

Under hermitian conjugation, they transform as $(\gamma^k)^\dagger = \gamma^k$ and $(\gamma^0)^\dagger = -\gamma^0$.

A Scalar-like Field

For this choice of γ-matrices, we may define Majorana and Dirac fields in terms of the scalar-like field

$$\phi(x) = \int \frac{d^3p}{2\sqrt{(2\pi)^3 p^0 (p^0+m)}} \left[\begin{pmatrix} I \\ I \end{pmatrix} A(\mathbf{p}) e^{ipx} + i \begin{pmatrix} -\sigma_2 \\ \sigma_2 \end{pmatrix} A^*(\mathbf{p}) e^{-ipx} \right] \qquad (6)$$

where $A(\mathbf{p})$ and $A^*(\mathbf{p})$ are the 2-vectors

$$A(\mathbf{p}) = \begin{pmatrix} a(\mathbf{p}, \frac{1}{2}) \\ a(\mathbf{p}, -\frac{1}{2}) \end{pmatrix} \quad \text{and} \quad A^*(\mathbf{p}) = \begin{pmatrix} a^\dagger(\mathbf{p}, \frac{1}{2}) \\ a^\dagger(\mathbf{p}, -\frac{1}{2}) \end{pmatrix} \qquad (7)$$

and the operators $a(\mathbf{p}, \pm\frac{1}{2})$ and $a^\dagger(\mathbf{p}, \pm\frac{1}{2})$ satisfy the anti-commutation relations

$$\begin{aligned} [a(\mathbf{p},\sigma), a^\dagger(\mathbf{p}',\sigma')]_+ &= \delta_{\sigma\,\sigma'}\,\delta(\mathbf{p}-\mathbf{p}') \\ [a(\mathbf{p},\sigma), a(\mathbf{p}',\sigma')]_+ &= 0 \\ [a^\dagger(\mathbf{p},\sigma), a^\dagger(\mathbf{p}',\sigma')]_+ &= 0. \end{aligned} \qquad (8)$$

They destroy and create spin-one-half particles of mass m, momentum $\mathbf{p}$, spin one-half in the $\pm\hat{\mathbf{z}}$ direction, and energy $p^0 = \sqrt{m^2 + \mathbf{p}^2}$.

These particles are their own anti-particles.

The 2×2 matrices I and σ_2 are

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{9}$$

The scalar-like field $\phi(x)$ is

$$\phi(x) = \int \frac{d^3p}{2\sqrt{(2\pi)^3 p^0(p^0+m)}} \left[\begin{pmatrix} a(\mathbf{p}, \frac{1}{2}) \\ a(\mathbf{p}, -\frac{1}{2}) \\ a(\mathbf{p}, \frac{1}{2}) \\ a(\mathbf{p}, -\frac{1}{2}) \end{pmatrix} e^{ipx} + \begin{pmatrix} -a^\dagger(\mathbf{p}, -\frac{1}{2}) \\ a^\dagger(\mathbf{p}, \frac{1}{2}) \\ a^\dagger(\mathbf{p}, -\frac{1}{2}) \\ -a^\dagger(\mathbf{p}, \frac{1}{2}) \end{pmatrix} e^{-ipx} \right]. \tag{10}$$

An equivalent formula for the scalar-like field $\phi(x)$ is

$$\phi(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 p^0(p^0+m)}} \sum_{\sigma=-\frac{1}{2}}^{\frac{1}{2}} \left[u(\sigma)\, a(\mathbf{p},\sigma) e^{ipx} + v(\sigma)\, a^\dagger(\mathbf{p},\sigma) e^{-ipx} \right] \tag{11}$$

in which the spinors $u(\sigma)$ and $v(\sigma)$ are

$$u(\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u(-\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \tag{12}$$

and

$$v(\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \qquad v(-\tfrac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{13}$$

They are the spinors of zero momentum of the Majorana and Dirac fields.

