19
Path Integrals

19.1 Path Integrals and Classical Physics
Since Richard Feynman invented them over 60 years ago, path integrals have been used with increasing frequency in high-energy and condensed-matter physics, in finance, and in biophysics (Kleinert, 2009). Feynman used them to express matrix elements of the time-evolution operator $\exp(-i t H/\hbar)$ in terms of the classical action. Others have used them to compute matrix elements of the Boltzmann operator $\exp(-H/kT)$ which in the limit of zero temperature projects out the ground state $|E_0\rangle$ of the system

$$\lim_{T \to 0} e^{-(H-E_0)/kT} = \lim_{T \to 0} \sum_{n=0}^{\infty} |E_n\rangle e^{-(E_n-E_0)/kT} \langle E_n| = |E_0\rangle \langle E_0|$$

(19.1)
a trick used in lattice gauge theory.
Path integrals magically express the quantum-mechanical probability amplitude for a process as a sum of exponentials $\exp(iS/\hbar)$ of the classical action $S$ of the various ways that process might occur.

19.2 Gaussian Integrals
The path integrals we can do are gaussian integrals of infinite order. So we begin by recalling the basic integral formula (5.174)

$$\int_{-\infty}^{\infty} \exp \left[ -ia \left( x - \frac{b}{2a} \right)^2 \right] dx = \sqrt{\frac{\pi}{ia}}$$

(19.2)
which holds for real \( a \) and \( b \), and also the one (5.175)

\[
\int_{-\infty}^{\infty} \exp \left[ - r \left( x - \frac{c}{2r} \right)^2 \right] \, dx = \sqrt{\frac{\pi}{r}}
\]

which is true for positive \( r \) and complex \( c \). Equivalent formulas for real \( a \) and \( b \), positive \( r \), and complex \( c \) are

\[
\int_{-\infty}^{\infty} \exp \left( - i ax^2 + ibx \right) \, dx = \sqrt{\frac{\pi}{ia}} \exp \left( \frac{b^2}{4a} \right)
\]

(19.4)

\[
\int_{-\infty}^{\infty} \exp \left( - r x^2 + cx \right) \, dx = \sqrt{\frac{\pi}{r}} \exp \left( \frac{c^2}{4r} \right).
\]

(19.5)

This last formula will be useful with \( x = p, r = \epsilon/(2m) \), and \( c = i \dot{q} \)

\[
\int_{-\infty}^{\infty} \exp \left( - \epsilon \frac{p^2}{2m} + i \epsilon \dot{q} p \right) \, dp = \sqrt{\frac{2\pi m}{\epsilon}} \exp \left( - \epsilon \frac{1}{2} m \dot{q}^2 \right)
\]

(19.6)

as will (19.4) with \( x = p, a = \epsilon/(2m) \), and \( b = \epsilon \dot{q} \)

\[
\int_{-\infty}^{\infty} \exp \left( - \epsilon \frac{p^2}{2m} + i \epsilon \dot{q} p \right) \, dp = \sqrt{\frac{2\pi m}{i\epsilon}} \exp \left( i \epsilon \frac{1}{2} m \dot{q}^2 \right).
\]

(19.7)

Doable path integrals are multiple gaussian integrals. One may show (exercise 19.1) that for positive \( r_1, \ldots, r_N \) and complex \( c_1, \ldots, c_N \), the integral (19.5) leads to

\[
\int_{-\infty}^{\infty} \exp \left( \sum_i -r_i x_i^2 + c_i x_i \right) \prod_{i=1}^{N} dx_i = \left( \prod_{i=1}^{N} \sqrt{\frac{\pi}{r_i}} \right) \exp \left( \frac{1}{4} \sum_i \frac{c_i^2}{r_i} \right).
\]

(19.8)

If \( R \) is the \( N \times N \) diagonal matrix with positive entries \( \{r_1, r_2, \ldots, r_N\} \), and \( X \) and \( C \) are \( N \)-vectors with real \( \{x_i\} \) and complex \( \{c_i\} \) entries, then this formula (19.8) in matrix notation is

\[
\int_{-\infty}^{\infty} \exp \left( -X^T RX + C^T X \right) \prod_{i=1}^{N} dx_i = \sqrt{\frac{\pi^N}{\det(R)}} \exp \left( \frac{1}{4} C^T R^{-1} C \right).
\]

(19.9)

Now every positive symmetric matrix \( S \) is of the form \( S = ORO^T \) for some positive diagonal matrix \( R \). So inserting \( R = O^T SO \) into the previous equation (19.9) and using the invariance of determinants under orthogonal transformations, we find

\[
\int_{-\infty}^{\infty} \exp \left( -X^T O^T SOX + C^T X \right) \prod_{i=1}^{N} dx_i = \sqrt{\frac{\pi^N}{\det(S)}} \exp \left[ \frac{1}{4} C^T O^T S^{-1} OC \right].
\]

(19.10)
The jacobian of the orthogonal transformations \( Y = OX \) and \( D = OC \) is unity, and so
\[
\int_{-\infty}^{\infty} \exp \left( - Y^T S Y + D^T Y \right) \prod_{i=1}^{N} dy_i = \sqrt{\frac{\pi^N}{\text{det}(S)}} \exp \left( \frac{1}{4} D^T S^{-1} D \right) \tag{19.11}
\]
in which \( S \) is a positive symmetric matrix, and \( D \) is a complex vector.

The other basic gaussian integral (19.4) leads for real \( S \) and \( D \) to (exercise 19.2)
\[
\int_{-\infty}^{\infty} \exp \left( - i Y^T S Y + i D^T Y \right) \prod_{i=1}^{N} dy_i = \sqrt{\frac{\pi^N}{\text{det}(iS)}} \exp \left( i \frac{1}{4} D^T S^{-1} D \right). \tag{19.12}
\]

The vector \( \overline{Y} \) that makes the argument \( - i Y^T S Y + i D^T Y \) of the exponential of this multiple gaussian integral (19.12) stationary is (exercise 19.3)
\[
\overline{Y} = \frac{1}{2} S^{-1} D. \tag{19.13}
\]

The exponential of that integral evaluated at its stationary point \( \overline{Y} \) is
\[
\exp \left( - i \overline{Y}^T S \overline{Y} + i D^T \overline{Y} \right) = \exp \left( i \frac{1}{4} D^T S^{-1} D \right). \tag{19.14}
\]

Thus, the multiple gaussian integral (19.12) is equal to its exponential evaluated at its stationary point \( \overline{Y} \), apart from a prefactor involving the determinant \( \text{det} iS \).

Similarly, the vector \( \overline{Y} \) that makes the argument \( - Y^T S Y + D^T Y \) of the exponential of the multiple gaussian integral (19.11) stationary is \( \overline{Y} = S^{-1} D/2 \), and that exponential evaluated at \( \overline{Y} \) is
\[
\exp \left( - Y^T S Y + D^T Y \right) = \exp \left( \frac{1}{4} D^T S^{-1} D \right). \tag{19.15}
\]

Once again, a multiple gaussian integral is simply its exponential evaluated at its stationary point \( \overline{Y} \), apart from a prefactor involving a determinant, \( \text{det} S \).

### 19.3 Quantum mechanics

In quantum mechanics, a system whose properties have been measured as completely as possible is said to be in a **pure state** and is represented by a vector. For example, a harmonic oscillator in its \( k \)th excited state is in a pure state described by a vector \( |k\rangle \). An electron moving with momentum
\( \mathbf{p} \) and spin \( \sigma \) is in a pure state represented by a vector \(|\mathbf{p}, \sigma\rangle\). A hydrogen atom at rest in its ground state is in a pure state \(|E_0\rangle\).

In classical mechanics, arbitrarily precise measurements of all physical quantities are in principle possible and their evolution in time is in principle precisely predictable. The \( n \) coordinates \( q = q_1, q_2, \ldots, q_n \) and momenta \( p = p_1, p_2, \ldots, p_n \) of a classical Hamilton system (section 17.1) are simultaneously and exactly measurable, and they obey Hamilton’s equations (17.2).

In quantum mechanics, arbitrarily precise measurements of some physical quantities are in principle impossible. Heisenberg’s commutation relations

\[
[q_i, p_k] = q_ip_k - p_kq_i = i\hbar\delta_{ik} \tag{19.16}
\]

imply (example 3.7) that the errors or variances (section 14.2) of simultaneous measurements of the \( j \)th position \( q_j \) and \( j \)th momentum must satisfy the uncertainty principle \( \Delta q_j \Delta p_j \geq \hbar/2 \).

Quantum mechanics gives us probabilities. These probabilities are the squared lengths of inner products of the vectors that represent the states of the system. The vector \(|q\rangle = |q_1, q_2, \ldots, q_n\rangle\), for example, is the state that represents the system at the point \( q \). If the system is in some state \(|s\rangle\), then the inner product \( \langle q|s\rangle = \langle s|q\rangle^* \) is a complex number, and the probability that the system is at position \( q \) is the square of the absolute value of that number \( P(q|s\rangle = \langle s|q\rangle\langle q|s\rangle = |\langle q|s\rangle|^2 \).

In classical physics, we can divide the states of the system into many complete sets of mutually exclusive states \( \ell \) and write the probability that the system is at position \( q \) as a sum of probabilities over all these states \( \ell \).

Each term in the sum is the probability that the system is at \( q \) if it’s in the state \( \ell \) times the probability that it’s in the state \( \ell \)

\[
P_{\text{classical}}(q) = \sum_{\ell} P_{\text{classical}}(q|\ell) P_{\text{classical}}(\ell). \tag{19.17}
\]

In classical physics, such formulas work for any property, not just position. Such formulas are among the axioms (14.8) of the classical theory of probability. They don’t work in quantum mechanics.

**Example 19.1 (Interference).** If a system is in a pure state \(|\psi\rangle\) that is a superposition \(|\psi\rangle = \cos \theta |s\rangle + \sin \theta |s'\rangle\) of two orthonormal states \(|s\rangle\) and \(|s'\rangle\), then in terms of \( P(s) = \cos^2 \theta \), \( P(q|s\rangle = |\langle q|s\rangle|^2 \), \( P(s') = \sin^2 \theta \), and \( P(q|s\rangle = |\langle q|s\rangle|^2 \), the probability that the system is at \( q \) is

\[
P(q) = |\langle q|\psi\rangle|^2 = |\cos \theta \langle q|s\rangle + \sin \theta \langle q|s'\rangle|^2
= \cos^2 \theta |\langle q|s\rangle|^2 + \sin^2 \theta |\langle q|s'\rangle|^2 + 2 \cos \theta \sin \theta \langle q|s\rangle \langle s'|q\rangle + \cos \theta \sin \theta \langle q|s'\rangle \langle s|q\rangle
= P(q|s\rangle P(s) + P(q|s'\rangle P(s') + \sin 2\theta \Re (\langle q|s\rangle \langle s'|q\rangle) . \tag{19.18}
\]
The first two terms are those of the classical formula (14.8). The third term is quantum mechanical and puzzling. It represents interference and violates the basic axioms of the classical theory of probability.

In the double-slit experiment, the probability that a photon gets to \( q \) is the probability that it goes through slit \( s \) times the probability that it then gets to \( q \) plus the probability that it goes through slit \( s' \) times the probability that it then gets to \( q \) plus an interference term, \( \sin 2 \theta \Re (\langle q|s\rangle \langle s'|q \rangle) \).

It evidently is not easy to separate a quantum-mechanical process into successive steps through mutually exclusive states. There always is, however, a unique set of states \( |n\rangle, |n'\rangle, \ldots \) such that classical formulas of probability theory like (19.17) do work.

The simplest case is when the system is in a pure state \( \psi \) as in the example (19.1). There the probability that the system is in the state \( \psi \) is unity \( P(\psi) = 1 \), and the probability that a system in the state \( \psi \) is at position \( q \) is \( P(q|\psi) = |\langle q|\psi \rangle|^2 \). So the classical rule (19.17) is trivially satisfied because the sum over states has only one term, \( P(q) = P(q|\psi) P(\psi) = |\langle q|\psi \rangle|^2 \).

The more general and more common case is when the system is in one of several mutually orthogonal (and normalized) pure states, \( n \), each with probability \( p_n \). In this more general case, the system is described by a density operator \( \rho \) that is a sum of the outer products \( |n\rangle\langle n| \) of the possible pure states of the system weighted by their probabilities \( p_n \)

\[
\rho = \sum_n p_n |n\rangle\langle n|.
\] (19.19)

Each of the pure states \( n \) is an eigenstate of the density operator with an eigenvalue equal to its weight, \( \rho|n\rangle = p_n |n\rangle \). These eigenvalues are the mean values \( p_n = \langle n|\rho|n\rangle \) of the density operator in the state \( n \). Since we know that the system is in state \( n \) with probability \( \langle n|\rho|n\rangle \), the probability that the system is in a state \( \phi \) is given by the axiom (14.8) of probability theory

\[
P(\phi) = \sum_n P(\phi|n) P(n) = \sum_n |\langle \phi|n \rangle|^2 \langle n|\rho|n\rangle = \text{Tr}(\rho|\phi\rangle\langle \phi|). \] (19.20)

So the special set of states to which we may apply a classical formula of probability theory like (14.8) are the eigenstates of the density operator of the system. This circle of ideas is consistent because the density operator is a sum of the outer products of the possible orthonormal pure states of the system weighted by their probabilities. In terms of the density operator \( \rho \), the probability that the system is at position \( q \) is

\[
P(q) = \sum_n P(q|n) P(n) = \sum_n |\langle q|n \rangle|^2 |\langle n|\rho|n\rangle|^2. \] (19.21)
19.4 Path Integrals in Imaginary Time

At the imaginary time \( t = i\beta \), the time-evolution operator \( \exp(-itH/\hbar) \) is \( \exp(-\beta H) \) in which the inverse temperature \( \beta = 1/kT \) is the reciprocal of Boltzmann’s constant \( k = 8.617 \times 10^{-5} \text{eV/K} \) times the absolute temperature \( T \). In the low-temperature limit, \( \exp(-\beta H) \) is a projection operator (19.1) on the ground state of the system. These path integrals in imaginary time are called euclidian path integrals.

Let us consider a quantum-mechanical system with hamiltonian

\[
H = \frac{p^2}{2m} + V(q)
\]

(19.22)
in which the commutator of the position \( q \) and momentum \( p \) operators is \( [q, p] = i \) in units in which \( \hbar = 1 \). For tiny \( \epsilon \), the corrections to the approximation

\[
\exp \left[ -\epsilon \left( \frac{p^2}{2m} + V(q) \right) \right] \approx \exp \left( -\epsilon \frac{p^2}{2m} \right) \exp \left( -\epsilon V(q) \right) + \mathcal{O}(\epsilon^2)
\]

(19.23)
are of second order in \( \epsilon \).

