A Hyperboloid

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The points of a cylindrical hyperboloid in 3-space obey the equation

$$z^2 = x^2 + y^2 - r^2 \quad \text{or} \quad r^2 = x^2 + y^2 - z^2. \tag{1}$$

A point on it is

$$p = r(\cosh\theta\cos\phi, \cosh\theta\sin\phi, \sinh\theta). \tag{2}$$

The (orthogonal) coordinate basis vectors are

$$\boldsymbol{e}_\theta = p_{,\theta} = r(\sinh\theta\cos\phi, \sinh\theta\sin\phi, \cosh\theta) \tag{3}$$

$$\boldsymbol{e}_\phi = p_{,\phi} = r(-\cosh\theta\sin\phi, \cosh\theta\cos\phi, 0). \tag{4}$$

The metric is $g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$ is

$$(g_{ij}) = r^2 \begin{pmatrix} \cosh^2\theta + \sinh^2\theta & 0 \\ 0 & \cosh^2\theta \end{pmatrix}. \tag{5}$$

Its inverse is

$$(g^{ij}) = r^{-2} \begin{pmatrix} 1/(\cosh^2\theta + \sinh^2\theta) & 0 \\ 0 & \cosh^{-2}\theta \end{pmatrix}. \tag{6}$$

The derivatives of the basis vectors are

$$\boldsymbol{e}_{\theta,\theta} = r(\cosh\theta\cos\phi, \cosh\theta\sin\phi, \sinh\theta) \tag{7}$$

$$\boldsymbol{e}_{\theta,\phi} = r(-\sinh\theta\sin\phi, \sinh\theta\cos\phi, 0) \tag{8}$$

$$\boldsymbol{e}_{\phi,\theta} = r(-\sinh\theta\sin\phi, \sinh\theta\cos\phi, 0) = \boldsymbol{e}_{\theta,\phi} \tag{9}$$

$$\boldsymbol{e}_{\phi,\phi} = r(-\cosh\theta\cos\phi, -\cosh\theta\sin\phi, 0). \tag{10}$$

The dual basis vectors $e^i = g^{ij} e_j$ are

$$\boldsymbol{e}^\theta = \frac{(\sinh\theta\cos\phi, \sinh\theta\sin\phi, \cosh\theta)}{r(\cosh^2\theta + \sinh^2\theta)} \tag{11}$$

$$\boldsymbol{e}^\phi = \frac{(-\cosh\theta\sin\phi, \cosh\theta\cos\phi, 0)}{r\cosh^2\theta}. \tag{12}$$

The six connections

$$\Gamma^i_{jk} = \Gamma^i_{kj} = \boldsymbol{e}^i \cdot \boldsymbol{e}_{j,k} \tag{13}$$

are

$$\Gamma^\theta_{\theta\theta} = \boldsymbol{e}^\theta \cdot \boldsymbol{e}_{\theta,\theta} = \frac{2\cosh\theta\sinh\theta}{\cosh^2\theta + \sinh^2\theta} \tag{14}$$

$$\Gamma^\theta_{\theta\phi} = \boldsymbol{e}^\theta \cdot \boldsymbol{e}_{\theta,\phi} = 0 \tag{15}$$

$$\Gamma^\theta_{\phi\phi} = \boldsymbol{e}^\theta \cdot \boldsymbol{e}_{\phi,\phi} = -\frac{\sinh\theta\cosh\theta}{\cosh^2\theta + \sinh^2\theta} \tag{16}$$

$$\Gamma^\phi_{\theta\theta} = \boldsymbol{e}^\phi \cdot \boldsymbol{e}_{\theta,\theta} = 0 \tag{17}$$

$$\Gamma^\phi_{\theta\phi} = \boldsymbol{e}^\phi \cdot \boldsymbol{e}_{\theta,\phi} = \tanh\theta \tag{18}$$

$$\Gamma^\phi_{\phi\phi} = \boldsymbol{e}^\phi \cdot \boldsymbol{e}_{\phi,\phi} = 0. \tag{19}$$

