19
Strings

19.1 The Infinities of Quantum Field Theory

Quantum field theory is plagued with infinities. Even exactly soluble theories of free fields have ground-state energies that diverge quarticly, as $+\Lambda^4$ for a theory of bosons and as $-\Lambda^4$ for a theory of fermions as $\Lambda \to \infty$. Theories with the same number of boson and fermion fields are less divergent, and theories with unbroken supersymmetry actually have ground-state energies that vanish.

One may be able to eliminate some of the infinities of a generic quantum field theory without spoiling the symmetries of its action density $\mathcal{L}$ by replacing $\mathcal{L}$ in the path integral by $\mathcal{L}' = \mathcal{L}/(1-\mathcal{L}/M^4)$ or $\mathcal{L}' = M^4 \left(\exp(\mathcal{L}/M^4) - 1\right)$. In euclidian space, the substitution

$$\mathcal{L}_e' = \frac{\mathcal{L}_e}{1 - \mathcal{L}_e/M^4} \tag{19.1}$$

with the understanding that $\mathcal{L}_e' = \infty$ when $|\mathcal{L}_e| > M^4$ almost certainly removes many infinities, but whether it preserves all the symmetries is less obvious. These substitutions change quantum field theory, but in the limit $M \to \infty$, one recovers standard quantum field theory.

A more physical approach is to represent elementary particles as objects that have finite size. Those that are one-dimensional are called **strings**.

19.2 The Nambu-Goto String Action

If we give up the idea of point particle, then the next simplest choice is a one-dimensional string. We'll use $0 \le \sigma \le \sigma_1$ and $\tau_i \le \tau \le \tau_f$ to parametrize the spacetime coordinates $X^\mu(\sigma, \tau)$ of the string. Nambu and Goto suggested using as the action the area

$$S = -\frac{T_0}{c} \int_{\tau_i}^{\tau_f} \int_0^{\sigma_1} \sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2} \, d\tau \, d\sigma \tag{19.2}$$

in which

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} \quad \text{and} \quad X'^\mu = \frac{\partial X^\mu}{\partial \sigma} \tag{19.3}$$

and a Lorentz metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, \dots)$ is used to form the inner products

$$\dot{X} \cdot X' = \dot{X}^\mu \eta_{\mu\nu} X^{\nu\prime} \quad \textit{etc.} \tag{19.4}$$

This action is the area swept out by a string of length σ_1 in time $\tau_f - \tau_i$.

If $\dot{X} d\tau = dt$ points in the time direction and $X' d\sigma = d\boldsymbol{r}$ points in a spatial direction, then it is easy to see that $\dot{X} \cdot X' = 0$, that $-\left(\dot{X}\right)^2 d\tau^2 = dt^2$, and that $(X')^2 \, d\sigma^2 = d\boldsymbol{r}^2$. So in this simple case, the action (19.2) is

$$S = -\frac{T_0}{c} \int_{t_i}^{t_f} \int_0^{r_1} dt \, dr = -\frac{T_0}{c} (t_f - t_i) r_1 \tag{19.5}$$

which is the area the string sweeps out. The other term within the square root ensures that the action is the area swept out for all $\dot{X}$ and X', and that it is invariant under arbitrary reparametrizations $\sigma \to \sigma'$ and $\tau \to \tau'$.

The equation of motion for the relativistic string follows from the requirement that the action (19.2) be stationary, $\delta S = 0$. Since

$$\delta \dot{X}^\mu = \delta \frac{\partial X^\mu}{\partial \tau} = \frac{\partial (X^\mu + \delta X^\mu)}{\partial \tau} - \frac{\partial X^\mu}{\partial \tau} = \frac{\partial \delta X^\mu}{\partial \tau} \tag{19.6}$$

and similarly

$$\delta X'^\mu = \frac{\partial \delta X^\mu}{\partial \sigma}, \tag{19.7}$$

we may express the change in the action in terms of derivatives of the Lagrange density

$$L = -\frac{T_0}{c} \sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2}. \tag{19.8}$$

as

$$\delta S = \int_{\tau_i}^{\tau_f} \int_0^{\sigma_1} \left[\frac{\partial L}{\partial \dot{X}^\mu} \frac{\partial \delta X^\mu}{\partial \tau} + \frac{\partial L}{\partial X'^\mu} \frac{\partial \delta X^\mu}{\partial \sigma} \right] d\tau \, d\sigma. \tag{19.9}$$

