In search of theories with finite energy

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Abstract

Recipes are presented for making quantum field theories whose ground states have finite energy. The recipe avoids the flaws of normal ordering.

I. THE PROBLEM

A hamiltonian of the form

$$H = \int d^3x \; Q^\dagger \, Q \tag{1}$$

$$2 + 2 = 4$$

has only nonnegative eigenvalues

$$H|E\rangle = E|E\rangle \tag{2}$$

because

$$E = \langle E|H|E\rangle = \int d^3x \; \langle E|Q^\dagger \, Q|E\rangle \geq 0 \tag{3}$$

in which for simplicity it was assumed that the state $|E\rangle$ is normalized to unity, $\langle E|E\rangle = 1$. So at least the nonnegative form (1) excludes energies that are infinitely negative.

The nonnegative form (1) has another advantage. The $E = 0$ eigenvalue equation

$$H|0\rangle = 0 \tag{4}$$

implies that

$$Q|0\rangle = 0 \tag{5}$$

which is easier to solve than the full $E = 0$ eigenvalue equation

$$H|0\rangle = \int d^3x \; Q^\dagger \, Q \; |0\rangle = 0. \tag{6}$$

But the ground state of the vacuum doesn't seem to have energy zero. Instead the energy density is $\rho_\Lambda = 5.83(16) \times 10^{-30}$ g cm^{-3} $= 5.83(16) \times 10^{-27}$ kg m^{-3}. So if instead of $Q|0\rangle = 0$, we have

$$Q|0\rangle = \sqrt{\epsilon}\,|0\rangle, \tag{7}$$

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then the average energy of the vacuum is

$$\langle 0|H|0\rangle = \langle 0| \int d^3x \; Q^\dagger \, Q \; |0\rangle = \epsilon V \tag{8}$$

in which V is the volume of the universe. So the energy density of the vacuum is

$$\rho = \frac{\epsilon\, V}{V} = \epsilon. \tag{9}$$

Eigenvalue equations like (6 and 8) are easier to solve than ones that are quadratic in Q.

The problem is that anticommuting operators are supposed to have eigenvalues that are Grassmann variables not complex numbers. So one would expect that the eigenvalues $\chi(x)$ of a fermionic operator $Q(x)$

$$Q(x)\,|\chi\rangle = \chi(x)|\chi\rangle \tag{10}$$

would be Grassmann with a vanishing anticommutator $\{\chi(x), \chi(y)\} = 0$.

Are there other possibilities? Let's consider one mode with operators a and $a^\dagger$

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad a^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{11}$$

Each of these matrices has only one eigenvalue $e = 0$

$$a|0\rangle = 0 \quad \text{and} \quad a^\dagger|1\rangle = 0. \tag{12}$$

Since a and $a^\dagger$ have only one eigenvalue each, they are defective matrices.

But $a + a^\dagger$ has two eigenvalues $e = \pm 1$, as does the complex linear combination $\alpha\, a + \beta a^\dagger$ whose two eigenvalues are

$$e^2 = \alpha\, \beta. \tag{13}$$

$e = \pm\sqrt{\alpha\beta}$. The explicit eigenvector equations are

$$(\alpha\, a + \beta a^\dagger)(x|0\rangle + y|1\rangle) = \alpha\, y\, |0\rangle + \beta\, x\, |1\rangle = \pm\sqrt{\alpha\beta}(x|0\rangle + y|1\rangle). \tag{14}$$

So $x = \pm\sqrt{\alpha/\beta}\, y$, and $y = \pm\sqrt{\beta/\alpha}\, x$. Normalization implies that

$$x^2 = \frac{\alpha}{\alpha + \beta} \quad \text{and} \quad y^2 = \frac{\beta}{\alpha + \beta}. \tag{15}$$

So altogether we have

$$(\alpha\, a + \beta a^\dagger) \left(\sqrt{\frac{\alpha}{\alpha + \beta}}|0\rangle \pm \sqrt{\frac{\beta}{\alpha + \beta}}|1\rangle \right) = \pm\sqrt{\alpha\beta} \left(\sqrt{\frac{\alpha}{\alpha + \beta}}|0\rangle \pm \sqrt{\frac{\beta}{\alpha + \beta}}|1\rangle \right). \tag{16}$$

The algebra for two modes $a, a^\dagger$ and $b, b^\dagger$ that anticommute

$$\{a, b\} = 0, \quad \{a, b^\dagger\} = 0, \quad \text{etc.} \tag{17}$$

is more complicated. We'll put a quanta to the left of b quanta:

$$a^\dagger|0,0\rangle = |1,0\rangle, \quad b^\dagger|0,0\rangle = |0,1\rangle, \quad \text{and} \quad a^\dagger b^\dagger|0,0\rangle = a^\dagger|0,1\rangle = |1,1\rangle. \tag{18}$$

An eigenstate of $\alpha a + \beta a^\dagger$ must satisfy

$$(\alpha a + \beta a^\dagger)\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big) = f\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big). \tag{19}$$

The αa terms in this equation are

$$\alpha a\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big) = \alpha y|0,0\rangle + \alpha w|0,1\rangle \tag{20}$$

while the $\beta a^\dagger$ terms are

$$\beta a^\dagger\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big) = \beta x|1,0\rangle + \beta z|1,1\rangle. \tag{21}$$

So the eigenvalue equation (20) for mode a is

$$\alpha y|0,0\rangle + \alpha w|0,1\rangle + \beta x|1,0\rangle + \beta z|1,1\rangle = f\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big). \tag{22}$$

So we need

$$\alpha y = fx, \quad \alpha w = fz, \quad \beta x = fy, \quad \beta z = fw \tag{23}$$

which imply that

$$f^2 = \alpha\beta \quad \text{and} \quad \frac{x}{y} = \sqrt{\frac{\alpha}{\beta}} = \frac{z}{w}. \tag{24}$$

The relation $f^2 = \alpha\beta$ echos $e^2 = \alpha\,\beta$ derived earlier (14).