To evaluate the matrix element \( \langle q_1 | \exp(-\epsilon H) | q_0 \rangle \), we insert the identity operator \( I \) in the form of an integral over the momentum eigenstates

\[
I = \int_{-\infty}^{\infty} |p'\rangle \langle p'| dp'
\]

(19.24)
and use the inner product \( \langle q_1 | p' \rangle = \exp(iq_1 p')/\sqrt{2\pi} \) so as to get as \( \epsilon \to 0 \)

\[
\langle q_1 | \exp(-\epsilon H) | q_0 \rangle = \int_{-\infty}^{\infty} \langle q_1 | \exp \left( -\epsilon \frac{p^2}{2m} \right) | p' \rangle \langle p' | \exp \left( -\epsilon V(q) \right) | q_0 \rangle dp'
\]

\[
= e^{-\epsilon V(q_0)} \int_{-\infty}^{\infty} \exp \left[ -\epsilon \frac{p'^2}{2m} + i p' (q_1 - q_0) \right] dp'.
\]

(19.25)
If we adopt the suggestive notation

\[
\frac{q_1 - q_0}{\epsilon} = \dot{q}_0
\]

(19.26)
and use the integral formula (19.6), then we find

\[
\langle q_1 | \exp(-\epsilon H) | q_0 \rangle = \frac{1}{2\pi} e^{-\epsilon V(q_0)} \int_{-\infty}^{\infty} \exp \left( -\epsilon \frac{p'^2}{2m} + \epsilon p' \dot{q}_0 \right) dp'
\]

\[
= \left( \frac{m}{2\pi\epsilon} \right)^{1/2} \exp \left\{ -\epsilon \left[ \frac{1}{2} m \dot{q}_0^2 + V(q_0) \right] \right\}
\]

(19.27)
in which \( q_1 \) enters through the notation (19.26).
The next step is to link two of these matrix elements together

\[
\langle q_2 | e^{-2\epsilon H} | q_0 \rangle = \int_{-\infty}^{\infty} \langle q_2 | e^{-\epsilon H} | q_1 \rangle \langle q_1 | e^{-\epsilon H} | q_0 \rangle dq_1
\]

\[
= \frac{m}{2\pi \epsilon} \int_{-\infty}^{\infty} \exp \left\{ -\epsilon \left[ \frac{1}{2} m \dot{q}_1^2 + V(q_1) + \frac{1}{2} m \dot{q}_2^2 + V(q_0) \right] \right\} dq_1.
\]

Linking three of these matrix elements together and using subscripts instead of primes, we have

\[
\langle q_3 | e^{-3\epsilon H} | q_0 \rangle = \int_{-\infty}^{\infty} \langle q_3 | e^{-\epsilon H} | q_2 \rangle \langle q_2 | e^{-\epsilon H} | q_1 \rangle \langle q_1 | e^{-\epsilon H} | q_0 \rangle dq_1 dq_2
\]

\[
= \left( \frac{m}{2\pi \epsilon} \right)^{3/2} \int_{-\infty}^{\infty} \exp \left\{ -\epsilon \sum_{j=0}^{2} \left[ \frac{1}{2} m \dot{q}_j^2 + V(q_j) \right] \right\} dq_2 dq_1.
\]

Boldly passing from 3 to \( n \) and suppressing some integral signs, we get

\[
\langle q_n | e^{-n\epsilon H} | q_0 \rangle = \int_{-\infty}^{\infty} \langle q_n | e^{-\epsilon H} | q_{n-1} \rangle \cdots \langle q_1 | e^{-\epsilon H} | q_0 \rangle dq_{n-1} \cdots dq_1
\]

\[
= \left( \frac{m}{2\pi \epsilon} \right)^{n/2} \int_{-\infty}^{\infty} \exp \left\{ -\epsilon \sum_{j=0}^{n-1} \left[ \frac{1}{2} m \dot{q}_j^2 + V(q_j) \right] \right\} dq_{n-1} \cdots dq_1.
\]

Writing \( d\beta \) for \( \epsilon \) and taking the limits \( \epsilon \to 0 \) and \( n \equiv \beta / \epsilon \to \infty \), we find that the matrix element \( \langle q_\beta | e^{-\beta H} | q_0 \rangle \) is a path integral of the exponential of the average energy multiplied by \(-\beta\)

\[
\langle q_\beta | e^{-\beta H} | q_0 \rangle = \int \exp \left[ -\int_0^\beta \frac{1}{2} m \dot{q}^2(\beta') + V(q(\beta')) d\beta' \right] Dq
\]

in which \( Dq \equiv (n m / 2\pi \beta h^2)^{n/2} dq_{n-1} \cdots dq_1 \) and \( \dot{q} \) is the \( \beta \)-derivative of the coordinate \( q(\beta) \). We sum over all paths \( q(t) \) that go from \( q(0) = q_0 \) at inverse temperature \( \beta = 0 \) to \( q(\beta) = q_\beta \) at inverse temperature \( \beta \).

In the limit \( \beta \to \infty \), the operator \( \exp(-\beta H) \) becomes proportional to a projection operator (19.1) on the ground state of the theory.

In three dimensions with \( \dot{q}(\beta) = dq(\beta)/d\beta \), and \( h \neq 1 \), the analog of equation (19.31) is (exercise 19.28)

\[
\langle q_\beta | e^{-\beta H} | q_0 \rangle = \int \exp \left[ -\int_0^\beta \frac{m}{2h^2} q^2(\beta') + V(q(\beta')) d\beta' \right] Dq
\]

where \( Dq \equiv (n m / 2\pi \beta h^2)^{3n/2} dq_{n-1} \cdots dq_1 \).

Path integrals in imaginary time are called euclidian mainly to distinguish
Path integrals, which represent matrix elements of the time-evolution operator \( \exp(-itH) \) in real time.

### 19.5 Path Integrals in Real Time

Path integrals in real time represent the time-evolution operator \( \exp(-itH) \). Using the integral formula (19.7), we find in the limit \( \varepsilon \to 0 \)

\[
\langle q_1 | e^{-i\varepsilon H} | q_0 \rangle = \int_{-\infty}^{\infty} \langle q_1 | \exp \left[ -i \varepsilon \frac{p^2}{2m} \right] | p' \rangle \langle p' | \exp \left[ -i \varepsilon V(q) \right] | q_0 \rangle \, dp'
\]

\[
= \frac{1}{2\pi} e^{-i\varepsilon V(q_0)} \int_{-\infty}^{\infty} \exp \left[ -i \varepsilon \frac{p'^2}{2m} + i p' (q_1 - q_0) \right] \, dp'
\]

\[
= \frac{1}{2\pi} e^{-i\varepsilon V(q_0)} \int_{-\infty}^{\infty} \exp \left[ -i \varepsilon \frac{p'^2}{2m} + i p' \dot{q}_0 \right] \, dp'
\]

\[
= \left( \frac{m}{2\pi i \varepsilon} \right)^{1/2} \exp \left[ i \varepsilon \left( -\frac{m q_0^2}{2} - V(q_0) \right) \right]. \tag{19.33}
\]

When we link together \( n \) of these matrix elements, we get the real-time version of (19.31)

\[
\langle q_n | e^{-i\varepsilon n H} | q_0 \rangle = \left( \frac{m}{2\pi i \varepsilon} \right)^{n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp \left[ i \varepsilon \sum_{j=0}^{n-1} \left[ \frac{1}{2} m \dot{q}_j^2 - V(q_j) \right] \right] \, dq_{n-1} \ldots dq_1. \tag{19.34}
\]

Writing \( dt \) for \( \varepsilon \) and taking the limits \( \varepsilon \to 0 \) and \( n\varepsilon \to t \), we find that the amplitude \( \langle q_t | e^{-iH/\hbar} | q_0 \rangle \) is the path integral

\[
\langle q_t | e^{-iH/\hbar} | q_0 \rangle = \int \exp \left[ \frac{i}{\hbar} \int_0^t \frac{1}{2} m q'^2 - V(q) \, dt' \right] Dq \tag{19.35}
\]

in which \( Dq \) differs from the one that appears in euclidian path integrals by the substitution \( \beta \to it/\hbar \)

\[
Dq = \lim_{n \to \infty} \left( \frac{nm}{2\pi i \hbar t} \right)^{n/2} dq_1 dq_2 \ldots dq_{n-1}. \tag{19.36}
\]

The integral in the exponent is the **classical action**

\[
S[q] = \int_0^t \frac{1}{2} m q'^2 - V(q) \, dt' \tag{19.37}
\]

for a process \( q(t') \) that runs from \( q(0) = q_0 \) to \( q(t) = q_t \). We sum over all such processes.
In three-dimensional space, the analog of (19.35) is

\[
\langle \mathbf{q}_t | e^{-itH} | \mathbf{q}_0 \rangle = \int \exp \left[ i \int_0^t \frac{1}{2} m \dot{\mathbf{q}}^2 - V(\mathbf{q}) \, dt' \right] D\mathbf{q}.
\] (19.38)

The units of action are energy × time, and the argument of the exponential must be dimensionless, so in arbitrary units the amplitude (19.35) is

\[
\langle \mathbf{q}_t | e^{-itH/\hbar} | \mathbf{q}_0 \rangle = \int e^{iS[\mathbf{q}] / \hbar} D\mathbf{q}.
\] (19.39)

When is this amplitude big? When is it tiny? Suppose there is a process \(\mathbf{q}(t) = \mathbf{q}_c(t)\) that goes from \(\mathbf{q}_c(0) = \mathbf{q}_0\) to \(\mathbf{q}_c(t) = \mathbf{q}_t\) in time \(t\) and that obeys the classical equation of motion (18.14–18.15)

\[
\frac{\delta S[\mathbf{q}_c]}{\delta \mathbf{q}_c} = m\ddot{\mathbf{q}}_c + V'(\mathbf{q}_c) = 0.
\] (19.40)

The action of such a classical process is stationary. A process \(\mathbf{q}_c(t) + dq(t)\) that differs from a stationary process \(\mathbf{q}_c(t)\) by a small deviation \(dq(t)\) has an action \(S[\mathbf{q}_c + dq]\) that differs from \(S[\mathbf{q}_c]\) only by terms of second order in \(\delta q\). Thus a stationary process has many neighboring processes that have the same action to within a fraction of \(\hbar\). These processes add with nearly the same phase to the path integral (19.39) and make a huge contribution to the amplitude \(\langle \mathbf{q}_t | e^{-itH/\hbar} | \mathbf{q}_0 \rangle\).

But if no classical process goes from \(\mathbf{q}_0\) to \(\mathbf{q}_t\) in time \(t\), then the nonclassical, nonstationary processes that go from \(\mathbf{q}_0\) to \(\mathbf{q}_t\) in time \(t\) have actions that differ from each other by large multiples of \(\hbar\). These amplitudes cancel each other, and the resulting amplitude is tiny. Thus the real-time path integral explains The principle of stationary action (section 12.39).

**Example 19.2** (Schrödinger’s equation). The amplitude \(\langle \mathbf{x}, t | \psi \rangle = \psi(\mathbf{x}, t)\) to get to \(\mathbf{x}\) in time \(t\) from somewhere \(\mathbf{x}_0\) at time 0 is a sum over all processes

\[
\psi(\mathbf{x}, t) = \sum e^{iS[\mathbf{x}] / \hbar} \psi(\mathbf{x}_0, 0).
\] (19.41)

So using our formulas (6.204 & 6.217) for the spacetime partial derivatives
\( \nabla S = \vec{p} \) and \( \partial S/\partial t = -H \), we have

\[
\begin{align*}
\frac{i\hbar}{\partial t} \frac{\partial \psi(\vec{x}, t)}{\partial t} &= \sum \frac{-\partial S}{\partial t} e^{iS/h} \psi(\vec{x}_0, 0) = \sum H e^{iS/h} \psi(\vec{x}_0, 0) \\
&= H \sum e^{iS/h} \psi(\vec{x}_0, 0) = H \psi(\vec{x}, t) = \left( \frac{\vec{p}^2}{2m} + V(\vec{x}) \right) \psi(\vec{x}, t) \\
&= \left( \frac{\vec{p}^2}{2m} + V(\vec{x}) \right) \sum e^{iS/h} \psi(\vec{x}_0, 0) \\
&= \left( \frac{(\nabla S)^2}{2m} + V(\vec{x}) \right) \sum e^{iS/h} \psi(\vec{x}_0, 0) \\
&= \left( -\frac{\hbar^2 \nabla^2}{2m} + V(\vec{x}) \right) \sum e^{iS/h} \psi(\vec{x}_0, 0) \\
&= \left( -\frac{\hbar^2 \nabla^2}{2m} + V(\vec{x}) \right) \psi(\vec{x}_0, 0)
\end{align*}
\]

which is Schrödinger’s equation. \( \square \)

19.6 Path Integral for a Free Particle

The amplitude for a free nonrelativistic particle to go from the origin to the point \( \vec{q} \) in time \( t \) is the path integral (19.38)

\[
\langle \vec{q} | e^{-iHt} | \vec{q} = 0 \rangle = \int e^{iS_0[\vec{q}]} D\vec{q} = \int \exp \left( i \int_0^t \frac{1}{2} m \dot{\vec{q}}^2(t') \, dt' \right) D\vec{q}. \tag{19.44}
\]

The classical path that goes from \( \vec{0} \) to \( \vec{q} \) in time \( t \) is \( \vec{q}_c(t') = (t'/t) \vec{q} \). The general path \( \vec{q}(t') \) over which we integrate is \( \vec{q}(t') = \vec{q}_c(t') + \delta \vec{q}(t') \). Since both \( \vec{q}(t') \) and \( \vec{q}_c(t') \) go from \( \vec{0} \) to \( \vec{q} \) in time \( t \), the arbitrary detour \( \delta \vec{q}(t') \) must be a loop that goes from \( \delta \vec{q}(0) = 0 \) to \( \delta \vec{q}(t) = 0 \) in time \( t \). The velocity \( \dot{\vec{q}} = \dot{\vec{q}}_c + \delta \dot{\vec{q}} \) is the sum of the constant classical velocity \( \dot{\vec{q}}_c = \vec{q}/t \) and the loop’s velocity \( \delta \dot{\vec{q}} \). The first-order change vanishes

\[
m \int_0^t \dot{\vec{q}}_c \cdot \frac{d\delta \vec{q}}{dt} \, dt = m \dot{\vec{q}}_c \cdot \int_0^t d\delta \vec{q} = m \dot{\vec{q}}_c \cdot [\delta \vec{q}(t) - \delta \vec{q}(0)] = 0 \tag{19.45}
\]

and so the action \( S_0[\vec{q}] \) is the classical action plus the loop action

\[
S_0[\vec{q}] = \frac{1}{2} m \int_0^t (\dot{\vec{q}}_c + \delta \dot{\vec{q}})^2 \, dt' = S_0[\vec{q}_c] + S_0[\delta \vec{q}]. \tag{19.46}
\]
The path integral therefore factorizes

$$\langle q | e^{-itH} | 0 \rangle = \int e^{iS_0[q]} Dq = \int e^{iS_0[q_c + \delta q]} D\delta q$$

$$= \int e^{iS_0[q_c]} e^{iS_0[\delta q]} D\delta q = e^{iS_0[q_c]} \int e^{iS_0[\delta q]} D\delta q$$  \hspace{1cm} (19.47)

into the phase of the classical action times a path integral over loops. The loop integral $L$ is independent of the spatial points $q$ and 0 and so can only depend upon the time interval, $L = L(t)$. Thus the amplitude is the phase of the classical action times a function $L(t)$ of the time

$$\langle q | e^{-itH} | 0 \rangle = e^{iS_0[q_c]} L(t).$$  \hspace{1cm} (19.48)

Since the classical velocity is $\dot{q}_c = q/t$, the classical action is

$$S_0[q_c] = \int_0^t \frac{m}{2} \dot{q}_c^2(t) \, dt = \frac{m}{2} \frac{q^2}{t}.$$  \hspace{1cm} (19.49)

So the amplitude is

$$\langle q | e^{-i(t_2-t_1)H} | 0 \rangle = e^{imq^2/2t} L(t).$$  \hspace{1cm} (19.50)

Because position eigenstates are orthogonal, this amplitude must reduce to a delta function as $t \to 0$

$$\lim_{t \to 0} \langle q | e^{-itH} | 0 \rangle = \langle q | 0 \rangle = \delta^3(q).$$  \hspace{1cm} (19.51)

One of the many representations of Dirac’s delta function is

$$\delta^3(q) = \lim_{t \to 0} \left( \frac{m}{2\pi i t} \right)^{3/2} e^{imq^2/2ht}. $$  \hspace{1cm} (19.52)

Thus $NL(t) = (m/2\pi i t)^{3/2}$ and

$$\langle q | e^{-iH/\hbar} | 0 \rangle = \left( \frac{m}{2\pi i \hbar} \right)^{3/2} e^{imq^2/2\hbar t}$$  \hspace{1cm} (19.53)

in arbitrary units. You can verify (exercise 19.6) this result by inserting a complete set of momentum dyadics $|p\rangle\langle p|$ into the left-hand side of the amplitude (19.53) and doing the resulting Fourier transform.