Its derivatives, which we'll call $\mathcal{P}^\tau_\mu$ and $\mathcal{P}^\sigma_\mu$, are

$$\mathcal{P}^\tau_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2}} \tag{19.10}$$

and

$$\mathcal{P}^\sigma_\mu = \frac{\partial L}{\partial X'^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2}}. \tag{19.11}$$

In terms of them, the change in the action is

$$\delta S = \int_{\tau_i}^{\tau_f}\!\!\int_0^{\sigma_1} \left[\frac{\partial}{\partial \tau}\left(\delta X^\mu \mathcal{P}^\tau_\mu\right) + \frac{\partial}{\partial \sigma}\left(\delta X^\mu \mathcal{P}^\sigma_\mu\right) - \delta X^\mu \left(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma}\right)\right] d\tau \, d\sigma. \tag{19.12}$$

The total τ-derivative integrates to a term involving the variation δX^μ which we make vanish at the initial and final values of τ. So we drop that term and find that the net change in the action is

$$\delta S = \int_{\tau_i}^{\tau_f} \left[\delta X^\mu \mathcal{P}^\sigma_\mu\right]_0^{\sigma_1} d\tau - \int_{\tau_i}^{\tau_f}\!\!\int_0^{\sigma_1} \delta X^\mu \left(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma}\right) d\tau \, d\sigma. \tag{19.13}$$

Thus the equations of motion for the string are

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0, \tag{19.14}$$

but the action is stationary only if

$$\int \delta X^\mu(\tau, \sigma_1)\mathcal{P}^\sigma_\mu(\tau, \sigma_1) - \delta X^\mu(\tau, 0)\mathcal{P}^\sigma_\mu(\tau, 0) \; d\tau = 0. \tag{19.15}$$

Closed strings automatically satisfy this condition. Open strings satisfy it if they obey for each end σ_* of the string, each spacetime dimension μ, and all times τ either the **free-endpoint** boundary condition

$$\mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0 \tag{19.16}$$

or the **Dirichlet** boundary condition

$$\delta X^\mu(\tau, \sigma_*) = 0. \tag{19.17}$$

But since $\delta X^0 \propto \delta\tau \neq 0$, only the free-endpoint condition $\mathcal{P}^\sigma_0(\tau, \sigma_*) = 0$ works for the time component for open strings. A **Dn-brane** is a space of n spatial dimensions to which an end $X^j(\tau, \sigma_*)$ of an open string is restricted but through which it can move.

Because the action density L is a homogeneous function (section 6.13) of degree 1 of the time (and space) derivatives of the fields X^μ, Euler's theorem (6.141) implies that the classical energy density (6.219) of the Nambu-Goto field vanishes independently of the equations of motion

$$E = \dot{X}^\mu \frac{\partial L}{\dot{X}^\mu} - L = 0. \tag{19.18}$$

19.3 Regge Trajectories

The quantity $\mathcal{P}^\tau_\mu(\tau, \sigma)$ defined as the derivative (19.10) is the momentum density of the string. The angular momentum M_{12} of a string rigidly rotating in the $1, 2$ plane is

$$M_{12}(\tau) = \int_0^{\sigma_1} X_1 \mathcal{P}^\tau_2(\tau, \sigma) - X_2 \mathcal{P}^\tau_1(\tau, \sigma)\, d\sigma. \tag{19.19}$$

In **static gauge**, $\tau = t = X^0(\tau, \sigma)/c$, and the σ and τ derivatives of $X = (X^0, \vec{X})$ are $X' = (0, \vec{X}')$ and $\dot{X} = (c, \dot{\vec{X}})$. In this gauge, the endpoints of an open string that is not attached to any brane have $\mathcal{P}^\sigma_\mu(t, \sigma_*) = 0$, and so move at the speed of light and transversely to the string. We choose σ so that the motion of every point on the string is transverse to the string, $\vec{X}' \cdot \dot{\vec{X}} = 0$, which in static gauge implies that $\dot{X} \cdot X' = 0$, and also so that

$$d\sigma = \frac{ds}{\sqrt{1 - v_\perp^2/c^2}} = \frac{dE}{T_0} \tag{19.20}$$

in which $v_\perp$ is the transverse velocity and E is the energy of the string (not that of its field). In this parametrization, an open unattached string obeys the wave equation

$$\ddot{\vec{X}} = c^2 \vec{X}'' \tag{19.21}$$

and the constraint

$$c^2 \left(\vec{X}'\right)^2 + \left(\dot{\vec{X}}\right)^2 = c^2, \tag{19.22}$$

and the free-endpoint boundary condition (19.16) is simply $\vec{X}' = 0$.