The two Christoffel matrices are

$$\Gamma_\theta = \begin{pmatrix} \Gamma^\theta_{\theta\theta} & 0 \\ 0 & \Gamma^\phi_{\theta\phi} \end{pmatrix} = \tanh\theta \begin{pmatrix} 2/(1+\tanh^2\theta) & 0 \\ 0 & 1 \end{pmatrix} \tag{20}$$

and

$$\Gamma_\phi = \begin{pmatrix} 0 & \Gamma^\theta_{\phi\phi} \\ \Gamma^\phi_{\phi\theta} & 0 \end{pmatrix} = \tanh\theta \begin{pmatrix} 0 & -1/(1+\tanh^2\theta) \\ 1 & 0 \end{pmatrix}. \tag{21}$$

Their commutator is

$$[\Gamma_\phi, \Gamma_\theta] = \tanh^2\theta \begin{pmatrix} 0 & 2(1+\tanh^2\theta)^{-2} - (1+\tanh^2\theta)^{-1} \\ 2(1+\tanh^2\theta)^{-1} - 1 & 0 \end{pmatrix} \tag{22}$$

Riemann's curvature tensor is

$$\begin{aligned} R^i_{mnk} &= (R_{nk})^i{}_m = [\partial_k + \Gamma_k, \partial_n + \Gamma_n]^i{}_m \\ &= (\Gamma_{n,k} - \Gamma_{k,n} + \Gamma_k \Gamma_n - \Gamma_n \Gamma_k)^i{}_m . \end{aligned} \tag{23}$$

The nonzero derivatives of the connections are

$$\Gamma_{\theta,\theta} = \begin{pmatrix} 2\operatorname{sech}^2(2\theta) & 0 \\ 0 & \operatorname{sech}^2\theta \end{pmatrix} \tag{24}$$

and

$$\Gamma_{\phi,\theta} = \begin{pmatrix} 0 & -\operatorname{sech}^2(2\theta) \\ \operatorname{sech}^2\theta & 0 \end{pmatrix}. \tag{25}$$

Since $R^i_{m\theta\theta} = R^i_{m\phi\phi} = 0$, and $R^\theta_{\theta\theta\phi} = R^\phi_{\phi\theta\phi} = 0$, the nonzero terms of the curvature tensor are

$$\begin{aligned} R^\phi_{\theta\theta\phi} &= [\partial_\phi + \Gamma_\phi, \partial_\theta + \Gamma_\theta]^\phi{}_\theta = -\left(\Gamma_{\phi,\theta}\right)^\phi{}_\theta + [\Gamma_\phi, \Gamma_\theta]^\phi{}_\theta \\ &= -\operatorname{sech}^2\theta + \tanh^2\theta\left(2(1+\tanh^2\theta)^{-1} - 1\right) = -\operatorname{sech}(2\theta) \\ R^\theta_{\phi\theta\phi} &= [\partial_\phi + \Gamma_\phi, \partial_\theta + \Gamma_\theta]^\theta{}_\phi = -\left(\Gamma_{\phi,\theta}\right)^\theta{}_\phi + [\Gamma_\phi, \Gamma_\theta]^\theta{}_\phi \\ &= \operatorname{sech}^2(2\theta) - \tanh^2\theta\left((1+\tanh^2\theta)^{-1} - 2(1+\tanh^2\theta)^{-2}\right) \\ &= \frac{1}{2}\left(\operatorname{sech}(2\theta) + \operatorname{sech}^2(2\theta)\right). \end{aligned} \tag{26}$$

The Ricci tensor is the contraction $R_{mk} = R^n_{mnk}$, and so

$$\begin{aligned} R_{\theta\theta} &= R^\theta_{\theta\theta\theta} + R^\phi_{\theta\phi\theta} = \operatorname{sech}(2\theta) \\ R_{\phi\phi} &= R^\theta_{\phi\theta\phi} + R^\phi_{\phi\phi\phi} = \frac{1}{2}\left(\operatorname{sech}(2\theta) + \operatorname{sech}^2(2\theta)\right). \end{aligned} \tag{27}$$

The curvature scalar is the contraction $R = g^{km} R_{mk}$, and so since $g^{\theta\theta} = r^{-2}/(\cosh^2\theta + \sinh^2\theta)$ and $g^{\phi\phi} = r^{-2}/\cosh^2\theta$, it is

$$\begin{aligned} R &= g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{1}{r^2}\left[\frac{\operatorname{sech}(2\theta)}{\cosh^2\theta + \sinh^2\theta} + \frac{\operatorname{sech}(2\theta) + \operatorname{sech}^2(2\theta)}{2\cosh^2\theta}\right] \\ &= \frac{2\operatorname{sech}^2(2\theta)}{r^2}. \end{aligned} \tag{28}$$