**Example 19.3** (The Bohm-Aharonov Effect). From our formula (12.283) for the action of a relativistic particle of mass $m$ and charge $q$, we infer (exercise 19.7) that the action a nonrelativistic particle in an electromagnetic field with no scalar potential is

$$S = \int_{x_1}^{x_2} \left[ \frac{1}{2}mv^2 + qA \right] \cdot dx.$$  \hspace{1cm} (19.54)
Now imagine that we shoot a beam of such particles past but not through a narrow cylinder in which a magnetic field is confined. The particles can go either way around the cylinder of area $S$ but cannot enter the region of the magnetic field. The difference in the phases of the amplitudes is the loop integral from the source to the detector and back to the source

$$\frac{\Delta S}{\hbar} = \oint \left[ \frac{mv}{2} + qA \right] \cdot \frac{dx}{\hbar} = \oint \frac{mv \cdot dx}{2\hbar} + \frac{q}{\hbar} \int_S B \cdot dS = \oint \frac{mv \cdot dx}{2\hbar} + \frac{q\Phi}{\hbar}$$

in which $\Phi$ is the magnetic flux through the cylinder.

### 19.7 Free Particle in Imaginary Time

If we mimic the steps of the preceding section (19.6) in which the hamiltonian is $H = p^2/(2m)$, set $\beta = 1/kT$, and use Dirac's delta function

$$\delta^3(q) = \lim_{t \to 0} \left( \frac{m}{2\pi \hbar t} \right)^{3/2} e^{-mq^2/2\hbar}$$

then we get

$$\langle q | e^{-\beta H} | 0 \rangle = \left( \frac{m}{2\pi \hbar^2 \beta} \right)^{3/2} \exp \left[ -\frac{mq^2}{2\hbar^2 \beta} \right] = \left( \frac{mkT}{2\pi \hbar^2} \right)^{3/2} e^{-mkTq^2/2\hbar}. \tag{19.57}$$

For $\beta = t/\hbar$, this is the solution (3.204 & 14.131) of the diffusion equation with diffusion constant $D = \hbar/(2m)$.

### 19.8 Harmonic Oscillator in Real Time

Biologists have mice; physicists have harmonic oscillators with hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2q^2}{2}. \tag{19.58}$$

For this hamiltonian, our formula (19.35) for the coordinate matrix elements of the time-evolution operator $\exp(-itH)$ is

$$\langle q_1 | e^{-itH} | q_0 \rangle = \int e^{iS[q]} Dq \tag{19.59}$$

with action

$$S[q] = \int_0^t \frac{1}{2} m\dot{q}^2(t') - \frac{1}{2} m\omega^2q^2(t') \, dt'.$$ \tag{19.60}$$

The classical solution $q_c(t) = q_0 \cos \omega t + \dot{q}_0 \sin(\omega t)/\omega$ in which $q_0 = q_c(0)$
and \( \dot{q}_0 = \dot{q}_c(0) \) are the initial position and velocity satisfies the classical equation of motion \( m\ddot{q}_c(t) = -\omega^2 q_c(t) \) and makes the action \( S[q] \) stationary

\[
\frac{d}{d\epsilon} S[q + \epsilon h] \bigg|_{\epsilon=0} = 0. \tag{19.61}
\]

We now apply the trick (19.45–19.47) we used for the free particle. We write an arbitrary process \( q(t) \) as the sum of the classical process \( q_c(t) \) and a loop \( \delta q(t) \) with \( \delta q(0) = \delta q(t) = 0 \). Since the action \( S[q] \) is quadratic in the variables \( q \) and \( \dot{q} \), the functional Taylor series (18.31) for \( S[q_c + \delta q] \) has only two terms

\[
S[q_c + \delta q] = S[q_c] + S[\delta q]. \tag{19.62}
\]

Thus we can write the path integral (19.59) as

\[
\langle q_t | e^{-itH} | q_0 \rangle = \int e^{iS[q_c]} Dq = \int e^{iS[q_c+\delta q]} D\delta q = \int e^{iS[q_c]} \int e^{iS[\delta q]} D\delta q. \tag{19.63}
\]

The remaining path integral over the loops \( \delta q \) does not involve the end points \( q_0 \) and \( q_t \) and so must be a function \( L(t) \) of the time \( t \) but not of \( q_0 \) or \( q_t \)

\[
\langle q_t | e^{-itH} | q_0 \rangle = e^{iS[q_c]} L(t). \tag{19.64}
\]

The action \( S[q_c] \) is (exercise 19.8)

\[
S[q_c] = \frac{m\omega}{2 \sin(\omega t)} \left[ q_0^2 + q_t^2 \right] \cos(\omega t) - 2q_0q_t. \tag{19.65}
\]

The action \( S[\delta q] \) of a loop

\[
\delta q(t') = \sum_{j=1}^{n-1} a_j \sin \frac{j\pi t'}{t} \tag{19.66}
\]

is (exercise 19.9)

\[
S[\delta q] = \sum_{j=1}^{n-1} \frac{mt}{4} a_j^2 \left[ \frac{(j\pi)^2}{t^2} - \omega^2 \right]. \tag{19.67}
\]
The path integral over the loops is then, apart from a constant jacobian $J$,

$$
\int e^{iS[q]} \delta q = J \left( \frac{nm}{2 \pi i t} \right)^{n/2} \prod_{j=1}^{n} \frac{\sin \left( j \pi \frac{a_j}{2} \right)}{j \pi} \prod_{j=1}^{n} \frac{\sin \left( j \pi \frac{q_j}{2} \right)}{j} \exp \left\{ \frac{i m t}{4} a_j^2 \left[ \frac{(j \pi)^2}{t^2} - \omega^2 \right] \right\} \prod_{j=1}^{n-1} da_j.
$$

(19.68)

Using the gaussian integral (19.2) and the infinite product (4.158), we get

$$
\int e^{iS[q]} \delta q = J n^{n/2} \sqrt{\frac{m}{2 \pi i t}} \prod_{j=1}^{n} \frac{\sqrt{2}}{j \pi} \left( 1 - \frac{\omega^2 t^2}{j^2} \right)^{-1/2}

= \sqrt{\frac{m \omega}{2 \pi i \sin \omega t}} \left( \lim_{n \to \infty} J n^{n/2} \prod_{j=1}^{n} \frac{\sqrt{2}}{j \pi} \right).
$$

(19.69)

From (19.64) and (19.65), we see that the number within the parentheses is unity because we then have (Feynman and Hibbs, 1965, ch. 3)

$$
\langle q_t | e^{-iHt/\hbar} | q_0 \rangle = \sqrt{\frac{m \omega}{2 \pi i \sin \omega t}} \exp \left[ \frac{m \omega}{2 \pi i \sin \omega t} \right] \exp \left[ \frac{m \omega}{2 \pi i \sin \omega t} \right] \exp \left[ \frac{m \omega}{2 \pi i \sin \omega t} \right]
$$

(19.70)

which agrees with the amplitude (19.53) in the limit $t \to 0$ (exercise 19.10).

### 19.9 Harmonic Oscillator in Imaginary Time

For the harmonic oscillator with hamiltonian (19.58), our formula (19.31) for euclidian path integrals becomes

$$
\langle q_\beta | e^{-\beta H} | q_0 \rangle = \int \exp \left\{ - \int_0^\beta \frac{1}{2} m q^2(\beta') + \frac{1}{2} m \omega^2 q^2(\beta') d\beta' \right\} Dq.
$$

(19.71)

The euclidian action, which is a $\beta$-integral of the energy of the oscillator,

$$
S_e[q] = \int_0^\beta \left[ \frac{1}{2} m q^2(\beta') + \frac{1}{2} m \omega^2 q^2(\beta') \right] d\beta'
$$

(19.72)
is purely quadratic, and so we may play a trick (18.32). We look for a path \( q_e(\beta) \) that makes the euclidian action (19.72) stationary

\[
\delta S_e[q_e][\hbar] = \frac{d}{de} \left. \int_0^\beta \left[ \frac{1}{2} m (\dot{q}_e(\beta') + \epsilon h(\beta'))^2 + \frac{1}{2} m \omega^2 (q_e(\beta') + \epsilon h(\beta'))^2 \right] d\beta' \right|_{e=0}
\]

\[
= \int_0^\beta m \ddot{q}_e(\beta') \dot{h}(\beta') + m \omega^2 q_e(\beta') h(\beta') d\beta'
\]

\[
= \int_0^\beta \left[ -m \ddot{q}_e(\beta') + m \omega^2 q_e(\beta') \right] h(\beta') d\beta' = 0. \quad (19.73)
\]

The path \( q_e(\beta) \) must satisfy the euclidian equation of motion

\[
\ddot{q}_e(\beta) = \omega^2 q_e(\beta) \quad (19.74)
\]

whose general solution is

\[
q_e(\beta) = A e^{\omega \beta} + B e^{-\omega \beta}. \quad (19.75)
\]

The path from \( q_e(0) = q_0 \) to \( q_e(\beta) = q_\beta \) must have

\[
A = \frac{q_\beta e^{-\omega \beta} - q_0 e^{-2\omega \beta}}{1 - e^{-2\omega \beta}} \quad \text{and} \quad B = q_0 - A. \quad (19.76)
\]

Its action \( S_e[q_e] \) is (exercise 19.12)

\[
S_e[q_e] = \frac{1}{2} m \omega \left[ A^2 \left( e^{2\omega \beta} - 1 \right) + B^2 \left( 1 - e^{-2\omega \beta} \right) \right]. \quad (19.77)
\]

Since the action is purely quadratic, the trick (18.32) tells us that the action \( S_e[q] \) of the arbitrary path \( q(\beta) = q_e(\beta) + \delta q(\beta) \) is the sum

\[
S_e[q] = S_e[q_e] + S_e[\delta q] \quad (19.78)
\]

in which the action \( S_e[\delta q] \) of the loop \( \delta q(\beta) \) depends upon \( \beta \) but not upon \( q_\beta \) or \( q_0 \). It follows then that for some loop function \( L(\beta) \) of \( \beta \) alone

\[
\langle q_\beta | e^{-\beta H} | q_0 \rangle = \exp \left( -S_e[q_e] \right) L(\beta)
\]

\[
= L(\beta) \exp \left\{ -\frac{1}{2} m \omega \left[ A^2 \left( e^{2\omega \beta} - 1 \right) + B^2 \left( 1 - e^{-2\omega \beta} \right) \right] \right\}. \quad (19.79)
\]

To study the ground state of the harmonic oscillator, we let \( \beta \to \infty \) in this equation. Inserting a complete set of eigenstates \( H | n \rangle = E_n | n \rangle \), we see that the limit of the left-hand side is

\[
\lim_{\beta \to \infty} \langle q_\beta | e^{-\beta H} | q_0 \rangle = \lim_{\beta \to \infty} \langle q_\beta | n \rangle e^{-\beta E_n} \langle n | q_0 \rangle = e^{-\beta E_0} \langle q_\beta | 0 \rangle \langle 0 | q_0 \rangle. \quad (19.80)
\]
Our formulas (19.76) for $A$ and $B$ say that $A \to q e^{-\omega t}$ and $B \to q_0$ as $\beta \to \infty$, and so in this limit by (19.79 & 19.80) we have

$$e^{-\beta E_0} |q_\beta \rangle \langle 0| q_0 \rangle = L(\beta) \exp \left[ -\frac{1}{2} m \omega (q_\beta^2 + q_0^2) \right]$$

(19.81)

from which may infer our earlier formula (18.44) for the wave function of the ground state

$$\langle q \rangle = \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} \exp \left[ -\frac{1}{2} m \omega q^2 \right]$$

(19.82)

in which the prefactor ensures the normalization

$$1 = \int_{-\infty}^{\infty} |\langle q \rangle|^2 dq.$$ 

(19.83)

euclidian path integrals help one study ground states.

19.10 Euclidian Correlation Functions

In the Heisenberg picture, the position operator $q(t)$ is

$$q(t) = e^{i t H / \hbar} q e^{-i t H / \hbar}$$

(19.84)

in which $q = q(0)$ is the position operator at time $t = 0$ or equivalently the position operator in the Schrödinger picture. The analogous operator in imaginary time $t = -i \hbar \beta = -i \hbar / (kT)$ is the euclidian position operator $e_\beta$ defined as

$$e_\beta = e^{\beta H} q e^{-\beta H}.$$ 

(19.85)

The euclidian-time-ordered product of two euclidian position operators is

$$\mathcal{T} \left[ e_\beta(\beta_1) e_\beta(\beta_2) \right] = \theta(\beta_1 - \beta_2) e_\beta(\beta_1) e_\beta(\beta_2) + \theta(\beta_2 - \beta_1) e_\beta(\beta_2) e_\beta(\beta_1)$$

(19.86)

in which $\theta(x) = (x + |x|)/2|x|$ is Heaviside’s function. We can use the method of section 19.4 to compute the matrix element of the euclidian-time-ordered product $\mathcal{T} \left[ e_\beta(\beta_1) e_\beta(\beta_2) \right]$ sandwiched between two factors of $\exp(-\beta H)$. For $\beta_1 \geq \beta_2$, this matrix element is

$$\langle q_\beta | e^{-\beta H} e_\beta(\beta_1) e_\beta(\beta_2) e^{-\beta H} | q_\beta \rangle = \langle q_\beta | e^{-(\beta-\beta_1)H} q e^{-(\beta-\beta_2)H} e^{-\beta H} | q_\beta \rangle.$$ 

(19.87)

Then instead of the path-integral formula (19.31), we get

$$\langle q_\beta | e^{-\beta H} \mathcal{T} \left[ e_\beta(\beta_1) e_\beta(\beta_2) \right] e^{-\beta H} | q_\beta \rangle = \int q(\beta_1) q(\beta_2) e^{-S_{\mathrm{euclidian}}[q_\beta, -\beta]} Dq$$ 

(19.88)
where as in (19.31) \( S_e[q, \beta, -\beta] \) is the **euclidian action**

\[
S_e[q, \beta, -\beta] = \int_{-\beta}^{\beta} \frac{1}{2} m q'^2(\beta') + V(q(\beta')) \, d\beta'
\]  

(19.89)

or the time integral of the energy. As in the path integral (19.31), the integration is over all paths that go from \( q(-\beta) = q_{-\beta} \) to \( q(\beta) = q_{\beta} \). The analog of the single-particle path integral (19.31) is

\[
\langle q_{\beta} | e^{-2\beta H} | q_{-\beta} \rangle = \int e^{-S_e[q, \beta, -\beta]} \, Dq
\]  

(19.90)

and the factors \((nm/2\pi\beta)^{n/2}\) cancel in the ratio of (19.88) to (19.90)

\[
\frac{\langle q_{\beta} | e^{-2\beta H} \mathcal{T} [q_e(\beta_1)q_e(\beta_2)] e^{-2\beta H} | q_{-\beta} \rangle}{\langle q_{\beta} | e^{-2\beta H} | q_{-\beta} \rangle} = \frac{\int q(\beta_1)q(\beta_2)e^{-S_e[q, \beta, -\beta]} \, Dq}{\int e^{-S_e[q, \beta, -\beta]} \, Dq}.
\]  

(19.91)

In the limit \( \beta \to \infty \), the operator \( \exp(-\beta H) \) projects out the ground state \( |0\rangle \) of the system

\[
\lim_{\beta \to \infty} e^{-\beta H} |q_{-\beta}\rangle = \lim_{\beta \to \infty} \sum_{n=0}^{\infty} e^{-\beta H} |n\rangle \langle n|q_{-\beta}\rangle = \lim_{\beta \to \infty} e^{-\beta E_0}|0\rangle \langle 0|q_{-\beta}\rangle
\]  

(19.92)

which we assume to be unique and normalized to unity. In the ratio (19.91), most of these factors cancel, leaving us with

\[
\langle 0 | \mathcal{T} [q_e(\beta_1)q_e(\beta_2)] | 0 \rangle = \frac{\int q(\beta_1)q(\beta_2)e^{-S_e[q, \infty, -\infty]} \, Dq}{\int e^{-S_e[q, \infty, -\infty]} \, Dq}.
\]  

(19.93)

More generally, the mean value in the ground state \( |0\rangle \) of any euclidian-time-ordered product of position operators \( q(t_i) \) is a ratio of path integrals

\[
\langle 0 | \mathcal{T} [q(\beta_1) \ldots q(\beta_n)] | 0 \rangle = \frac{\int q(\beta_1) \ldots q(\beta_n) e^{-S_e[q]} \, Dq}{\int e^{-S_e[q]} \, Dq}
\]  

(19.94)

in which \( S_e[q] \) stands for \( S_e[q, \infty, -\infty] \). Why do we need the time-ordered product \( \mathcal{T} \) on the LHS? Because successive factors of \( \exp(-(\beta_k - \beta_l)H) \) lead to the path integral of \( \exp(-S_e[q]) \). Why don’t we need \( \mathcal{T} \) on the RHS? Because the \( q(\beta_i)'s \) are real numbers which commute with each other.
The result (19.94) is important because it can be generalized to all quantum theories, including field theories.