In particular, a string rotating in the 1-2 plane has coordinates

$$\vec{X}(t, \sigma) = \frac{\sigma_1}{\pi} \cos\frac{\pi\sigma}{\sigma_1} \left(\cos\frac{\pi c t}{\sigma_1},\ \sin\frac{\pi c t}{\sigma_1}\right), \tag{19.23}$$

with $X^k = 0$ for $k > 2$, and momentum densities

$$\vec{\mathcal{P}}^\tau(t,\sigma) = \frac{T_0}{c}\frac{\partial \vec{X}}{\partial t} = \frac{T_0}{c}\cos\frac{\pi\sigma}{\sigma_1}\left(-\sin\frac{\pi ct}{\sigma_1},\ \cos\frac{\pi ct}{\sigma_1}\right). \tag{19.24}$$

Its angular momentum (19.19) is

$$M_{12} = \frac{\sigma_1}{\pi}\frac{T_0}{c}\int_0^{\sigma_1}\cos^2\frac{\pi\sigma}{\sigma_1}d\sigma = \frac{\sigma_1^2\,T_0}{2\pi c}. \tag{19.25}$$

In our parametrization, the energy of the string is $E = \sigma_1 T_0$. Thus the angular momentum $J = |M_{12}|$ of a classical relativistic string is proportional to the square of its total energy

$$J = \frac{E^2}{2\pi c T_0}. \tag{19.26}$$

This rule is obeyed by many meson and baryon resonances. The nucleon and five baryon resonances fit it with nearly the same value of the string tension

$$T_0 \approx 0.92 \quad \text{GeV/fm} \tag{19.27}$$

as shown by Figs. 19.1, which displays the **Regge trajectories** of six N and Δ resonances on a single curve.

The energy E and angular momentum J of a Kerr black hole obey a relation

$$J = \frac{E^2}{c^5/(aG)} = \frac{E^2}{2\pi c T} \tag{19.28}$$

like that (19.26) of the baryon resonances with string tension

$$T = \frac{c^4}{2\pi a G} \tag{19.29}$$

in which G is Newton's constant. This string tension is higher by at least 37 orders of magnitude since the dimensionless spin parameter a must be less than unity and $\hbar c/G = 1.4906 \times 10^{38}\,\text{GeV}^2/\hbar c$.

A string theory of hadrons took off in 1968 when Gabriel Veneziano published his amplitude for $\pi + \pi$ scattering as a sum of three Euler beta functions (Veneziano, 1968). But after eight years of intense work, this effort was largely abandoned with the discovery of quarks at SLAC and the promise of QCD as a theory of the strong interactions. In 1974, Joël Scherk and John H. Schwarz proposed increasing the string tension by 38 orders of magnitude so as to use strings to make a quantum theory that included gravity (Scherk and Schwarz, 1974). They identified the graviton as an excitation of the closed string.