The Klein-Gordon Equation

In the definition (6) of the scalar-like field $\phi(x)$, the energy p^0 is $\sqrt{m^2 + \mathbf{p}^2}$, and so

$$m^2 + p^2 = m^2 + \mathbf{p}^2 - (p^0)^2 = 0. \tag{14}$$

Thus the scalar-like field $\phi(x)$ satisfies the Klein-Gordon equation

$$\begin{aligned}(m^2 + \partial_0^2 - \nabla^2)\phi(x) &= (m^2 - \eta^{ab}\partial_a\partial_b)\phi(x) \\ &= (m^2 + p^2)\phi(x) = 0.\end{aligned} \tag{15}$$

The Majorana Field

The Majorana field $\chi(x)$ is obtained from derivatives of the scalar-like field $\phi(x)$:

$$\chi(x) = (m - \gamma^a \partial_a)\phi(x). \tag{16}$$

Because the scalar-like field $\phi(x)$ satisfies the Klein-Gordon equation (15) and because the γ-matrices satisfy the anti-commutation relations (4), the Majorana field $\chi(x)$ satisfies the Dirac equation:

$$\begin{aligned}
(\gamma^a \partial_a + m)\,\chi(x) &= (\gamma^a \partial_a + m)\,(m - \gamma^a \partial_a)\,\phi(x) \\
&= \left(m^2 - \gamma^a \gamma^b \partial_a \partial_b\right)\phi(x) \\
&= \left(m^2 - \tfrac{1}{2}[\gamma^a, \gamma^b]_+ \partial_a \partial_b\right)\phi(x) \\
&= \left(m^2 - \eta^{ab} \partial_a \partial_b\right)\phi(x) = 0.
\end{aligned} \tag{17}$$

Spinors

It follows from Eqs.(11 & 16) that the explicit form of the Majorana field $\chi(x)$ is

$$\begin{aligned}
\chi(x) &= (m - \gamma^a \partial_a)\phi(x) \\
&= \int \frac{d^3p}{\sqrt{2(2\pi)^3 p^0 (p^0 + m)}} \\
&\quad \times \sum_\sigma \left[(m - i\gamma^a p_a)\; u(\sigma)\, a(\mathbf{p},\sigma) e^{ipx} + (m + i\gamma^a p_a)\; v(\sigma)\, a^\dagger(\mathbf{p},\sigma) e^{-ipx} \right] \\
&= \int \frac{d^3p}{(2\pi)^{3/2}} \sum_\sigma \left[u(\mathbf{p},\sigma)\, a(\mathbf{p},\sigma) e^{ipx} + v(\mathbf{p},\sigma)\, a^\dagger(\mathbf{p},\sigma) e^{-ipx} \right] \qquad (18)
\end{aligned}$$

where the spinors $u(\mathbf{p},\sigma)$ and $v(\mathbf{p},\sigma)$ are

$$u(\mathbf{p},\sigma) = \frac{(m - i\gamma^a p_a)\; u(\sigma)}{\sqrt{2p^0(p^0+m)}} \qquad \text{and} \qquad v(\mathbf{p},\sigma) = \frac{(m + i\gamma^a p_a)\; v(\sigma)}{\sqrt{2p^0(p^0+m)}}. \qquad (19)$$

The Dirac Field

Suppose there are two spin-one-half particles of the same mass m described by the two operators $a_1(\mathbf{p}, \sigma)$ and $a_2(\mathbf{p}, \sigma)$ which satisfy the anti-commutation relations

$$[a_i(\mathbf{p}, \sigma), a_j^\dagger(\mathbf{p}', \sigma')]_+ = \delta_{ij}\, \delta_{\sigma\sigma'} \delta^3(\mathbf{p} - \mathbf{p}'). \tag{20}$$