### 19.11 Finite-Temperature Field Theory

Matrix elements of the operator \( \exp(-\beta H) \) where \( \beta = 1/kT \) tell us what a system is like at temperature \( T \). In the low-temperature limit, they describe the ground state of the system.

Quantum mechanics imposes upon \( n \) coordinates \( q_i \) and conjugate momenta \( \pi_k \) the commutation relations

\[
[q_i, p_k] = i \delta_{i,k} \quad \text{and} \quad [q_i, q_k] = [p_i, p_k] = 0. \tag{19.95}
\]

In quantum field theory, we associate a coordinate \( q_x \equiv \phi(x) \) and a conjugate momentum \( p_x \equiv \pi(x) \) with each point \( x \) of space and impose upon them the very similar commutation relations

\[
[\phi(x), \pi(y)] = i \delta(x - y) \quad \text{and} \quad [\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0. \tag{19.96}
\]

Just as in quantum mechanics the time derivative of a coordinate is its commutator with a Hamiltonian \( \dot{q}_i = i[H, q_i] \), so too in quantum field theory the time derivative of a field \( \dot{\phi}(x, t) \equiv \phi(x) \) is \( \dot{\phi}(x) = i[H, \phi(x)] \). A typical Hamiltonian for a single scalar field \( \phi \) is

\[
H = \int \left[ \frac{1}{2} \pi^2(x) + \frac{1}{2} (\nabla \phi(x))^2 + \frac{1}{2} m^2 \phi^2(x) + P(\phi(x)) \right] d^3x \tag{19.97}
\]

in which \( P \) is a quartic polynomial.

Since quantum field theory is just the quantum mechanics of many variables, we can use the methods of sections 19.4 & 19.5 to write matrix elements of \( \exp(-\beta H) \) as path integrals. We define a potential

\[
V(\phi(x)) = \frac{1}{2} (\nabla \phi(x))^2 + \frac{1}{2} m^2 \phi^2(x) + P(\phi(x)) \tag{19.98}
\]

and write the Hamiltonian \( H \) as

\[
H = \int \left[ \frac{1}{2} \pi^2(x) + V(\phi(x)) \right] d^3x. \tag{19.99}
\]

Like \( |q'\rangle \) and \( |p'\rangle \), the states \( |\phi'\rangle \) and \( |\pi'\rangle \) are eigenstates of the hermitian operators \( \phi(x, 0) \) and \( \pi(x, 0) \)

\[
\phi(x, 0)|\phi'\rangle = \phi'(x)|\phi'\rangle \quad \text{and} \quad \pi(x, 0)|\pi'\rangle = \pi'(x)|\pi'\rangle. \tag{19.100}
\]
The analog of $\langle q'|p' \rangle$ is

$$\langle \phi'|\pi' \rangle = f \exp \left[ i \int \phi'(x)\pi'(x) d^3 x \right] \quad (19.101)$$

in which $f$ is a factor which eventually will cancel.

Repeating our derivation of Eq. (19.27) with

$$D\pi' \equiv \prod_x d\pi'(x) \quad (19.102)$$

we find in the limit $\epsilon \to 0$

$$\langle \phi_1 | \exp(-\epsilon H)|\phi_0 \rangle = \int \langle \phi_1 | \exp \left( -\epsilon \int \pi^2(x) d^3 x \right) |\pi' \rangle \times \langle \pi' | \exp \left( -\epsilon \int V(\phi(x)) d^3 x \right) |\phi_0 \rangle \, D\pi'$$

$$= |f|^2 \exp \left( -\epsilon \int V(\phi_0(x)) d^3 x \right) \times \int \exp \left[ \int -\frac{1}{2} \dot{\phi}^2(x) + i\pi'(x)[\phi_1(x) - \phi_0(x)] d^3 x \right] D\pi'. \quad (19.103)$$

Using the abbreviation

$$\dot{\phi}_0(x) \equiv \frac{\phi_1(x) - \phi_0(x)}{\epsilon} \quad (19.104)$$

and the integral formula (19.6), we get

$$\langle \phi_1 | \exp(-\epsilon H)|\phi_0 \rangle = f' \exp \left\{ -\epsilon \int \left[ \frac{1}{2} \dot{\phi}_0^2(x) + V(\phi(x)) \right] d^3 x \right\}.$$  

Putting together $n = \beta/\epsilon$ such terms, integrating over the intermediate states $|\phi_j \rangle$ from $j = 1$ to $j = n - 1$ and absorbing the normalizing factors into $D\phi$, we get

$$\langle \phi_\beta | e^{-\beta H} |\phi_0 \rangle = \int_{\phi_0}^{\phi_\beta} \exp \left[ -\int_0^\beta \int \frac{1}{2} \phi'^2(x) + V(\phi(x)) \, d^3 x \, d\beta' \right] D\phi. \quad (19.105)$$

Replacing the potential $V(\phi)$ with its definition (19.98), we have

$$\langle \phi_\beta | e^{-\beta H} |\phi_0 \rangle = \int_{\phi_0}^{\phi_\beta} \exp \left[ -\int_0^\beta \left( \frac{1}{2} \phi'^2 + (\nabla \phi)^2 + m^2 \phi^2 \right) + P(\phi) \, d^3 x \, d\beta' \right] D\phi$$

(19.106)

in which the limits $\phi_0$ and $\phi_\beta$ remind us that we are to integrate over all fields $\phi(x,t)$ that run from $\phi(x,0) = \phi_0(x)$ to $\phi(x,\beta) = \phi_\beta(x)$.
In terms of the energy density
\[
\mathcal{H}(\phi) \equiv \frac{1}{2} \left[ (\partial_a \phi)^2 + m^2 \phi^2 \right] + P(\phi)
\] (19.107)
in which \( \partial_a = (\partial/\partial \beta, \nabla) \), the path integral (19.106) is
\[
\langle \phi_\beta | e^{-\beta H} | \phi_0 \rangle = \int_{\phi_0}^{\phi_\beta} \exp \left[ -\int_0^\beta \mathcal{H}(\phi) \, d^3 x \, d\beta' \right] D\phi.
\] (19.108)

The partition function \( Z(\beta) \)—defined as the trace \( Z(\beta) \equiv \text{Tr} \, e^{-\beta H} \) over all the states of the system—is an integral over all periodic \( (\phi_\beta = \phi_0) \) fields
\[
Z(\beta) \equiv \text{Tr} \, e^{-\beta H} = \int_{\phi_0}^{\phi_\beta} \langle \phi | e^{-\beta H} | \phi \rangle \, D\phi = \int_{\phi_0}^{\phi_\beta} \exp \left[ -\int_0^\beta \mathcal{H}(\phi) \, d^3 x \, d\beta' \right] D\phi.
\] (19.109)

Because the squares of the four derivatives \( \partial_a \) occur with the same sign, finite-temperature field theory is called euclidian quantum field theory.

Like the definition (19.85) of the euclidian position operator \( q_e(\beta) \), the euclidian field operator \( \phi_e(x) \) is defined as
\[
\phi_e(x, \beta) = e^{\beta H} \phi(x, 0) e^{-\beta H}.
\] (19.110)

The euclidian-time-ordered product (19.86) of two fields is
\[
\mathcal{T} \{ \phi_e(x_1, \beta_1) \phi_e(x_2, \beta_2) \} = \theta(\beta_1 - \beta_2) e^{\beta_1 H} \phi_e(x_1, 0) e^{-(\beta_1 - \beta_2) H} \phi_e(x_2, 0) e^{-\beta_2 H}
+ \theta(\beta_2 - \beta_1) e^{\beta_2 H} \phi_e(x_2, 0) e^{-(\beta_2 - \beta_1) H} \phi_e(x_1, 0) e^{-\beta_1 H}.
\]

The logic of equations (19.85–19.94) leads us to write its mean value in a system described by a maximum-entropy-at-inverse-temperature-\( \beta \) density operator \( \rho = e^{-\beta H} / Z(\beta) \) as the ratio
\[
\langle \mathcal{T} \{ \phi_e(x_1) \phi_e(x_2) \} \rangle = \frac{\text{Tr} \{ \rho \, \mathcal{T} \{ \phi_e(x_1) \phi_e(x_2) \} \}}{\text{Tr} \{ e^{-\beta H} \}}
= \frac{\int_{\phi_0}^{\phi_\beta} \phi(x_1) \phi(x_2) \exp \left[ -\int_0^\beta \mathcal{H}(\phi) \, d^3 x \, d\beta' \right] D\phi}{\int_{\phi_0}^{\phi_\beta} \exp \left[ -\int_0^\beta \mathcal{H}(\phi) \, d^3 x \, d\beta' \right] D\phi}
\] (19.111)
in which several factors have canceled.

In the zero-temperature \( (\beta \to \infty) \) limit, the density operator \( \rho \) becomes the projection operator \( |0\rangle \langle 0| \) on the ground state, and mean-value formulas
like (19.111) become

\[ \langle 0 | \mathcal{T} [\phi_e(x_1) \ldots \phi_e(x_n)] | 0 \rangle = \frac{\int \phi(x_1) \ldots \phi(x_n) \exp \left[ -\int \mathcal{H}(\phi) \, d^4x \right] D\phi}{\int \exp \left[ -\int \mathcal{H}(\phi) \, d^4x \right] D\phi} \]

(19.112)

in which Hamilton’s density \( \mathcal{H}(\phi) \) is integrated over all of euclidian spacetime and over all fields that are periodic on the infinite time interval. Statistical field theory and lattice gauge theory use formulas like (19.111 & 19.112).

### 19.12 Real-Time Field Theory

We can apply the derivation of section 19.11 to amplitudes in real time. In the amplitude (19.103), we replace \( -\varepsilon H \) by \( -i\varepsilon H \)
and follow the logic of sections (19.5 & 19.11). We find in the limit \( \varepsilon \to 0 \) with \( \dot{\phi}_0 \equiv (\dot{\phi}_1 - \phi_0)/\varepsilon \)

\[ \langle \phi_1 | e^{-i\varepsilon H} | \phi_0 \rangle = \int \langle \phi_1 | e^{-i\varepsilon \int \pi^2/2 \, d^3x} | \pi' \rangle \langle \pi' | e^{-i\varepsilon \int V(\phi) \, d^3x} | \phi_0 \rangle D\pi' \]

\[ = |f|^2 e^{-i\varepsilon \int V(\phi) \, d^3x} \int e^{-i\varepsilon \pi'^2/2 + i\pi'(\dot{\phi}_1 - \phi_0) \, d^3x} D\pi' \]

\[ = f' \exp \left[ i\varepsilon \int \frac{1}{2} \dot{\phi}_0^2 - V(\phi_0) \, d^3x \right]. \]

(19.113)

Putting together \( 2t/\varepsilon \) similar factors and integrating over all the intermediate states \( |\phi_j \rangle \langle \phi_j| \), we arrive at the path integral

\[ \langle \phi_t | e^{-i2tH} | \phi_{-t} \rangle = \int_{\phi_{-t}}^{\phi_t} \exp \left[ i\int \frac{1}{2} \dot{\phi}^2(x) - V(\phi(x)) \, d^4x \right] D\phi \]

(19.114)

in which we integrate over all fields \( \phi(x) \) that run from \( \phi_{-t}(x,-t) \) to \( \phi_t(x,t) \).

After expanding the definition (19.98) of the potential \( V(\phi) \), we have

\[ \langle \phi_t | e^{-i2tH} | \phi_{-t} \rangle = \int_{\phi_{-t}}^{\phi_t} \exp \left[ i\int \frac{1}{2} \left[ \dot{\phi}^2 - (\nabla \phi)^2 - m^2 \phi^2 \right] - P(\phi) \, d^4x \right] D\phi. \]

(19.115)

This amplitude is a path integral

\[ \langle \phi_t | e^{-i2tH/h} | \phi_{-t} \rangle = \int_{\phi_{-t}}^{\phi_t} e^{iS[\phi]/h} D\phi \]

(19.116)

of phases \( \exp(iS[\phi]/h) \) that are exponentials of the classical action

\[ S[\phi] = \int \frac{1}{2} \left[ \dot{\phi}^2 - (\nabla \phi)^2 - m^2 \phi^2 \right] - P(\phi) \, d^4x. \]

(19.117)
The time dependence of the Heisenberg field operator $\phi(x,t)$ is

$$\phi(x,t) = e^{itH/\hbar}\phi(x,0)e^{-itH/\hbar}. \tag{19.118}$$

The time-ordered product of two fields, as in (19.86), is the sum

$$\mathcal{T}[\phi(x_1)\phi(x_2)] = \delta(x_1^0 - x_2^0)\phi(x_1)\phi(x_2) + \delta(x_2^0 - x_1^0)\phi(x_2)\phi(x_1). \tag{19.119}$$

Between two factors of $\exp(-iH)$, it is for $t_1 > t_2$

$$e^{-it_2H}\mathcal{T}[\phi(x_1)\phi(x_2)]e^{-it_1H} = e^{-i(t_1-t_2)H}\phi(x_1,0)e^{-i(t_1-t_2)H}\phi(x_2,0)e^{-i(t_2-t_1)H}. \tag{19.120}$$

So by the logic that led to the path-integral formulas (19.111) and (19.116), we can write a matrix element of the time-ordered product (19.119) as

$$\langle \phi_t|e^{-itH}\mathcal{T}[\phi(x_1)\phi(x_2)]e^{-itH}|\phi_{-t}\rangle = \int_{\phi_{-t}}^{\phi_t} \phi(x_1)\phi(x_2) e^{iS[\phi]} D\phi$$

(19.120)

in which we integrate over fields that go from $\phi_{-t}$ at time $-t$ to $\phi_t$ at time $t$. The time-ordered product of any combination of fields is then

$$\langle \phi_t|e^{-itH}\mathcal{T}[\phi(x_1)\ldots\phi(x_n)]e^{-itH}|\phi_{-t}\rangle = \int \phi(x_1)\ldots\phi(x_n) e^{iS[\phi]} D\phi.$$ 

(19.121)

Like the position eigenstates $|q\\rangle$ of quantum mechanics, the eigenstates $|\phi\\rangle$ are states of infinite energy that overlap many states, whereas we often are interested in the ground state $|0\rangle$ or in states of a few particles. To form such matrix elements, we multiply both sides of equations (19.116 & 19.121) by $\langle 0|\phi_t\rangle\langle \phi_{-t}|0\rangle$ and integrate over $\phi_{-t}$ and $\phi_t$. Since the ground state is a normalized eigenstate of the Hamiltonian $H|0\rangle = E_0|0\rangle$ with eigenvalue $E_0$, we find from the path-integral formulas (19.116)

$$\int\langle 0|\phi_t\rangle\langle \phi_{-t}|e^{-i2tH}\phi_{-t}|0\rangle D\phi_t D\phi_{-t} = \langle 0|e^{-i2tH}|0\rangle$$

(19.122)

and (19.121) (suppressing the differentials $D\phi_t D\phi_{-t}$)

$$e^{-2itE_0}\langle 0|\mathcal{T}[\phi(x_1)\ldots\phi(x_n)]|0\rangle = \int\langle 0|\phi_t\rangle\phi(x_1)\ldots\phi(x_n) e^{iS[\phi]}\langle \phi_{-t}|0\rangle D\phi.$$ 

(19.123)
The mean value in the ground state of a time-ordered product of field operators is then a ratio of these path integrals

\[ \langle 0 | T [ \phi(x_1) \ldots \phi(x_n)] | 0 \rangle = \frac{\int \langle 0 | \phi_t \phi(x_1) \ldots \phi(x_n) e^{iS[\phi]} \phi_{-t} | 0 \rangle D\phi}{\int \langle 0 | \phi_t e^{iS[\phi]} \phi_{-t} | 0 \rangle D\phi} \]  

(19.124)

in which factors involving \( E_0 \) have canceled and the integration is over all fields that go from \( \phi(x, -t) = \phi_{-t}(x) \) to \( \phi(x, t) = \phi_t(x) \) and over \( \phi_{-t}(x) \) and \( \phi_t(x) \).