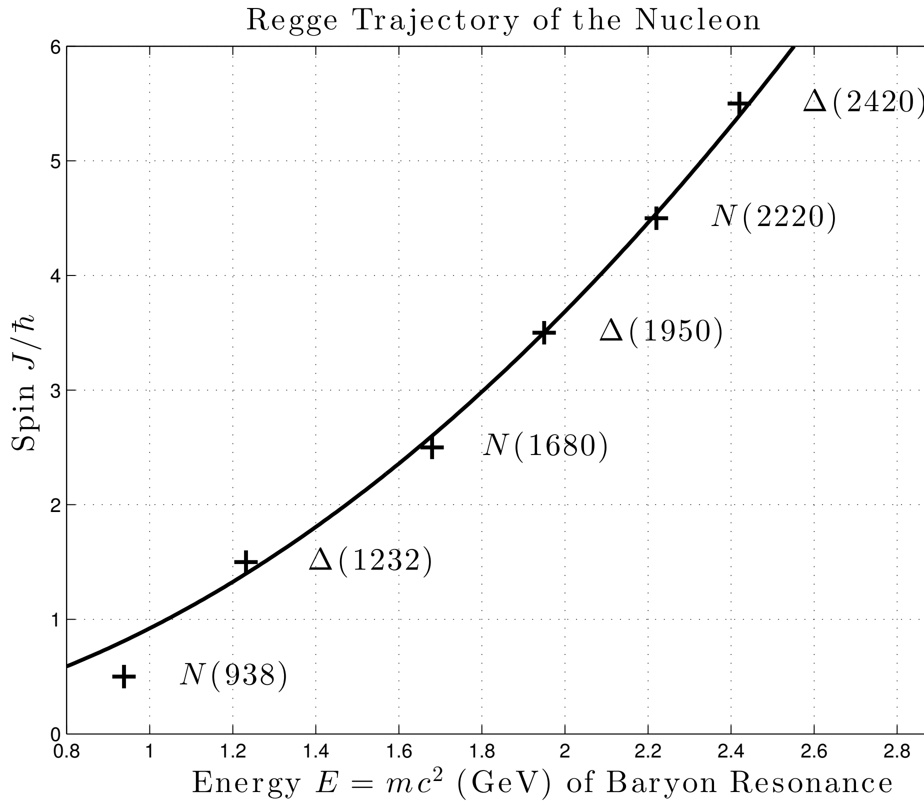

Figure 19.1 The angular momentum and energy of the nucleon and five baryon resonances approximately fit the curve $J/\hbar = T_0\, E^2$ with string tension $T_0 = 0.92$ GeV/fm.

19.4 Light-cone coordinates

In Dirac's light-cone coordinates x^+, x^-, x^2, x^3, the new coordinates x^+ and x^- are

$$x^+ = \frac{1}{\sqrt{2}}\left(x^0 + x^1\right) \quad \text{and} \quad x^- = \frac{1}{\sqrt{2}}\left(x^0 - x^1\right) \tag{19.30}$$

in which x^1 is one of the three spatial coordinates, x^1, x^2, and x^3. One may choose either x^+ or x^- as the light-cone time coordinate. The convention is to choose x^+ as the light-cone variable corresponding to the time x^0. The light-cone components of the momentum 4-vector are

$$p^+ = \frac{1}{\sqrt{2}}\left(p^0 + p^1\right) \quad \text{and} \quad p^- = \frac{1}{\sqrt{2}}\left(p^0 - p^1\right). \tag{19.31}$$

The (specially) invariant squared distance ds^2 is

$$ds^2 = \boldsymbol{dx}\cdot\boldsymbol{dx} - (dx^0)^2 = (dx^2)^2 + (dx^3)^2 - dx^- dx^+ - dx^+ dx^-, \tag{19.32}$$

and similarly the invariant $p \cdot x$ is

$$p \cdot x = \boldsymbol{p}\cdot\boldsymbol{x} - p^0 x^0 = p^2 x^2 + p^3 x^3 - p^- x^+ - p^+ x^-. \tag{19.33}$$

Just as in quantum mechanics we identify $i\hbar\,\partial_t$ with the energy E so that

$$i\hbar\,\frac{\partial}{\partial x^0}\, e^{i(\boldsymbol{p}\cdot\boldsymbol{x}-Et)/\hbar} = \frac{E}{c}\, e^{i(\boldsymbol{p}\cdot\boldsymbol{x}-Et)/\hbar}, \tag{19.34}$$

so too in light-cone coordinates, we have

$$i\hbar\,\frac{\partial}{\partial x^+} e^{i(p^2x^2+p^3x^3-p^-x^+-p^+x^-)/\hbar} = p^-\, e^{i(p^2x^2+p^3x^3-p^-x^+-p^+x^-)/\hbar}. \tag{19.35}$$

So p^- is the light-cone version of E/c.