Then by following Eqs.(6--17) and defining two 2-vectors $A_i(\mathbf{p}, \sigma)$ as in (7), we may construct two scalar-like fields

$$\phi_i(x) = \int \frac{d^3p}{2\sqrt{(2\pi)^3 p^0 (p^0 + m)}} \left[\begin{pmatrix} I \\ I \end{pmatrix} A_i(\mathbf{p}) e^{ipx} + i \begin{pmatrix} -\sigma_2 \\ \sigma_2 \end{pmatrix} A_i^*(\mathbf{p}) e^{-ipx} \right] \tag{21}$$

and from them two Majorana fields

$$\chi_i(x) = (m - \gamma^a \partial_a) \phi_i(x) \tag{22}$$

that satisfy the Dirac equation

$$(\gamma^a \partial_a + m)\, \chi_i(x) = 0. \tag{23}$$

Anti-Particles

Because the two fields $\phi_i(x)$ are of the same mass, we may combine them into a complex, scalar-like field

$$\Phi(x) = \frac{1}{\sqrt{2}}\left[\phi_1(x) + i\phi_2(x)\right]. \tag{24}$$

The complex operators

$$a(\mathbf{p}, \sigma) = \frac{1}{\sqrt{2}}\left[a_1(\mathbf{p}, \sigma) + i a_2(\mathbf{p}, \sigma)\right] \tag{25}$$

and

$$a^c(\mathbf{p}, \sigma) = \frac{1}{\sqrt{2}}\left[a_1(\mathbf{p}, \sigma) - i a_2(\mathbf{p}, \sigma)\right], \tag{26}$$

destroy particles that are each other's anti-particles.

From the complex 2-vectors

$$A(\mathbf{p}) = \frac{1}{\sqrt{2}}\left[A_1(\mathbf{p}) + iA_2(\mathbf{p})\right] = \begin{pmatrix} a(\mathbf{p}, \frac{1}{2}) \\ a(\mathbf{p}, -\frac{1}{2}) \end{pmatrix} \tag{27}$$

and

$$A^c(\mathbf{p}) = \frac{1}{\sqrt{2}}\left[A_1(\mathbf{p}) - iA_2(\mathbf{p})\right] = \begin{pmatrix} a^c(\mathbf{p}, \frac{1}{2}) \\ a^c(\mathbf{p}, -\frac{1}{2}) \end{pmatrix} \tag{28}$$

with

$$A^{c*}(\mathbf{p}) = \frac{1}{\sqrt{2}}\left[A_1(\mathbf{p}) - iA_2(\mathbf{p})\right]^* = \frac{1}{\sqrt{2}}\left[A_1^*(\mathbf{p}) + iA_2^*(\mathbf{p})\right] = \begin{pmatrix} a^{c\,\dagger}(\mathbf{p}, \frac{1}{2}) \\ a^{c\,\dagger}(\mathbf{p}, -\frac{1}{2}) \end{pmatrix}, \tag{29}$$

we can make a complex, scalar-like field $\Phi(x)$

$$\Phi(x) = \int \frac{d^3p}{2\sqrt{(2\pi)^3 p^0(p^0+m)}} \left[\begin{pmatrix} I \\ I \end{pmatrix} A(\mathbf{p}) e^{ipx} + i \begin{pmatrix} -\sigma_2 \\ \sigma_2 \end{pmatrix} A^{c*}(\mathbf{p}) e^{-ipx}\right]. \tag{30}$$

The Dirac Field

The Dirac field is then

$$\begin{aligned}
\psi(x) &= (m - \gamma^a \partial_a)\, \Phi(x) \\
&= (m - \gamma^a \partial_a)\, \frac{1}{\sqrt{2}} \left[\phi_1(x) + i\phi_2(x)\right] \\
&= \frac{1}{\sqrt{2}} \left[\chi_1(x) + i\chi_2(x)\right].
\end{aligned} \tag{31}$$

It satisfies the Dirac equation

$$(\gamma^a \partial_a + m)\, \psi(x) = 0 \tag{32}$$

because the Majorana fields χ_1 and χ_2 do. It follows from Eqs.(18, 30, & 31) that the explicit form of the Dirac field is

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sum_\sigma \left[u(\mathbf{p}, \sigma)\, a(\mathbf{p}, \sigma) e^{ipx} + v(\mathbf{p}, \sigma)\, a^{c\,\dagger}(\mathbf{p}, \sigma) e^{-ipx} \right] \tag{33}$$

where the spinors are the same as for the Majorana field, Eq.(19).