### 19.13 Perturbation Theory

Field theories with hamiltonians that are quadratic in their fields like

\[ H_0 = \int \frac{1}{2} \left[ \pi^2(x) + (\nabla \phi(x))^2 + m^2 \phi^2(x) \right] d^3x \]  

(19.125)

are soluble. Their fields evolve in time as

\[ \phi(x, t) = e^{itH_0} \phi(x, 0)e^{-itH_0}. \]  

(19.126)

The mean value in the ground state of \( H_0 \) of a time-ordered product of these fields is by (19.124) a ratio of path integrals

\[ \langle 0 | T [ \phi(x_1) \ldots \phi(x_n)] | 0 \rangle = \frac{\int \langle 0 | \phi_t \phi(x_1) \ldots \phi(x_n) e^{iS_0[\phi]} \phi' | 0 \rangle D\phi}{\int \langle 0 | \phi'' e^{iS_0[\phi]} \phi' | 0 \rangle D\phi} \]  

(19.127)

in which the action \( S_0[\phi] \) is quadratic in the fields

\[ S_0[\phi] = \int \frac{1}{2} \left[ \dot{\phi}^2(x) - (\nabla \phi(x))^2 - m^2 \phi^2(x) \right] d^4x \]

\[ = \int \frac{1}{2} \left[ - \partial_a \phi(x) \partial^a \phi(x) - m^2 \phi^2(x) \right] d^4x. \]  

(19.128)

So the path integrals in the ratio (19.127) are gaussian and doable.

The Fourier transforms

\[ \tilde{\phi}(p) = \int e^{-ipx} \phi(x) d^4x \quad \text{and} \quad \phi(x) = \int e^{ipx} \tilde{\phi}(p) \frac{d^4p}{(2\pi)^4} \]  

(19.129)

turn the spacetime derivatives in the action into a quadratic form

\[ S_0[\phi] = -\frac{1}{2} \int |\tilde{\phi}(p)|^2 (p^2 + m^2) \frac{d^4p}{(2\pi)^4} \]  

(19.130)
in which $p^2 = P^2 - P^0_0$, and $\tilde{\phi}(-p) = \tilde{\phi}^*(p)$ by (3.25) since the field $\phi$ is real.

The initial $\langle \phi_{-\epsilon}|0 \rangle$ and final $\langle 0|\phi_{\epsilon} \rangle$ wave functions produce the $i\epsilon$ in the Feynman propagator (5.255). Although its exact form doesn’t matter here, the wave function $\langle \phi|0 \rangle$ of the ground state of $H_0$ is the exponential (18.53)

$$\langle \phi|0 \rangle = c \exp \left[ -\frac{1}{2} \int |\tilde{\phi}(p)|^2 \sqrt{p^2 + m^2} \frac{d^3p}{(2\pi)^3} \right]$$  \hspace{1cm} (19.131)

in which $\tilde{\phi}(p)$ is the spatial Fourier transform of the eigenvalue $\phi(x)$

$$\tilde{\phi}(p) = \int e^{-ip\cdot x} \phi(x) \, d^3x$$  \hspace{1cm} (19.132)

and $c$ is a normalization factor that will cancel in ratios of path integrals.

Apart from $-2i \ln c$ which we will not keep track of, the wave functions $\langle \phi_{-\epsilon}|0 \rangle$ and $\langle 0|\phi_{\epsilon} \rangle$ add to the action $S_0[\phi]$ the term

$$\Delta S_0[\phi] = \frac{i}{2} \int \sqrt{p^2 + m^2} \left( |\tilde{\phi}(p, t)|^2 + |\tilde{\phi}(p, -t)|^2 \right) \frac{d^3p}{(2\pi)^3}$$  \hspace{1cm} (19.133)

in which we envision taking the limit $t \to \infty$ with $\phi(x, t) = \phi_t(x)$ and $\phi(x, -t) = \phi_{-t}(x)$. The identity (Weinberg, 1995, pp. 386–388)

$$f(+\infty) + f(-\infty) = \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} f(t) \, e^{-\epsilon|t|} \, dt$$  \hspace{1cm} (19.134)

allows us to write $\Delta S_0[\phi]$ as

$$\Delta S_0[\phi] = \lim_{\epsilon \to 0^+} \frac{i\epsilon}{2} \int \sqrt{p^2 + m^2} \int_{-\infty}^{\infty} |\tilde{\phi}(p, t)|^2 e^{-\epsilon|t|} \, dt \frac{d^3p}{(2\pi)^3}$$  \hspace{1cm} (19.135)

To first order in $\epsilon$, the change in the action is (exercise 19.15)

$$\Delta S_0[\phi] = \lim_{\epsilon \to 0^+} \frac{i\epsilon}{2} \int \sqrt{p^2 + m^2} \int_{-\infty}^{\infty} |\tilde{\phi}(p, t)|^2 \, dt \frac{d^3p}{(2\pi)^3}$$

$$= \lim_{\epsilon \to 0^+} \frac{i\epsilon}{2} \int \sqrt{p^2 + m^2} |\tilde{\phi}(p)|^2 \frac{d^4p}{(2\pi)^4}.$$  \hspace{1cm} (19.136)

Thus the modified action is

$$S_0[\phi, \epsilon] = S_0[\phi] + \Delta S_0[\phi] = -\frac{1}{2} \int |\tilde{\phi}(p)|^2 \left( p^2 + m^2 - i\epsilon \sqrt{p^2 + m^2} \right) \frac{d^4p}{(2\pi)^4}$$

$$= -\frac{1}{2} \int |\tilde{\phi}(p)|^2 \left( p^2 + m^2 - i\epsilon \right) \frac{d^4p}{(2\pi)^4}$$  \hspace{1cm} (19.137)

since the square root is positive. In terms of the modified action, our formula
(19.127) for the time-ordered product is the ratio

$$\langle 0 | \mathcal{T} [\phi(x_1) \ldots \phi(x_n)] | 0 \rangle = \frac{\int \phi(x_1) \ldots \phi(x_n) e^{iS_0[\phi, \epsilon]} D\phi}{\int e^{iS_0[\phi, \epsilon]} D\phi}. \quad (19.138)$$

We can use this formula (19.138) to express the mean value in the vacuum $|0\rangle$ of the time-ordered exponential of a spacetime integral of $j(x)\phi(x)$, in which $j(x)$ is a classical (c-number, external) current, as the ratio

$$Z_0[j] \equiv \langle 0 | \mathcal{T} \left\{ \exp \left[ i \int j(x) \phi(x) d^4x \right] \right\} | 0 \rangle$$

$$= \frac{\int \exp \left[ i \int j(x) \phi(x) d^4x \right] e^{iS_0[\phi, \epsilon]} D\phi}{\int e^{iS_0[\phi, \epsilon]} D\phi}. \quad (19.139)$$

Since the state $|0\rangle$ is normalized, the mean value $Z_0[0]$ is unity,

$$Z_0[0] = 1. \quad (19.140)$$

If we absorb the current into the action

$$S_0[\phi, \epsilon, j] = S_0[\phi, \epsilon] + \int j(x) \phi(x) d^4x \quad (19.141)$$

then in terms of the current’s Fourier transform

$$\tilde{j}(p) = \int e^{-ipx} j(x) d^4x \quad (19.142)$$

the modified action $S_0[\phi, \epsilon, j]$ is (exercise 19.16)

$$S_0[\phi, \epsilon, j] = -\frac{1}{2} \int \left[ (\tilde{\phi}(p))^2 (p^2 + m^2 - i\epsilon) - \tilde{j}^*(p)\tilde{\phi}(p) - \tilde{\phi}^*(p)\tilde{j}(p) \right] \frac{d^4p}{(2\pi)^4}. \quad (19.143)$$

Changing variables to

$$\tilde{\psi}(p) = \tilde{\phi}(p) - \tilde{j}(p)/(p^2 + m^2 - i\epsilon) \quad (19.144)$$

we write the action $S_0[\phi, \epsilon, j]$ as (exercise 19.17)

$$S_0[\phi, \epsilon, j] = -\frac{1}{2} \int \left[ (\tilde{\psi}(p))^2 (p^2 + m^2 - i\epsilon) - \frac{\tilde{j}^*(p)\tilde{j}(p)}{(p^2 + m^2 - i\epsilon)} \right] \frac{d^4p}{(2\pi)^4}$$

$$= S_0[\psi, \epsilon] + \frac{1}{2} \int \left[ \frac{\tilde{j}^*(p)\tilde{j}(p)}{(p^2 + m^2 - i\epsilon)} \right] \frac{d^4p}{(2\pi)^4}. \quad (19.145)$$
And since $D\phi = D\psi$, our formula (19.139) gives simply (exercise 19.18)

$$Z_0[j] = \exp \left( \frac{i}{2} \int \frac{|\tilde{j}(p)|^2}{p^2 + m^2 - i\epsilon} \, d^4p \right). \quad (19.146)$$

Going back to position space, one finds (exercise 19.19)

$$Z_0[j] = \exp \left[ \frac{i}{2} \int j(x) \Delta(x - x') \, j(x') \, d^4x \, d^4x' \right] \quad (19.147)$$
in which $\Delta(x - x')$ is Feynman’s propagator (5.255)

$$\Delta(x - x') = \int \frac{e^{ip(x-x')}}{p^2 + m^2 - i\epsilon} \, d^4p. \quad (19.148)$$

The functional derivative (chapter 18) of $Z_0[j]$, defined by (19.139), is

$$\frac{1}{i} \frac{\delta Z_0[j]}{\delta j(x)} = \langle 0 | T \left[ \phi(x) \exp \left( i \int j(x') \phi(x') \, d^4x' \right) \right] | 0 \rangle \quad (19.149)$$

while that of equation (19.147) is

$$\frac{1}{i} \frac{\delta Z_0[j]}{\delta j(x)} = Z_0[j] \int \Delta(x - x') \, j(x') \, d^4x'. \quad (19.150)$$

Thus the second functional derivative of $Z_0[j]$ evaluated at $j = 0$ gives

$$\langle 0 | T \left[ \phi(x) \phi(x') \right] | 0 \rangle = \left. \frac{\delta^2 Z_0[j]}{\delta j(x) \delta j(x')} \right|_{j=0} = -i \Delta(x - x'). \quad (19.151)$$

Similarly, one may show (exercise 19.20) that

$$\langle 0 | T \left[ \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \right] | 0 \rangle = \left. \frac{\delta^4 Z_0[j]}{\delta j(x_1) \delta j(x_2) \delta j(x_3) \delta j(x_4)} \right|_{j=0}$$

$$= - \Delta(x_1 - x_2) \Delta(x_3 - x_4) - \Delta(x_1 - x_3) \Delta(x_2 - x_4)$$
$$- \Delta(x_1 - x_4) \Delta(x_2 - x_3). \quad (19.152)$$

Suppose now that we add a potential $V = P(\phi)$ to the free hamiltonian (19.125). Scattering amplitudes are matrix elements of the time-ordered exponential $T \exp \left[ -i \int P(\phi) \, d^4x \right]$. Our formula (19.138) for the mean value in the ground state $|0\rangle$ of the free hamiltonian $H_0$ of any time-ordered product of fields leads us to

$$\langle 0 | T \left\{ \exp \left[ -i \int P(\phi) \, d^4x \right] \right\} | 0 \rangle = \frac{\int \exp \left[ -i \int P(\phi) \, d^4x \right] e^{iS_0[\phi, \epsilon]} \, D\phi}{\int e^{iS_0[\phi, \epsilon]} \, D\phi} \quad (19.153)$$
Using (19.151 & 19.152), we can cast this expression into the magical form
\[ \langle 0 | \mathcal{T} \left\{ \exp \left[ -i \int P(\phi) \, d^4x \right] \right\} | 0 \rangle = \exp \left[ -i \int P \left( \frac{\delta}{i\delta j(x)} \right) \, d^4x \right] Z_0[j] \bigg|_{j=0} . \]  
(19.154)

The generalization of the path-integral formula (19.138) to the ground state |Φ⟩ of an interacting theory with action \( S \) is
\[ \langle \Omega | \mathcal{T} [\phi(x_1) \ldots \phi(x_n)] | \Omega \rangle = \frac{\int \phi(x_1) \ldots \phi(x_n) e^{iS[\phi,\epsilon]} \, D\phi}{\int e^{iS[\phi,\epsilon]} \, D\phi} \]  
(19.155)
in which a term like \( ie\phi^2 \) is added to make the modified action \( S[\phi,\epsilon] \).

These are some of the techniques one uses to make states of incoming and outgoing particles and to compute scattering amplitudes (Weinberg, 1995, 1996; Srednicki, 2007; Zee, 2010).

### 19.14 Application to Quantum Electrodynamics

In the Coulomb gauge \( \nabla \cdot A = 0 \), the QED hamiltonian is
\[ H = H_m + \int \left[ \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \times A)^2 - A \cdot j \right] \, d^3x + V_C \]  
(19.156)
in which \( H_m \) is the matter hamiltonian, and \( V_C \) is the Coulomb term
\[ V_C = \frac{1}{2} \int \frac{j_0(x,t) \cdot j_0(y,t)}{4\pi |x-y|} \, d^3x \, d^3y. \]  
(19.157)
The operators \( A \) and \( \pi \) are canonically conjugate, but they satisfy the Coulomb-gauge conditions
\[ \nabla \cdot A = 0 \quad \text{and} \quad \nabla \cdot \pi = 0. \]  
(19.158)

One may show (Weinberg, 1995, pp. 413–418) that in this theory, the analog of equation (19.155) is
\[ \langle \Omega | \mathcal{T} [\mathcal{O}_1 \ldots \mathcal{O}_n] | \Omega \rangle = \frac{\int \mathcal{O}_1 \ldots \mathcal{O}_n e^{iS_C} \delta[\nabla \cdot A] \, DA \, D\psi}{\int e^{iS_C} \delta[\nabla \cdot A] \, DA \, D\psi} \]  
(19.159)
in which the Coulomb-gauge action is
\[ S_C = \int \frac{1}{2} \dot{A}^2 - \frac{1}{2} (\nabla \times A)^2 + A \cdot j + \mathcal{L}_m \, d^4x - \int V_C \, dt \]  
(19.160)
and the functional delta function

\[ \delta(\nabla \cdot \mathbf{A}) = \prod_x \delta(\nabla \cdot \mathbf{A}(x)) \]  

(19.161)

enforces the Coulomb-gauge condition. The term \( \mathcal{L}_m \) is the action density of the matter field \( \psi \).