19.5 Light-cone gauge

In very symmetrical theories, one often imposes a gauge condition to simplify the equations of motion and the quantization of fields. The Coulomb gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ is an example in quantum electrodynamics. The complicated equations (19.2 –19.17) that follow from the Nambu-Goto action simplify when one requires the string parameters σ and τ to obey the gauge conditions

$$n \cdot X = \beta\alpha'(n\cdot p)\tau \ \text{ and } \ n\cdot p = \frac{2\pi}{\beta} n \cdot \mathcal{P}^\tau \tag{19.36}$$

where n is a unit vector, $\beta = 2$ for open strings, and $\beta = 1$ for closed strings (Zwiebach, 2009, chapter 9). These conditions imply the constraints

$$\left(\dot{X} \pm X'\right)^2 = 0, \tag{19.37}$$

the momentum densities

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} X^{\mu\prime} \ \text{ and } \ \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu, \tag{19.38}$$

and the wave equations

$$\ddot{X}^\mu - X^{\mu\prime\prime} = 0 \tag{19.39}$$

where now $\sigma \in [0, \pi]$ and τ are dimensionless and the string tension is $T_0 = 1/(2\pi\alpha')$.

Open strings possess the expansion

$$X^\mu(\tau, \sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\,\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}\alpha_n^\mu \cos n\sigma. \tag{19.40}$$

Closed strings have left- and right-moving waves indicated by barred and unbarred amplitudes; their expansion is

$$X^\mu(\tau, \sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\,\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}\left(\alpha_n^\mu e^{in\sigma} + \bar{\alpha}_n^\mu e^{-in\sigma}\right). \tag{19.41}$$

Among this class of gauges (19.36) is the light-cone gauge

$$X^+ = \frac{X^0 + X^1}{\sqrt{2}} = \beta\alpha' p^+ \tau \quad \text{and} \quad p^+ = \frac{p^0 + p^1}{\sqrt{2}} = \frac{2\pi}{\beta}\,\mathcal{P}^{\tau +} \tag{19.42}$$

in which $X^{+\prime} = 0$ and $\dot{X}^+ = \beta\alpha' p^+$. In this gauge, the action is a sum over the transverse coordinates $J = 2, 3, \ldots d$

$$S = \frac{1}{4\pi\alpha'}\int d\tau \int_0^\pi d\sigma \left(\dot{X}^J\dot{X}^J - X'^J X'^J\right). \tag{19.43}$$

The coefficients of open strings satisfy

$$x_0^+ = \alpha_n^+ = 0 \quad \text{for} \quad n \neq 0 \tag{19.44}$$

and

$$\sqrt{2\alpha'}\,\alpha_n^- = \frac{1}{2p^+}\sum_{p=-\infty}^{\infty}\alpha_{n-p}^J\alpha_p^J \equiv \frac{1}{2p^+}L_n^\perp \tag{19.45}$$

in which $L_n^\perp$ is a transverse Virasoro mode. The string now is expressed entirely in terms of its transverse modes α_n, and classically its mass squared is

$$M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty}|\alpha_n^\ell|^2. \tag{19.46}$$

19.6 Quantized strings

In the light-cone gauge, the transverse coordinates X^J of open strings obey the equal-time commutation relations

$$[X^I(\tau, \sigma), \mathcal{P}^{\tau J}(\tau, \sigma')] = i\eta^{IJ}\,\delta(\sigma - \sigma') \quad \text{and} \quad [x_0^-(\tau), p^+(\tau)] = -i, \tag{19.47}$$

while the transverse coordinates $X^J(\tau,\sigma)$ commute at equal times, as do the transverse momenta $\mathcal{P}^{\tau J}(\tau,\sigma)$. The amplitude operators α_n^J obey the commutation relations

$$[\alpha_m^I, \alpha_n^J] = m\,\eta^{IJ}\delta_{m+n,0} \quad \text{and} \quad [x_0^I, p^J] = i\eta^{IJ}. \tag{19.48}$$

Lorentz invariance implies that the number of spacetime dimensions is $D = 26$ and that the hamiltonian is

$$H = L_0^\perp - 1. \tag{19.49}$$

The theory has a tachyon, that is, a particle that moves faster than light.