Other Conventions

We have defined Majorana and Dirac fields in terms of Weinberg's choice of γ-matrices. If one uses a different set of γ-matrices

$$\gamma^{a'} = S\gamma^a S^{-1}, \tag{34}$$

then the fields should be multiplied from the left by the matrix S:

$$\Phi'(x) = S\,\Phi(x), \quad \psi'(x) = S\,\psi(x), \quad etc. \tag{35}$$

Very Light Fermions

In the $m \to 0$ limit, it follows from their formulas (19) that the spinors $u(\mathbf{p},\sigma)$ and $v(\mathbf{p},\sigma)$ for $\mathbf{p} = p\,\hat{\mathbf{z}}$ are

$$\begin{aligned} u(p\,\hat{\mathbf{z}},\tfrac{1}{2}) &= \frac{i}{\sqrt{2}}\left(\gamma^0 - \gamma^3\right) u(\mathbf{0},\tfrac{1}{2}) = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} u(\mathbf{0},\tfrac{1}{2}) \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \end{aligned} \qquad (36)$$

$$u(p\,\hat{\mathbf{z}},-\tfrac{1}{2}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v(p\,\hat{\mathbf{z}},\tfrac{1}{2}) = -\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \& \quad v(p\,\hat{\mathbf{z}},-\tfrac{1}{2}) = -\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \qquad (37)$$

So the upper (lower) two components of a Majorana or Dirac field (the $(\frac{1}{2},0)$ ($(0,-\frac{1}{2})$) part) can only destroy particles of helicity $-\frac{1}{2}$ ($+\frac{1}{2}$).

Neutrinos

The upper two components of a Majorana field (the $(\frac{1}{2}, 0)$ part) can only create particles of helicity $+\frac{1}{2}$. The lower two components of a Majorana field (the $(0, -\frac{1}{2})$ part) can only create particles of helicity $-\frac{1}{2}$. The upper two components of a Dirac field (the $(\frac{1}{2}, 0)$ part) can only create antiparticles of helicity $+\frac{1}{2}$. The lower two components of a Dirac field (the $(0, -\frac{1}{2})$ part) can only create antiparticles of helicity $-\frac{1}{2}$.

The Standard Muddle represents neutrinos by fields with only the upper two components (the $(\frac{1}{2}, 0)$ part). Nobody knows whether these fields are Majorana or Dirac.

If Majorana, then the neutrino field uses $u(p\,\hat{\mathbf{z}}, -\frac{1}{2})\,a(p\,\hat{\mathbf{z}}, -\frac{1}{2})$ to destroy neutrinos of helicity $-\frac{1}{2}$ and uses $v(p\,\hat{\mathbf{z}}, \frac{1}{2})\,a^\dagger(p\,\hat{\mathbf{z}}, \frac{1}{2})$ to create neutrinos of helicity $\frac{1}{2}$. The hermitian adjoint of that field creates neutrinos of helicity $-\frac{1}{2}$ and destroys neutrinos of helicity $\frac{1}{2}$.

If Dirac, then the neutrino field uses $u(p\,\hat{\mathbf{z}}, -\frac{1}{2})\,a(p\,\hat{\mathbf{z}}, -\frac{1}{2})$ to destroy neutrinos of helicity $-\frac{1}{2}$ and uses $v(p\,\hat{\mathbf{z}}, \frac{1}{2})\,a^{c\dagger}(p\,\hat{\mathbf{z}}, \frac{1}{2})$ to create antineutrinos of helicity $\frac{1}{2}$. The hermitian adjoint of that field creates neutrinos of helicity $-\frac{1}{2}$ and destroys antineutrinos of helicity $\frac{1}{2}$.

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