Tricks are available. We introduce a new field \( A^0(x) \) and consider the factor

\[ F = \int \exp \left[ i \int \frac{1}{2} \left( \nabla A^0 + \nabla \Delta^{-1} j^0 \right)^2 \, d^4 x \right] \, DA^0 \]  

(19.162)

which is just a number independent of the charge density \( j^0 \) since we can cancel the \( j^0 \) term by shifting \( A^0 \). By \( \Delta^{-1} \), we mean \( -1/4\pi |x - y| \). By integrating by parts, we can write the number \( F \) as (exercise 19.21)

\[ F = \int \exp \left[ i \int \frac{1}{2} \left( \nabla A^0 \right)^2 - A^0 j^0 - \frac{1}{2} j^0 \Delta^{-1} j^0 \, d^4 x \right] \, DA^0 \]  

(19.163)

So when we multiply the numerator and denominator of the amplitude (19.159) by \( F \), the awkward Coulomb term \( V_C \) cancels, and we get

\[ \langle \Omega | T [\mathcal{O}_1 \ldots \mathcal{O}_n] | \Omega \rangle = \frac{\int \mathcal{O}_1 \ldots \mathcal{O}_n e^{iS'} \delta(\nabla \cdot \mathbf{A}) \, DA \, D\psi}{\int e^{iS'} \delta(\nabla \cdot \mathbf{A}) \, DA \, D\psi} \]  

(19.164)

where now \( DA \) includes all four components \( A^\mu \) and

\[ S' = \int \frac{1}{2} \dot{\mathbf{A}}^2 - \frac{1}{2} (\nabla \times \mathbf{A})^2 + \frac{1}{2} \left( \nabla A^0 \right)^2 + \mathbf{A} \cdot \mathbf{j} - A^0 j^0 + \mathcal{L}_m \, d^4 x. \]  

(19.165)

Since the delta-function \( \delta(\nabla \cdot \mathbf{A}) \) enforces the Coulomb-gauge condition, we can add to the action \( S' \) the term \( (\nabla \cdot \dot{\mathbf{A}}) A^0 \) which is \( -\dot{\mathbf{A}} \cdot \nabla A^0 \) after we integrate by parts and drop the surface term. This extra term makes the action gauge invariant

\[ S = \int \frac{1}{2} (\dot{\mathbf{A}} - \nabla A^0)^2 - \frac{1}{2} (\nabla \times \mathbf{A})^2 + \mathbf{A} \cdot \mathbf{j} - A^0 j^0 + \mathcal{L}_m \, d^4 x \]  

(19.166)

\[ = \int -\frac{1}{4} F_{ab} F^{ab} + A^a j_b + \mathcal{L}_m \, d^4 x. \]
Thus at this point we have

$$\langle \Omega | T [ \mathcal{O}_1 \ldots \mathcal{O}_n ] | \Omega \rangle = \frac{\int \mathcal{O}_1 \ldots \mathcal{O}_n e^{i S} \delta [ \nabla \cdot A ] D A D \psi}{\int e^{i S} \delta [ \nabla \cdot A ] D A D \psi}$$

(19.167)

in which $S$ is the gauge-invariant action (19.166), and the integral is over all fields. The only relic of the Coulomb gauge is the gauge-fixing delta functional $\delta [ \nabla \cdot A ]$.

We now make the gauge transformation

$$A'_b(x) = A_b(x) + \partial_b \Lambda(x) \quad \text{and} \quad \psi'(x) = e^{iq\Lambda(x)} \psi(x)$$

(19.168)

in the numerator and also, using a different gauge transformation $\Lambda'$, in the denominator of the ratio (19.167) of path integrals. Since we are integrating over all gauge fields, these gauge transformations merely change the order of integration in the numerator and denominator of that ratio. They are like replacing $\int_{-\infty}^{\infty} f(x) \, dx$ by $\int_{-\infty}^{\infty} f(y) \, dy$. They change nothing, and so

$$\langle \Omega | T [ \mathcal{O}_1 \ldots \mathcal{O}_n ] | \Omega \rangle = \langle \Omega | T [ \mathcal{O}_1 \ldots \mathcal{O}_n ] | \Omega' \rangle$$

(19.169)

in which the prime refers to the gauge transformation (19.168).

We’ve seen that the action $S$ is gauge invariant. So is the measure $D A D \psi$, and we now restrict ourselves to operators $\mathcal{O}_1 \ldots \mathcal{O}_n$ that are gauge invariant. So in the right-hand side of equation (19.169), the replacement of the fields by their gauge transforms affects only the term $\delta [ \nabla \cdot A ]$ that enforces the Coulomb-gauge condition

$$\langle \Omega | T [ \mathcal{O}_1 \ldots \mathcal{O}_n ] | \Omega \rangle = \frac{\int \mathcal{O}_1 \ldots \mathcal{O}_n e^{i S} \delta [ \nabla \cdot A + \Delta \Lambda ] D A D \psi}{\int e^{i S} \delta [ \nabla \cdot A + \Delta \Lambda ] D A D \psi}.$$  

(19.170)

We now have two choices. If we integrate over all gauge functions $\Lambda(x)$ in both the numerator and the denominator of this ratio (19.170), then apart from over-all constants that cancel, the mean value in the vacuum of the time-ordered product is the ratio

$$\langle \Omega | T [ \mathcal{O}_1 \ldots \mathcal{O}_n ] | \Omega \rangle = \frac{\int \mathcal{O}_1 \ldots \mathcal{O}_n e^{i S} D A D \psi}{\int e^{i S} D A D \psi}$$

(19.171)

in which we integrate over all matter fields, gauge fields, and gauges. That is, we do not fix the gauge.
The analogous formula for the euclidian time-ordered product is

$$\langle \Omega | T_e [O_1 \ldots O_n] | \Omega \rangle = \frac{\int O_1 \ldots O_n e^{-S_e} DA D\psi}{\int e^{-S_e} DA D\psi} \quad (19.172)$$

in which the euclidian action $S_e$ is the spacetime integral of the energy density. This formula is quite general; it holds in nonabelian gauge theories and is important in lattice gauge theory.

Our second choice is to multiply the numerator and the denominator of the ratio (19.170) by the exponential $\exp\left[-i\frac{1}{2} \alpha \int (\Delta \Lambda)^2 d^4x\right]$ and then integrate over $\Lambda(x)$ separately in the numerator and denominator. This operation just multiplies the numerator and denominator by the same constant factor, which cancels. But if before integrating over all gauge transformations, we shift $\Lambda$ so that $\Delta \Lambda$ changes to $\Delta \Lambda - \bar{A}^0$, then the exponential factor is $\exp\left[-i\frac{1}{2} \alpha \int (\bar{A}^0 - \Delta \Lambda)^2 d^4x\right]$. Now when we integrate over $\Lambda(x)$, the delta function $\delta(\nabla \cdot A + \Delta \Lambda)$ replaces $\Delta \Lambda$ by $-\nabla \cdot A$ in the inserted exponential, converting it to $\exp\left[-i\frac{1}{2} \alpha \int (\bar{A}^0 + \nabla \cdot A)^2 d^4x\right]$. This term changes the gauge-invariant action (19.166) to the gauge-fixed action

$$S_\alpha = \int -\frac{1}{4} F_{ab} F^{ab} - \frac{\alpha}{2} (\partial_b A^b)^2 + A^b j_b + \mathcal{L}_m d^4x. \quad (19.173)$$

This Lorentz-invariant, gauge-fixed action is much easier to use than the Coulomb-gauge action (19.160) with the Coulomb potential (19.157). We can use it to compute scattering amplitudes perturbatively. The mean value of a time-ordered product of operators in the ground state $|0\rangle$ of the free theory is

$$\langle 0 | T [O_1 \ldots O_n] | 0 \rangle = \frac{\int O_1 \ldots O_n e^{iS_\alpha} DA D\psi}{\int e^{iS_\alpha} DA D\psi}. \quad (19.174)$$

By following steps analogous to those the led to (19.148), one may show (exercise 19.22) that in Feynman’s gauge, $\alpha = 1$, the photon propagator is

$$\langle 0 | T [A_\mu(x) A_\nu(y)] | 0 \rangle = -i \delta_{\mu\nu}(x-y) = -i \int \frac{\eta_{\mu\nu}}{q^2 - i\epsilon} e^{iq \cdot (x-y)} \frac{d^4q}{(2\pi)^4}. \quad (19.175)$$
19.15 Fermionic Path Integrals

In our brief introduction (1.11–1.12) and (1.44–1.46), to Grassmann variables, we learned that because
\[
\theta^2 = 0
\]  
the most general function \( f(\theta) \) of a single Grassmann variable \( \theta \) is
\[
f(\theta) = a + b \theta.  \tag{19.177}
\]
So a complete integral table consists of the integral of this linear function
\[
\int f(\theta) \, d\theta = \int (a + b \theta) \, d\theta = a \int d\theta + b \int \theta \, d\theta.  \tag{19.178}
\]
This equation has two unknowns, the integral \( \int d\theta \) of unity and the integral \( \int \theta \, d\theta \) of \( \theta \). We choose them so that the integral of \( f(\theta + \zeta) \)
\[
\int f(\theta + \zeta) \, d\theta = \int (a + b (\theta + \zeta)) \, d\theta = (a + b \zeta) \int d\theta + b \int \theta \, d\theta \tag{19.179}
\]
is the same as the integral (19.178) of \( f(\theta) \). Thus the integral \( \int d\theta \) of unity must vanish, while the integral \( \int \theta \, d\theta \) of \( \theta \) can be any constant, which we choose to be unity. Our complete table of integrals is then
\[
\int d\theta = 0 \quad \text{and} \quad \int \theta \, d\theta = 1.  \tag{19.180}
\]

The anticommutation relations for a fermionic degree of freedom \( \psi \) are
\[
\{ \psi, \psi^\dagger \} = \psi \psi^\dagger + \psi^\dagger \psi = 1 \quad \text{and} \quad \{ \psi, \psi \} = \{ \psi^\dagger, \psi^\dagger \} = 0.  \tag{19.181}
\]
Because \( \psi \) has \( \psi^\dagger \), it is conventional to introduce a variable \( \theta^* = \theta^\dagger \) that anti-commutes with itself and with \( \theta \)
\[
\{ \theta^*, \theta^* \} = \{ \theta^*, \theta \} = \{ \theta, \theta \} = 0. \tag{19.182}
\]
The logic that led to (19.180) now gives
\[
\int d\theta^* = 0 \quad \text{and} \quad \int \theta^* \, d\theta^* = 1.  \tag{19.183}
\]

We define the reference state \(|0\rangle \) as \(|0\rangle \equiv \psi |s\rangle \) for a state \(|s\rangle \) that is not annihilated by \( \psi \). Since \( \psi^2 = 0 \), the operator \( \psi \) annihilates the state \(|0\rangle \)
\[
\psi |0\rangle = \psi^2 |s\rangle = 0.  \tag{19.184}
\]
The effect of the operator \( \psi \) on the state
\[
|\theta\rangle = \exp \left( \psi^\dagger \theta - \frac{1}{2} \theta^* \theta \right) |0\rangle = \left( 1 + \psi^\dagger \theta - \frac{1}{2} \theta^* \theta \right) |0\rangle \tag{19.185}
\]
\[ \psi |\theta\rangle = \psi (1 + \psi^\dagger \theta - \frac{1}{2} \theta^* \theta) |0\rangle = \psi \psi^\dagger \theta |0\rangle = (1 - \psi^\dagger \psi) \theta |0\rangle = \theta |0\rangle \]  
(19.186)

while that of \( \theta \) on \( |\theta\rangle \) is

\[ \theta |\theta\rangle = \theta (1 + \psi^\dagger \theta - \frac{1}{2} \theta^* \theta) |0\rangle = \theta |0\rangle . \]  
(19.187)

The state \( |\theta\rangle \) therefore is an eigenstate of \( \psi \) with eigenvalue \( \theta \)

\[ \psi |\theta\rangle = \theta |\theta\rangle . \]  
(19.188)

The bra corresponding to the ket \( |\zeta\rangle \) is

\[ \langle \zeta | = \langle 0 | (1 + \zeta^* \psi - \frac{1}{2} \zeta^* \zeta) \]  
(19.189)

and the inner product \( \langle \zeta | \theta \rangle \) is (exercise 19.23)

\[
\langle \zeta | \theta \rangle = \langle 0 | \left( 1 + \zeta^* \psi - \frac{1}{2} \zeta^* \zeta \right) \left( 1 + \psi^\dagger \theta - \frac{1}{2} \theta^* \theta \right) |0\rangle \\
= \langle 0 | 1 + \zeta^* \psi \psi^\dagger \theta - \frac{1}{2} \zeta^* \zeta - \frac{1}{2} \theta^* \theta + \frac{1}{4} \zeta^* \theta \theta^* \theta |0\rangle \\
= \langle 0 | 1 + \zeta^* \theta - \frac{1}{2} \zeta^* \zeta - \frac{1}{2} \theta^* \theta + \frac{1}{4} \zeta^* \theta \theta^* \theta |0\rangle \\
= \exp \left[ \zeta^* \theta - \frac{1}{2} (\zeta^* \zeta + \theta^* \theta) \right]. \]  
(19.190)

**Example 19.4 (A gaussian integral).** For *any* number \( c \), we can compute the integral of \( \exp(c \theta^* \theta) \) by expanding the exponential

\[
\int e^{c \theta^* \theta} d\theta^* d\theta = \int (1 + c \theta^* \theta) d\theta^* d\theta = \int (1 - c \theta \theta^*) d\theta^* d\theta = -c. \]  
(19.191)

The identity operator for the space of states

\[ c |0\rangle + d |1\rangle \equiv c |0\rangle + d \psi^\dagger |0\rangle \]  
(19.192)

is (exercise 19.24) the integral

\[ I = \int |\theta\rangle \langle \theta | d\theta^* d\theta = |0\rangle \langle 0 | + |1\rangle \langle 1 | \]  
(19.193)

in which the differentials anti-commute with each other and with other fermionic variables: \{ \( d\theta, d\theta^* \} = 0, \{ d\theta, \theta \} = 0, \{ d\theta, \psi \} = 0, \) and so forth.

The case of several Grassmann variables \( \theta_1, \theta_2, \ldots, \theta_n \) and several Fermi operators \( \psi_1, \psi_2, \ldots, \psi_n \) is similar. The \( \theta_k \) anticommute among themselves

\[ \{ \theta_i, \theta_j \} = \{ \theta_i, \theta_j^* \} = \{ \theta_i^*, \theta_j^* \} = 0 \]  
(19.194)
while the $\psi_k$ satisfy
\[
\{\psi_k, \psi^\dagger_k\} = \delta_{kk} \quad \text{and} \quad \{\psi_k, \psi_l\} = \{\psi^\dagger_k, \psi^\dagger_l\} = 0. \tag{19.195}
\]
The reference state $|0\rangle$ is
\[
|0\rangle = \left( \prod_{k=1}^{n} \psi_k \right) |s\rangle \tag{19.196}
\]
in which $|s\rangle$ is any state not annihilated by any $\psi_k$ (so the resulting $|0\rangle$ isn’t zero). The direct-product state
\[
|\theta\rangle \equiv \exp \left( \sum_{k=1}^{n} \hat{\psi}^\dagger_k \theta_k - \frac{1}{2} \hat{\vartheta}^\ast_k \theta_k \right) |0\rangle = \left[ \prod_{k=1}^{n} \left( 1 + \hat{\psi}^\dagger_k \theta_k - \frac{1}{2} \hat{\vartheta}^\ast_k \theta_k \right) \right] |0\rangle \tag{19.197}
\]
is (exercise 19.25) a simultaneous eigenstate of each $\psi_k$
\[
\psi_k |\theta\rangle = \theta_k |\theta\rangle. \tag{19.198}
\]
It follows that
\[
\psi_l \psi_k |\theta\rangle = \psi_l \theta_k |\theta\rangle = -\theta_k \psi_l |\theta\rangle = -\theta_k \theta_l |\theta\rangle = \theta_l \theta_k |\theta\rangle \tag{19.199}
\]
and so too $\psi_k \psi_l |\theta\rangle = \theta_k \theta_l |\theta\rangle$. Since the $\psi$’s anticommute, their eigenvalues must also
\[
\theta_l \theta_k |\theta\rangle = \psi_l \psi_k |\theta\rangle = -\psi_k \psi_l |\theta\rangle = -\theta_k \theta_l |\theta\rangle \tag{19.200}
\]
(even if they commuted with the $\psi$’s in which case we’d have $\psi_l \psi_k |\theta\rangle = \theta_k \theta_l |\theta\rangle = -\psi_k \psi_l |\theta\rangle = -\theta_k \theta_l |\theta\rangle$).