String theory is a quantum field theory in two dimensions τ and σ. Like other quantum field theories, it has divergences which string theorists interpret by using Ser's series expansion (4.93 & 5.105) of Riemann's zeta function to say that

$$\begin{aligned}\sum_{n=1}^{\infty} n = \zeta(-1) &= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k(k+1)^2\binom{n}{k}\\ &= -\frac{1}{2}\sum_{n=0}^{2}\frac{1}{n+1}\frac{d}{dx}\left\{x\frac{d}{dx}\left[x(1-x)^n\right]\right\}\Bigg|_{x=1}\\ &= -\frac{1}{2}\left(1-\frac{3}{2}+\frac{2}{3}\right) = -\frac{1}{12}.\end{aligned} \tag{19.50}$$

The relation $1 = -\frac{1}{2}(D-2)\sum_n n$, then says that open bosonic strings make sense in $D = 26$ dimensions.

19.7 Superstrings

In light-cone gauge, one can add fermionic variables $\psi_1^\mu(\tau,\sigma)$ and $\psi_2^\mu(\tau,\sigma)$ to the action in a supersymmetric way

$$\begin{aligned}S = \frac{1}{4\pi\alpha'}\int d\tau \int_0^\pi d\sigma\left(\dot{X}^J\dot{X}^J - X'^J X'^J\right)\\ + \frac{1}{4\pi\alpha'}\int d\tau\int_0^\pi d\sigma\left[\psi_1^J(\partial_\tau+\partial_\sigma)\psi_1^J + \psi_2^J(\partial_\tau-\partial_\sigma)\psi_2^J\right].\end{aligned} \tag{19.51}$$

The tachyon then goes away, and the number of spacetime dimensions drops from 26 to 10. Although string theory requires renormalization, it does give finite scattering amplitudes.

There are five distinct superstring theories — types I, IIA, and IIB; $E_8 \otimes E_8$ heterotic; and $SO(32)$ heterotic. All five may be related to a single theory in 11 dimensions called **M-theory**, which is not a string theory. M-theory contains membranes (2-branes) and 5-branes, which are not D-branes.

19.8 Covariant and Polyakov actions

Other actions offer other advantages. The covariant action

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \left(\partial_\tau X^\mu \partial_\tau X_\mu - \partial_\sigma X^\mu \partial_\sigma X_\mu\right) \tag{19.52}$$

offers manifest Lorentz invariance and simple momentum operators

$$\mathcal{P}_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = \frac{1}{2\pi\alpha'} \dot{X}_\mu. \tag{19.53}$$

The commutation relations are

$$[X^\mu(\tau,\sigma), \mathcal{P}^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\,\delta(\sigma - \sigma'), \tag{19.54}$$

but the 00 commutation relation has a minus sign that requires constraints on the physical states, reminiscent of the Gupta-Bleuler formalism.

Polyakov's action is

$$S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-h}\, h^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu\, \eta_{\mu\nu} \tag{19.55}$$

in which $h = \det(h_{\alpha\beta})$ and $h^{\alpha\beta}$ is the inverse of the 2×2 matrix $h_{\alpha\beta}$. The Minkowski metric $\eta_{\mu\nu}$ is fixed and diagonal, but the metric $h_{\alpha\beta}$ is dynamical and plays in the two dimensions τ, σ a role like that of $g_{\mu\nu}$ in general relativity.

19.9 D-branes or P-branes

One may satisfy Dirichlet boundary conditions (19.17) by requiring the ends of a string to be attached to a spatial manifold, called a **D-brane** after Dirichlet. These branes should be called P-branes after Polchinski. If the manifold to which the string is stuck has p dimensions, then it's called a **D*p*-brane**. Figure (19.2) shows a string whose ends are free to move only within a D2-brane.

Dp-branes offer a natural way to explain the extra six dimensions required in a universe of superstrings. One imagines that the ends of all the strings are