An eigenstate of $\gamma b + \delta b^\dagger$ must satisfy

$$(\gamma b + \delta b^\dagger)\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big) = g\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big). \tag{25}$$

The γb terms in this equation are

$$\gamma b\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big) = \gamma z|0,0\rangle - \gamma w|1,0\rangle \tag{26}$$

while the $\delta b^\dagger$ terms are

$$\delta b^\dagger\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big) = \delta x|0,1\rangle - \delta y|1,1\rangle. \tag{27}$$

So the eigenvalue equation (26) for mode b is

$$\gamma z|0,0\rangle - \gamma w|1,0\rangle + \delta x|0,1\rangle - \delta y|1,1\rangle = g\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big). \tag{28}$$

So we need

$$\gamma z = gx, \quad -\gamma w = gy, \quad \delta x = gz, \quad -\delta y = gw \tag{29}$$

which imply that

$$g^2 = \gamma\,\delta \quad \text{and} \quad \frac{x}{z} = \sqrt{\frac{\gamma}{\delta}} = -\frac{y}{w}. \tag{30}$$

The relation $g^2 = \gamma\delta$ echos $e^2 = \alpha\,\beta$ derived earlier (14).

Are the two sets of eigenvalue conditions (25 and 31) consistent with each other? No. The a conditions (25) imply that

$$\frac{x}{y} = \frac{z}{w} \tag{31}$$

while the b conditions (31) imply that

$$\frac{x}{y} = -\frac{z}{w}. \tag{32}$$

The inconsistency arises from the incompatibility of the operators $\alpha a + \beta a^\dagger$ and $\gamma b + \delta b^\dagger$ which do not commute although and because they anticommute

$$\{\alpha a + \beta a^\dagger, \gamma b + \delta b^\dagger\} = 0. \tag{33}$$

That's why there's no eigenvector of both $\alpha a + \beta a^\dagger$ and $\gamma b + \delta b^\dagger$.

Yet every square matrix has eigenvectors and eigenvalues (section 1.27 of PM). That's why we were able to find an eigenvector (17) of $\alpha a + \beta a^\dagger$.

That's also why we can find an eigenvector of $A = (\alpha a + \beta a^\dagger)(\gamma b + \delta b^\dagger)$ and therefore of $-A = (\gamma b + \delta b^\dagger)(\alpha a + \beta a^\dagger)$. To do that, we carry out these steps:

$$\begin{aligned}(\alpha a + \beta a^\dagger)(\gamma b + \delta b^\dagger)&\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big)\\ &= e\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big) \qquad (34)\\ (\gamma b + \delta b^\dagger)(\alpha a + \beta a^\dagger)&\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big)\\ &= -\,e\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big). \qquad (35)\end{aligned}$$

The γb terms in the first equation are

$$\gamma b\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big) = \gamma z|0,0\rangle - \gamma w|1,0\rangle \tag{36}$$

while the $\delta b^\dagger$ terms are

$$\delta b^\dagger\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big) = \delta x|0,1\rangle - \delta y|1,1\rangle. \tag{37}$$

So we find from the first equation

$$\begin{aligned}(\alpha a + \beta a^\dagger)(\gamma b + \delta b^\dagger)&\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big)\\ &= (\alpha a + \beta a^\dagger)\big(\gamma z|0,0\rangle - \gamma w|1,0\rangle + \delta x|0,1\rangle - \delta y|1,1\rangle\big)\\ &= -\alpha\gamma w|0,0\rangle - \alpha\delta y|0,1\rangle + \beta\gamma z|1,0\rangle + \beta\delta x|1,1\rangle\\ &= e\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big).\end{aligned} \tag{38}$$

The αa terms in the second equation are

$$\alpha a\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big) = \alpha y|0,0\rangle + \alpha w|0,1\rangle \tag{39}$$

while the $\beta a^\dagger$ terms are

$$\beta a^\dagger\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big) = \beta x|1,0\rangle + \beta z|1,1\rangle. \tag{40}$$

So we find from the second equation

$$\begin{aligned}(\gamma b + \delta b^\dagger)(\alpha a + \beta a^\dagger)&\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big)\\ &= (\gamma b + \delta b^\dagger)\big(\alpha y|0,0\rangle + \alpha w|0,1\rangle + \beta x|1,0\rangle + \beta z|1,1\rangle\big)\\ &= \gamma\alpha w|0,0\rangle - \gamma\beta z|1,0\rangle + \delta\alpha y|0,1\rangle - \delta\beta x|1,1\rangle\\ &= -e\big(x|0,0\rangle + y|1,0\rangle + z|0,1\rangle + w|1,1\rangle\big).\end{aligned} \tag{41}$$

So we have two sets of conditions:

$$-\alpha\gamma w = ex, \quad -\alpha\delta y = ez, \quad \beta\gamma z = ey, \quad \beta\delta x = ew \tag{42}$$

and

$$\gamma\alpha w = -ex, \quad \gamma\beta z = ey, \quad \delta\alpha y = -ez, \quad \delta\beta x = ew \tag{43}$$

which are consistent since $\alpha, \beta, \gamma,$ and δ are complex numbers. So we have a solution to both equations (35 and 36) which are actually the same equation:

$$e^2 = -\alpha\beta\gamma\delta. \tag{44}$$