The inner product $\langle \zeta | \theta \rangle$ is
\[
\langle \zeta | \theta \rangle = \langle 0 | \left[ \prod_{k=1}^{n} \left( 1 + \zeta_k^\ast \zeta_k - \frac{1}{2} \zeta_k^\ast \zeta_k \right) \right] \left[ \prod_{l=1}^{n} \left( 1 + \hat{\psi}^\dagger_l \theta_l - \frac{1}{2} \hat{\vartheta}^\ast_l \theta_l \right) \right] |0\rangle
= \exp \left[ \sum_{k=1}^{n} \zeta_k^\ast \theta_k - \frac{1}{2} \left( \zeta_k^\ast \zeta_k + \theta_k^\ast \theta_k \right) \right] = e^{\zeta^\dagger \theta - (\zeta^\ast \zeta + \theta^\ast \theta)/2}. \tag{19.201}
\]

The identity operator is
\[
I = \int |\theta\rangle \langle \theta| \prod_{k=1}^{n} d\theta^\ast_k d\theta_k. \tag{19.202}
\]

**Example 19.5** (Gaussian Grassmann Integral). For any $2 \times 2$ matrix $A$, we may compute the gaussian integral
\[
g(A) = \int e^{-\theta^\dagger A \theta} d\theta^\ast_1 d\theta_1 d\theta^\ast_2 d\theta_2 \tag{19.203}
\]
by expanding the exponential. The only terms that survive are the ones that
have exactly one of each of the four variables $\theta_1$, $\theta_2$, $\theta^*_1$, and $\theta^*_2$. Thus, the
integral is the determinant of the matrix $A$

$$g(A) = \int \frac{1}{2} \left( \theta^*_k A_{kl} \theta_l \right)^2 d\theta^*_1 d\theta_1 d\theta^*_2 d\theta_2$$

$$= \int \left( \theta^*_1 A_{11} \theta_1 \theta^*_2 A_{22} \theta_2 + \theta^*_1 A_{12} \theta_1 \theta^*_2 A_{21} \theta_2 \right) d\theta^*_1 d\theta_1 d\theta^*_2 d\theta_2$$

$$= A_{11} A_{22} - A_{12} A_{21} = \det A.$$  \hspace{1cm} (19.204)

The natural generalization to $n$ dimensions

$$\int e^{-\theta^t A \theta} \prod_{k=1}^n d\theta^*_k d\theta_k = \det A$$  \hspace{1cm} (19.205)

is true for any $n \times n$ matrix $A$. If $A$ is invertible, then the invariance of
Grassmann integrals under translations implies that

$$\int e^{-\theta^t A \theta + \theta^t \zeta + \zeta^t \theta} \prod_{k=1}^n d\theta^*_k d\theta_k = \int e^{-\theta^t A (\theta + A^{-1} \zeta) + \theta^t (\theta + A^{-1} \zeta)} \prod_{k=1}^n d\theta^*_k d\theta_k$$

$$= \int e^{-\theta^t A \theta + \theta^t \zeta + \zeta^t A^{-1} \zeta} \prod_{k=1}^n d\theta^*_k d\theta_k$$

$$= \int e^{-\theta^t A \theta + \theta^t \zeta + \zeta^t A^{-1} \zeta} \prod_{k=1}^n d\theta^*_k d\theta_k$$

$$= \int e^{-\theta^t A \theta + \theta^t \zeta + \zeta^t A^{-1} \zeta} \prod_{k=1}^n d\theta^*_k d\theta_k$$

$$= \det A \ e^{\zeta^t A^{-1} \zeta}.$$  \hspace{1cm} (19.206)

The values of $\theta$ and $\theta^t$ that make the argument $-\theta^t A \theta + \theta^t \zeta + \zeta^t \theta$ of the exponential stationary are $\overline{\theta} = A^{-1} \zeta$ and $\overline{\theta^t} = \zeta^t A^{-1}$. So a gaussian Grassmann integral is equal to its exponential evaluated at its stationary point, apart from a prefactor involving the determinant $\det A$. This result is a fermionic echo of the bosonic results (19.13–19.15). \hfill \Box

One may further extend these definitions to a Grassmann field $\chi_m(x)$ and
an associated Dirac field $\psi_m(x)$. The $\chi_m(x)$’s anticommute among themselves and with all fermionic variables at all points of spacetime

$$\{\chi_m(x), \chi_n(x')\} = \{\chi^*_m(x), \chi^*_n(x')\} = \{\chi^*_m(x), \chi^*_n(x')\} = 0$$  \hspace{1cm} (19.207)
and the Dirac field $\psi_m(x)$ obeys the equal-time anticommutation relations

$$\{\psi_m(x,t),\psi_n^\dagger(x',t)\} = \delta_{mn} \delta(x-x') \quad (n,m = 1,\ldots,4)$$

$$\{\psi_m(x,t),\psi_n(x',t)\} = \{\psi_m^\dagger(x,t),\psi_n^\dagger(x',t)\} = 0.$$  \hfill (19.208)

As in (19.100 & 19.196), we use eigenstates of the field $\psi$ at $t = 0$. If $|0\rangle$ is defined in terms of a state $|s\rangle$ that is not annihilated by any $\psi_m(x,0)$ as

$$|0\rangle = \left[\prod_{m,x} \psi_m(x,0)\right] |s\rangle$$  \hfill (19.209)

then (exercise 19.26) the state

$$|\chi\rangle = \exp\left(\int \sum_m \psi_m^\dagger(x,0) \chi_m(x) - \frac{1}{2} \chi_m^*(x) \chi_m(x) \, d^3x\right) |0\rangle$$

$$= \exp\left(\int \psi^\dagger \chi - \frac{1}{2} \chi^\dagger \chi \, d^3x\right) |0\rangle$$  \hfill (19.210)

is an eigenstate of the operator $\psi_m(x,0)$ with eigenvalue $\chi_m(x)$

$$\psi_m(x,0)|\chi\rangle = \chi_m(x)|\chi\rangle.$$  \hfill (19.211)

The inner product of two such states is (exercise 19.27)

$$\langle \chi'|\chi\rangle = \exp\left[\int \chi'^\dagger \chi - \frac{1}{2} \chi'^\dagger \chi' - \frac{1}{2} \chi^\dagger \chi \, d^3x\right].$$  \hfill (19.212)

The identity operator is the integral

$$I = \int \langle \chi'|\chi\rangle \, D\chi^* D\chi$$ \hfill (19.213)

in which

$$D\chi^* D\chi \equiv \prod_{m,x} dx_m^* d\chi_m(x).$$ \hfill (19.214)

The hamiltonian for a free Dirac field $\psi$ of mass $m$ is the spatial integral

$$H_0 = \int \bar{\psi} (\gamma \cdot \nabla + m) \psi \, d^3x$$ \hfill (19.215)

in which $\bar{\psi} \equiv i\psi^\dagger \gamma^0$ and the gamma matrices (10.322) satisfy

$$\{\gamma^a,\gamma^b\} = 2 \eta^{ab}$$ \hfill (19.216)

where $\eta$ is the $4 \times 4$ diagonal matrix with entries $(-1,1,1,1)$. Since $\psi|\chi\rangle =$
\[ \chi | \psi \rangle \text{ and } \langle \chi' | \psi^\dagger = \langle \chi' | \chi^\dagger \text{, the quantity } \langle \chi' | \exp(-iH_0) | \chi \rangle \text{ is by (19.212)} \]

\[ \langle \chi' | e^{-iH_0} | \chi \rangle = \langle \chi' | \chi \rangle \exp \left[ -i \epsilon \int \mathcal{L} \left( \gamma \cdot \nabla + m \right) \chi \, d^3 x \right] \quad (19.217) \]

\[ = \exp \left[ \epsilon \int \left[ \frac{1}{2} (\chi^\dagger - \chi^\dagger) \chi - \frac{1}{2} \chi^\dagger (\chi' - \chi) - i \epsilon \overline{\chi} \left( \gamma \cdot \nabla + m \right) \chi \right] \, d^3 x \right] \]

\[ = \exp \left\{ \epsilon \int \left[ \frac{1}{2} \chi^\dagger \chi - \frac{1}{2} \chi^\dagger \chi - i \epsilon \overline{\chi} \left( \gamma \cdot \nabla + m \right) \chi \right] \, d^3 x \right\} \]

in which \( \chi^\dagger - \chi^\dagger = \epsilon \chi^\dagger \) and \( \chi' - \chi = \epsilon \chi \). Everything within the square brackets is multiplied by \( \epsilon \), so we may replace \( \chi^\dagger \) by \( \chi^\dagger \) and \( \overline{\chi}' \) by \( \overline{\chi} \) so as to write to first order in \( \epsilon \)

\[ \langle \chi' | e^{-iH_0} | \chi \rangle = \exp \left[ \epsilon \int \frac{1}{2} \chi^\dagger \chi - \frac{1}{2} \chi^\dagger \chi - i \epsilon \overline{\chi} \left( \gamma \cdot \nabla + m \right) \chi \, d^3 x \right] \quad (19.218) \]

in which the dependence upon \( \chi' \) is through the time derivatives.

Putting together \( n = 2t/\epsilon \) such matrix elements, integrating over all intermediate-state dyadics \( |\chi \rangle \langle \chi| \), and using our formula (19.223), we find

\[ \langle \chi | e^{-2itH_0} | \chi_{-t} \rangle = \int \exp \left[ \epsilon \int \left[ \frac{1}{2} \chi^\dagger \chi - \frac{1}{2} \chi^\dagger \chi - i \epsilon \overline{\chi} \left( \gamma \cdot \nabla + m \right) \chi \right] \, d^4 x \right] D\chi^* D\overline{\chi}. \]

(19.219)

Integrating \( \chi^\dagger \chi \) by parts and dropping the surface term, we get

\[ \langle \chi | e^{-2itH_0} | \chi_{-t} \rangle = \int \exp \left[ \epsilon \int \left[ - \chi^\dagger \chi - i \epsilon \overline{\chi} \left( \gamma \cdot \nabla + m \right) \chi \right] \, d^4 x \right] D\chi^* D\overline{\chi}. \]

(19.220)

Since \( - \chi^\dagger \chi = - i \epsilon \overline{\chi}^0 \gamma^0 \chi \), the argument of the exponential is

\[ i \int - \chi^\dagger \chi - i \epsilon \overline{\chi} \left( \gamma \cdot \nabla + m \right) \chi \, d^4 x = i \int - \overline{\chi} \left( \gamma^\mu \partial_\mu + m \right) \chi \, d^4 x. \]

(19.221)

We then have

\[ \langle \chi | e^{-2itH_0} | \chi_{-t} \rangle = \int \exp \left( i \int \mathcal{L}_0(\chi) \, d^4 x \right) D\chi^* D\overline{\chi} \quad (19.222) \]

in which \( \mathcal{L}_0(\chi) = - \overline{\chi} \left( \gamma^\mu \partial_\mu + m \right) \chi \) is the action density (10.324) for a free Dirac field. Thus the amplitude is a path integral with phases given by the classical action \( S_0[\chi] \)

\[ \langle \chi | e^{-2itH_0} | \chi_{-t} \rangle = \int e^{i \int \mathcal{L}_0(\chi) \, d^4 x} D\chi^* D\overline{\chi} = \int e^{iS_0[\chi]} D\chi^* D\overline{\chi} \quad (19.223) \]

and the integral is over all fields that go from \( \chi(x, -t) = \chi_{-t}(x) \) to \( \chi(x, t) = \chi_t(x) \). Any normalization factor will cancel in ratios of such integrals.
Since Fermi fields anticommute, their time-ordered product has an extra minus sign
\[
\mathcal{T} [\bar{\psi}(x_1)\psi(x_2)] = \theta(x_1^0 - x_2^0) \bar{\psi}(x_1) \psi(x_2) - \theta(x_2^0 - x_1^0) \psi(x_2) \bar{\psi}(x_1). \tag{19.224}
\]

The logic behind our formulas (19.121) and (19.127) for the time-ordered product of bosonic fields now leads to an expression for the time-ordered product of \(2n\) Dirac fields (with \(D^{\mu 0}\) and \(D^{\mu} \) and so forth suppressed)
\[
\langle 0 | \mathcal{T} [\bar{\psi}(x_1) \ldots \psi(x_{2n})] | 0 \rangle = \frac{\int \langle 0 | \chi''(x_1) \ldots \chi(x_{2n}) e^{iS_0[\chi]} \langle \chi' | 0 \rangle D\chi \ast D\chi}{\int \langle 0 | \chi'' \rangle e^{iS_0[\chi]} \langle \chi' | 0 \rangle D\chi \ast D\chi}.
\tag{19.225}
\]
As in (19.138), the effect of the inner products \(\langle 0 | \chi'' \rangle\) and \(\langle \chi' | 0 \rangle\) is to insert \(\epsilon\)-terms which modify the Dirac propagators
\[
\langle 0 | \mathcal{T} [\bar{\psi}(x_1) \ldots \psi(x_{2n})] | 0 \rangle = \frac{\int \bar{\chi}(x_1) \ldots \chi(x_{2n}) e^{iS_0[\chi, e]} D\chi \ast D\chi}{\int e^{iS_0[\chi, e]} D\chi \ast D\chi}.
\tag{19.226}
\]

Imitating (19.139), we introduce a Grassmann external current \(\zeta(x)\) and define a fermionic analog of \(Z_0[j]\)
\[
Z_0[\zeta] \equiv \langle 0 | \mathcal{T} \left[ e^{i d^4 x (\bar{\zeta} \psi + \bar{\psi} \zeta)} \right] | 0 \rangle = \frac{\int e^{i S_0[\chi, e]} D\chi \ast D\chi}{\int e^{iS_0[\chi, e]} D\chi \ast D\chi}. \tag{19.227}
\]

**Example 19.6** (Feynman’s fermion propagator). Since
\[
i (\gamma^\mu \partial_\mu + m) \Delta(x - y) \equiv \int \frac{d^4 p}{(2\pi)^4} e^{ip(x-y)} \frac{-i(-i\gamma^\mu p_\mu + m)}{p^2 + m^2 - i\epsilon}
= \int \frac{d^4 p}{(2\pi)^4} e^{ip(x-y)} (i\gamma^\mu p_\mu + m) \frac{(-i\gamma^\mu p_\mu + m)}{p^2 + m^2 - i\epsilon}
= \int \frac{d^4 p}{(2\pi)^4} e^{ip(x-y)} \frac{p^2 + m^2}{p^2 + m^2 - i\epsilon} = \delta^4(x - y),
\tag{19.228}
\]
the function \(\Delta(x - y)\) is the inverse of the differential operator \(i(\gamma^\mu \partial_\mu + m)\).
Thus the Grassmann identity (19.206) implies that $Z_0[\zeta]$ is

$$
\langle 0 | T \left[ e^{i\bar{\psi} + \bar{\phi} d^4 x} \right] | 0 \rangle = \frac{\int e^{i \bar{\phi} \cdot \gamma \cdot \partial \cdot \phi + \bar{\phi} \cdot \gamma \cdot \phi} \, D^4 \phi \, D^{\phi}}{\int e^{i S_0[\phi, \bar{\phi}]} \, D^4 \phi \, D^{\phi}}
$$

$$
= \exp \left[ \int \bar{\phi}(x) \Delta(x - y) \phi(y) \, d^4 x \, d^4 y \right].
$$

(19.229)

Differentiating we get

$$
\langle 0 | T \left[ \psi(x) \overline{\psi}(y) \right] | 0 \rangle = \Delta(x - y) = -i \int \frac{d^4 p}{(2\pi)^4} e^{i p(x - y)} \frac{-i \gamma^\mu p^\mu + m}{p^2 + m^2 - i\epsilon}.
$$

(19.230)

19.16 Application to nonabelian gauge theories

The action of a generic non-abelian gauge theory is

$$
S = \int -\frac{1}{4} F_{a\mu} F_{a}^{\mu\nu} - \overline{\psi} (\gamma^\mu D_\mu + m) \psi \, d^4 x
$$

(19.231)

in which the Maxwell field is

$$
F_{a\mu} \equiv \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + g f_{a\mu\nu} A_{a\nu}
$$

(19.232)

and the covariant derivative is

$$
D_\mu \psi \equiv \partial_\mu \psi - ig t_a A_{a\mu} \psi.
$$

(19.233)

Here $g$ is a coupling constant, $f_{a\mu\nu}$ is a structure constant (10.69), and $t_a$ is a generator (10.57) of the Lie algebra (section 10.16) of the gauge group.

One may show (Weinberg, 1996, pp. 14–18) that the analog of equation (19.167) for quantum electrodynamics is

$$
\langle \Omega | T [O_1 \ldots O_n] | \Omega \rangle = \int O_1 \ldots O_n e^{iS} \delta[A_{a3}] \, D A \, D \psi
$$

(19.234)

in which the functional delta function

$$
\delta[A_{a3}] \equiv \prod_{x,b} \delta(A_{a3}(x))
$$

(19.235)

enforces the axial-gauge condition, and $D\psi$ stands for $D^4 \psi$. 