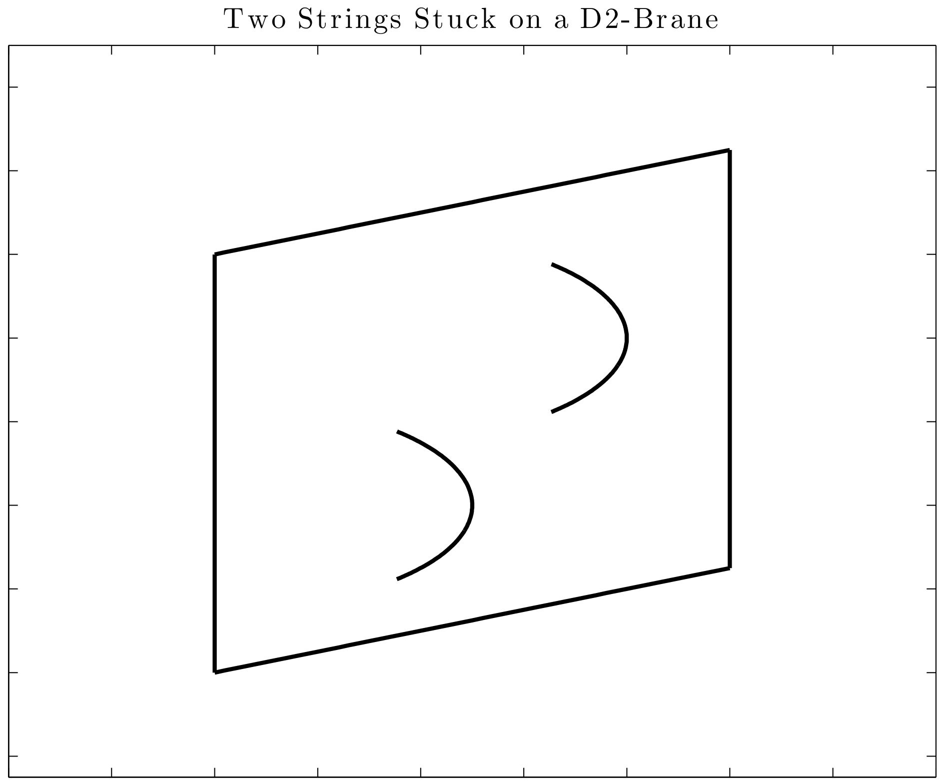

Figure 19.2 Two strings stuck on a D2-brane.

free to move only in our four-dimensional spacetime; the strings are stuck on a D3-brane, which is the three-dimensional space of our physical universe. The tension of the superstring then keeps it from wandering far enough into the extra six spatial dimensions for us ever to have noticed.

19.10 String-String Scattering

Strings interact by joining and by breaking. Figure 19.3 shows two open strings joining to form one open string and then breaking into two open strings. Figure 19.4 shows two closed strings joining to form one closed string and then breaking into two closed strings. The interactions of strings do not occur at points.

Because strings are extended objects, their scattering amplitudes are finite.

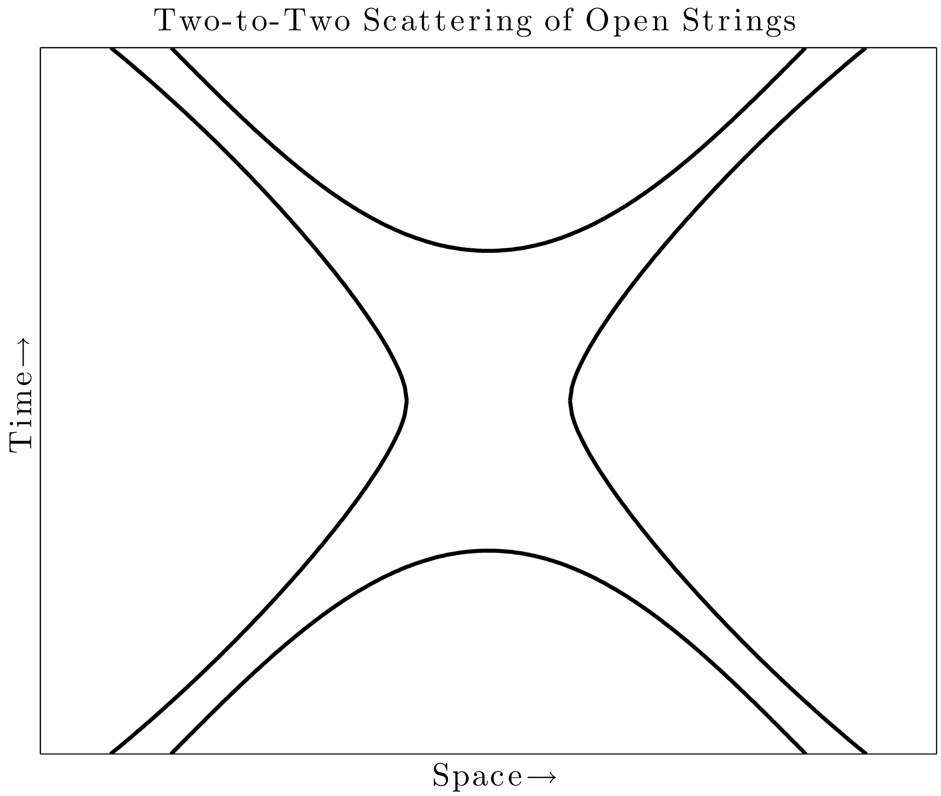

Figure 19.3 A spacetime diagram of the scattering of two open strings into two open strings. Scattering amplitudes are finite in string theory.