Initially, physicists had trouble computing nonabelian amplitudes beyond the lowest order of perturbation theory. Then DeWitt showed how to compute to second order (DeWitt, 1967), and Faddeev and Popov, using path integrals, showed how to compute to all orders (Faddeev and Popov, 1967).

### 19.17 The Faddeev-Popov trick

The path-integral tricks of Faddeev and Popov are described in (Weinberg, 1996, pp. 19–27). We will use gauge-fixing functions \( G_a(x) \) to impose a gauge condition on our non-abelian gauge fields \( A_a^\mu(x) \). For instance, we can use \( G_a(x) = A_3^a(x) \) to impose an axial gauge or \( G_a(x) = i\partial_\mu A_\mu^a(x) \) to impose a Lorentz-invariant gauge.

Under an infinitesimal gauge transformation (12.489)

\[
A_\mu^a = A_\mu^a - \partial_\mu \lambda_a - g f_{abc} A_{\mu}^b \lambda_c
\]

the gauge fields change, and so the gauge-fixing functions \( G_b(x) \), which depend upon them, also change. The Jacobian \( J \) of that change at \( \lambda = 0 \) is

\[
J = \det \left( \frac{\delta G_b^\lambda(x)}{\delta \lambda_b(y)} \right) \bigg|_{\lambda=0} \equiv \frac{DG^\lambda}{D\lambda} \bigg|_{\lambda=0}
\]

and it typically involves the delta function \( \delta^4(x-y) \).

Let \( B[G] \) be any functional of the gauge-fixing functions \( G_b(x) \) such as

\[
B[G] = \prod_{x,a} \delta(G_a(x)) = \prod_{x,a} \delta(A_3^a(x))
\]

in an axial gauge or

\[
B[G] = \exp \left[ \frac{i}{2} \int (G_a(x))^2 \, d^4x \right] = \exp \left[ -\frac{i}{2} \int (\partial_\mu A_\mu^a(x))^2 \, d^4x \right]
\]

in a Lorentz-invariant gauge.

We want to understand functional integrals like (19.240)

\[
\langle \Omega | T \left[ \mathcal{O}_1 \ldots \mathcal{O}_n \right] | \Omega \rangle = \frac{\int \mathcal{O}_1 \ldots \mathcal{O}_n \ e^{iS} \ B[G] \ J \, DAD\psi}{\int e^{iS} \ B[G] \ J \, DAD\psi}
\]

in which the operators \( \mathcal{O}_k \), the action functional \( S[A] \), and the differentials \( DAD\psi \) (but not the gauge-fixing functional \( B[G] \) or the Jacobian \( J \)) are gauge invariant. The axial-gauge formula (19.234) is a simple example in
which $B[G] = \delta[A_{a3}]$ enforces the axial-gauge condition $A_{a3}(x) = 0$ and the determinant $J = \det(\delta_{ab}\partial_3\delta(x - y))$ is a constant that cancels.

If we translate the gauge fields by gauge transformations $\Lambda$ and $\Lambda'$, then the ratio (19.240) does not change

$$\langle \Omega | T[O_1 \ldots O_n] | \Omega \rangle = \frac{\int \mathcal{O}_1^\Lambda \ldots \mathcal{O}_n^\Lambda e^{iS^\Lambda} B[G^\Lambda] J^\Lambda DAD\psi^\Lambda}{\int e^{iS^\Lambda} B[G^\Lambda] J^\Lambda DAD\psi^\Lambda}$$

(19.241)

any more than $\int f(y) dy$ is different from $\int f(x) dx$. Since the operators $O_k$, the action functional $S[A]$, and the differentials $DAD\psi$ are gauge invariant, most of the $\Lambda$-dependence goes away

$$\langle \Omega | T[O_1 \ldots O_n] | \Omega \rangle = \frac{\int \mathcal{O}_1 \ldots \mathcal{O}_n e^{iS} B[G] JAD\psi}{\int e^{iS} B[G] JAD\psi}.$$  

(19.242)

Let $\Lambda\lambda$ be a gauge transformation $\Lambda$ followed by an infinitesimal gauge transformation $\lambda$. The jacobian $J^\Lambda$ is a determinant of a product of matrices which is a product of their determinants

$$J^\Lambda = \det \left( \frac{\delta G_{a}^{\Lambda \lambda}(x)}{\delta \lambda_b(y)} \right)_{\lambda=0} = \det \left( \int \frac{\delta G_{a}^{\Lambda \lambda}(x)}{\delta \lambda_c(z)} \frac{\delta \lambda_c(z)}{\delta \lambda_b(y)} \, dz \right)_{\lambda=0}$$

$$= \det \left( \frac{\delta G_{a}^{\Lambda \lambda}(x)}{\delta \lambda_b(y)} \right)_{\lambda=0} \det \left( \frac{\delta \lambda_c(z)}{\delta \lambda_b(y)} \right)_{\lambda=0}$$

$$= \det \left( \frac{\delta G_{a}^{\Lambda \lambda}(x)}{\delta \lambda_b(y)} \right) \det \left( \frac{\delta \lambda_c(z)}{\delta \lambda_b(y)} \right)_{\lambda=0} \equiv \left. \frac{DG^\Lambda D\Lambda}{D\Lambda D\Lambda} \right|_{\lambda=0}. \quad (19.243)$$

Now we integrate over the gauge transformations $\Lambda$ (and $\Lambda'$) with weight function $\rho(\Lambda) = (D\Lambda\lambda/D\lambda)|_{\lambda=0}^{-1}$ and find, since the ratio (19.242) is $\Lambda$-
Thus the mean-value in the vacuum of a time-ordered product of gauge-invariant operators is a ratio of path integrals over all gauge fields without any gauge fixing. No matter what gauge condition $G$ or gauge-fixing functional $B[G]$ we use, the resulting gauge-fixed ratio (19.240) is equal to the ratio (19.244) of path integrals over all gauge fields without any gauge fixing. All gauge-fixed ratios (19.240) give the same time-ordered products, and so we can use whatever gauge condition $G$ or gauge-fixing functional $B[G]$ is most convenient.

The analogous formula for the euclidian time-ordered product is

$$\langle \Omega | T [O_1 \ldots O_n] | \Omega \rangle = \frac{\int O_1 \ldots O_n e^{iS} B[G^\Lambda] D\Lambda' D\Lambda D\psi}{\int e^{iS} B[G^\Lambda] D\Lambda' D\Lambda D\psi}$$

where the euclidian action $S_e$ is the spacetime integral of the energy density. This formula is the basis for lattice gauge theory.

The path-integral formulas (19.171 & 19.172) derived for quantum electrodynamics therefore also apply to nonabelian gauge theories.
Path Integrals

to the gauge-fixed expression (19.240) for the time-ordered product
\[ \langle \Omega | \mathcal{T} [O_1 \ldots O_n] | \Omega \rangle = \frac{\int \mathcal{O}_1 \ldots \mathcal{O}_n \, e^{iS} \, B[G] \, J \, DA \, D\psi}{\int e^{iS} \, B[G] \, J \, DA \, D\psi} \quad (19.246) \]

set \( G_b(x) = -i \partial_{\mu} A_b^\mu(x) \) and use (19.239) as the gauge-fixing functional \( B[G] \)
\[ B[G] = \exp \left[ \frac{i}{2} \int (G_a(x))^2 \, d^4x \right] = \exp \left[ -\frac{i}{2} \int (\partial_{\mu} A_a^\mu(x))^2 \, d^4x \right]. \quad (19.247) \]

This functional adds to the action density the term \(- (\partial_{\mu} A_a^\mu)^2/2\) which leads to a gauge-field propagator like the photon's (19.175)
\[ \langle 0 | \mathcal{T} \left[ A_\mu^a(x) A_\nu^b(y) \right] | 0 \rangle = -i \delta_{ab} \delta_{\mu\nu} (x - y) = -i \int \frac{\eta_{\mu\nu} \delta_{ab}}{q^2 - i\epsilon} \quad e^{iq(x - y)} \, \frac{d^4q}{(2\pi)^4}. \quad (19.248) \]

What about the determinant \( J \)? Under an infinitesimal gauge transformation (19.236), the gauge field becomes
\[ A^{\lambda}_{a\mu} = A_{a\mu} - \partial_{\mu} \lambda_a - g f_{abc} A_{b\mu} \lambda_c \quad (19.249) \]
and so \( G^{\lambda}_a(x) = i \partial^{\mu} A^{\lambda}_{a\mu}(x) \) is
\[ G^{\lambda}_a(x) = i \partial^{\mu} A_{a\mu}(x) + i \partial^{\mu} \int [-\delta_{ac} \partial_{\mu} - g f_{abc} A_{b\mu}(x)] \delta^4(x - y) \lambda_c(y) d^4y. \quad (19.250) \]

The jacobian \( J \) then is the determinant (19.237) of the matrix
\[ \left( \frac{\delta G^{\lambda}_a(x)}{\delta \lambda_c(y)} \right) \bigg|_{\lambda=0} = -i \delta_{ac} \Box \delta^4(x - y) - ig f_{abc} \frac{\partial}{\partial x^\mu} \left[ A_b^\mu(x) \delta^4(x - y) \right] \quad (19.251) \]
that is
\[ J = \det \left( -i \delta_{ac} \Box \delta^4(x - y) - ig f_{abc} \frac{\partial}{\partial x^\mu} \left[ A_b^\mu(x) \delta^4(x - y) \right] \right). \quad (19.252) \]

But we’ve seen (19.205) that a determinant can be written as a fermionic path integral
\[ \det A = \int e^{-\theta^i A^i} \prod_{k=1}^n \, d\theta^*_k \, d\theta_k. \quad (19.253) \]

So we can write the jacobian \( J \) as
\[ J = \int \exp \left[ \int i \omega^*_a \Box \omega_a + ig f_{abc} \omega^*_a \partial_{\mu} (A_b^\mu \omega_c) \, d^4x \right] \, D\omega^* \, D\omega \quad (19.254) \]
19.19 Integrating over the momenta

which contributes the terms $-\partial_\mu \omega_a^* \partial^\mu \omega_a$ and

$$-\partial_\mu \omega_a^* g f_{abc} A^a_\mu \omega_c = \partial_\mu \omega_a^* g f_{abc} A^a_\mu \omega_b$$  \hspace{1cm} (19.255)

to the action density.

Thus we can do perturbation theory by using the modified action density

$$\mathcal{L}' = -\frac{1}{4} F_{\mu\nu} F^\mu_\nu - \frac{1}{2} (\partial_\mu A_\nu)^2 - \partial_\mu \omega_a^* \partial^\mu \omega_a + \partial_\mu \omega_a^* g f_{abc} A^a_\mu \omega_b - \bar{\psi} (\mathcal{D} + m) \psi$$  \hspace{1cm} (19.256)

in which $\mathcal{D} \equiv \gamma^\mu D_\mu = \gamma^\mu (\partial_\mu - i g t^a A_\mu a)$. The ghost field $\omega$ is a mathematical device, not a physical field describing real particles, which would be spinless fermions violating the spin-statistics theorem (example 10.23).

19.19 Integrating over the momenta

When the hamiltonian is quadratic in the momenta like (19.22) and (19.97), one easily integrates over the momenta and converts the hamiltonian into the lagrangian. It may happen, however, that the hamiltonian is so complicated a function of the momenta that one can’t integrate over the momenta. In such cases, the partition function for a scalar field $\phi(x)$ is a path integral over both $\phi$ and $\pi$

$$Z(\beta) = \int \exp \left\{ \int_0^\beta \int [i \dot{\phi}(x) \pi(x) - H(\phi, \pi)] \, dt \, d^3x \right\} D\phi D\pi. \hspace{1cm} (19.257)$$

For some values of $\phi$, the exponential is not positive, and so is not a probability distribution for $\phi$ and $\pi$. The Monte Carlo methods of chapter 15 are designed to work with probability distributions, not with weight functions that assume values that are negative or complex. This is one aspect of the sign problem.

The integral over the momenta

$$P[\phi] = \int \exp \left\{ \int_0^\beta \int [i \dot{\phi}(x) \pi(x) - H(\phi, \pi)] \, dt \, d^3x \right\} D\pi \hspace{1cm} (19.258)$$

is a probability distribution. So one can numerically integrate over the momenta, make a look-up table for $P[\phi]$, and then apply the usual Monte Carlo method to the probability functional $P[\phi]$ (Amdahl and Cahill, 2016).

Further reading

Quantum Field Theory (Srednicki, 2007), The Quantum Theory of Fields
I, II, & III (Weinberg, 1995, 1996, 2005), and Quantum Field Theory in a Nutshell (Zee, 2010) all provide excellent treatments of path integrals.

**Exercises**

19.1 Derive the multiple gaussian integral (19.8) from (5.175).
19.2 Derive the multiple gaussian integral (19.12) from (5.174).
19.3 Show that the vector $Y$ that makes the argument of the multiple gaussian integral (19.12) stationary is given by (19.13), and that the multiple gaussian integral (19.12) is equal to its exponential evaluated at its stationary point $Y$ apart from a prefactor involving $\det iS$. 
19.4 Repeat the previous exercise for the multiple gaussian integral (19.11).
19.5 Compute the double integral (19.29) for the case $V(q, j) = 0$. 
19.6 Insert into the LHS of (19.53) a complete set of momentum dyadics $|pih, p|$, use the inner product $\langle q|p \rangle = \exp(iqp/\hbar)/\sqrt{2\pi\hbar}$, do the resulting Fourier transform, and so verify the free-particle path integral (19.53). 
19.7 By taking the nonrelativistic limit of the formula (12.283) for the action of a relativistic particle of mass $m$ and charge $q$, derive the expression (19.54) for the action a nonrelativistic particle in an electromagnetic field with no scalar potential. 
19.8 Show that for the hamiltonian (19.58) of the simple harmonic oscillator the action $S[q_c]$ of the classical path is (19.65). 
19.9 Show that the harmonic-oscillator action of the loop (19.66) is (19.67).
19.10 Show that the harmonic-oscillator amplitude (19.70) for $q' = 0$ and $q'' = q$ reduces as $t \to 0$ to the one-dimensional version of the free-particle amplitude (19.53).
19.11 Work out the path-integral formula for the amplitude for a mass $m$ initially at rest to fall to the ground from height $h$ in a gravitational field of local acceleration $g$ to lowest order and then including loops up to an overall constant. Hint: use the technique of section 19.8.
19.12 Show that the action (19.72) of the stationary solution (19.75) is (19.77).
19.14 Derive identity (19.134). Split the time integral at $t = 0$ into two halves, use

$$\epsilon e^{\pm \epsilon t} = \pm \frac{d}{dt} e^{\pm \epsilon t}$$

(19.259)

and then integrate each half by parts.
19.15 Derive the third term in equation (19.136) from the second term.
19.16 Use (19.141) and the Fourier transform (19.142) of the external current $j$ to derive the formula (19.143) for the modified action $S_0[\phi, \epsilon, j]$.  
19.17 Derive equation (19.145) from equations (19.143) and (19.144).  
19.18 Derive the formula (19.146) for $Z_0[j]$ from the expression (19.145) for $S_0[\phi, \epsilon, j]$.  
19.20 Derive equation (19.152) from the formula (19.147) for $Z_0[j]$.  
19.21 Show that the time integral of the Coulomb term (19.157) is the term that is quadratic in $j^0$ in the number $F$ defined by (19.162).  
19.22 By following steps analogous to those the led to (19.148), derive the formula (19.175) for the photon propagator in Feynman’s gauge.  
19.23 Derive expression (19.190) for the inner product $\langle \zeta | \theta \rangle$.  
19.25 Derive the eigenvalue equation (19.198) from the definition (19.196 & 19.197) of the eigenstate $|\theta \rangle$ and the anticommutation relations (19.194 & 19.195).  
19.26 Derive the eigenvalue relation (19.211) for the Fermi field $\psi_m(\mathbf{x}, t)$ from the anticommutation relations (19.207 & 19.208) and the definitions (19.209 & 19.210).  
19.27 Derive the formula (19.212) for the inner product $\langle \chi' | \chi \rangle$ from the definition (19.210) of the ket $|\chi \rangle$.  
19.28 Without setting $\hbar = 1$, imitate the derivation of the path-integral formula (19.31) and derive its three-dimensional version (19.32).