19.11 Riemann Surfaces and Moduli

A **homeomorphism** is a map that is one to one and continuous with a continuous inverse. A **Riemann surface** is a two-dimensional real manifold whose open sets U_α are mapped onto open sets of the complex plane $\mathbb{C}$ by **homeomorphisms** z_α whose transition functions $z_\alpha \circ z_\beta^{-1}$ are analytic on the images of the intersections $U_\alpha \cap U_\beta$. Two Riemann surfaces are **equivalent** if they are related by a continuous analytic map that is one to one and onto.

A parameter that distinguishes a Riemann surface from other, inequivalent Riemann surfaces is called a **modulus**. Some Riemann surfaces have several moduli; others have one modulus; others none at all. Some moduli are continuous parameters; others are discrete.

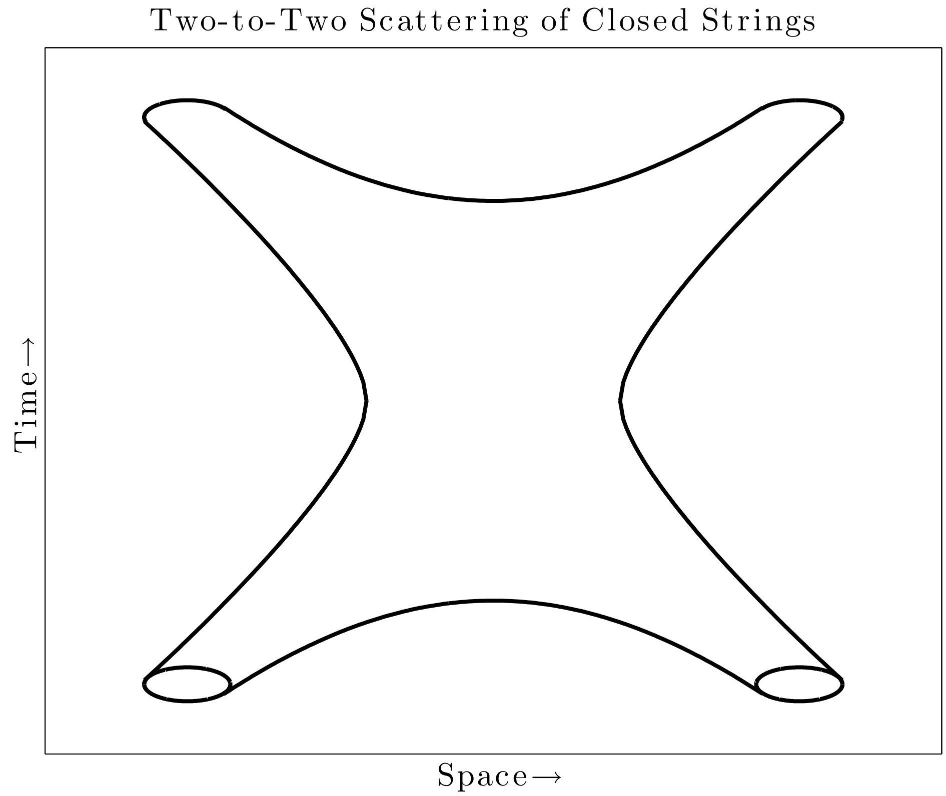

Figure 19.4 A spacetime diagram of the scattering of two closed closed strings into two closed closed strings. Scattering amplitudes are finite in string theory.

Further Reading

The excellent textbook *A First Course in String Theory* (Zwiebach, 2009) is a good way to learn more about field theory and string theory.

Exercises

19.1 Derive formulas (19.10) and (19.11).
19.2 Derive equation (19.12) from (19.9–19.11).