17

The Renormalization Group

17.1 Quantum field theory as an interpolating theory

Probably because they describe point particles, quantum field theories are rife with infinities. Unknown physics at very short distance scales, removes these infinities. Since these infinities really are absent, we can cancel them consistently in **renormalizable** theories by a procedure called **renormalization**. One starts with an action that contains infinite, unknown constants, such as $e_0 = e + \delta e$, in which e is the charge of the proton. One then uses δe and other infinite parameters to cancel unwanted infinite terms as they appear in perturbation theory.

Alternatively, we may consider quantum field theory as a way to interpolate between the value of a physical quantity as given by experiment at one energy scale and its value at other energies.

For instance, in the theory of a scalar field ϕ described by the action density

$$\mathcal{L} = -\frac{1}{2}\partial_\nu\phi\,\partial^\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4, \tag{17.1}$$

the amplitude for the elastic scattering of two bosons of initial four-momenta p_1 and p_2 into two of final momenta p_1' and p_2' is

$$\begin{aligned} A = g - \frac{g^2}{16\pi^2}\int_0^\infty k^3 dk \int_0^1 dx \Big\{ & [k^2 + m^2 - sx(1-x)]^{-2} \\ & + [k^2 + m^2 - tx(1-x)]^{-2} + [k^2 + m^2 - ux(1-x)]^{-2} \Big\} \end{aligned} \tag{17.2}$$

to one-loop order (Weinberg, 1995, section 12.2). In this formula, s, t, and

u are the Mandelstam variables $s = -(p_1 + p_2)^2$, $t = -(p_1 - p_1')^2$, and $u = -(p_1 - p_2')^2$.

The amplitude $A(s,t,u)$ diverges logarithmically as $k \to \infty$, but its derivatives with respect to s, t, and u are convergent. So we can write it as its value $A_0 = A(s_0, t_0, u_0)$ at some point (s_0, t_0, u_0) plus convergent integrals of its derivatives with respect to s, t, and u

$$A(s,t,u) - A_0 = -\frac{g^2}{16\pi^2}\int_0^\infty k^3 dk \int_0^1 dx \left[\int_{s_0}^s ds' \frac{2x(1-x)}{[k^2 + m^2 - s'x(1-x)]^3} + \int_{t_0}^t dt' \frac{2x(1-x)}{[k^2 + m^2 - t'x(1-x)]^3} + \int_{u_0}^u du' \frac{2x(1-x)}{[k^2 + m^2 - u'x(1-x)]^3} \right]. \tag{17.3}$$

And as long as we finesse the poles in k by keeping the Mandelstam variables off-shell and negative, the integrals over k are elementary, as are those over s', t', and u'

$$\begin{aligned} A(s,t,u) - A_0 = & -\frac{g^2}{32\pi^2}\int_0^1 x(1-x)\left[\int_{s_0}^s \frac{ds'}{m^2 - s'x(1-x)} + \int_{t_0}^t \frac{dt'}{m^2 - t'x(1-x)} + \int_{u_0}^u \frac{du'}{m^2 - u'x(1-x)}\right] dx \\ = & -\frac{g^2}{32\pi^2}\int_0^1 dx \left\{ \ln\left[\frac{m^2 - s_0x(1-x)}{m^2 - sx(1-x)}\right] + \ln\left[\frac{m^2 - t_0x(1-x)}{m^2 - tx(1-x)}\right] + \ln\left[\frac{m^2 - u_0x(1-x)}{m^2 - ux(1-x)}\right] \right\}. \end{aligned} \tag{17.4}$$

If we choose as the renormalization point $s_0 = t_0 = u_0 = -4\mu^2/3$, then we get the usual result (Weinberg, 1995, 1996, sections 12.2, 18.1–2).

17.2 The Renormalization Group in Quantum Field Theory

We now use our expression (17.4) to define the **running coupling constant** g_μ at energy scale μ as the amplitude $A_0 = A(s_0, t_0, u_0)$ at $s_0 = t_0 = u_0 = -\mu^2$

$$g_\mu \equiv A(s_0, t_0, u_0) = A(-\mu^2). \tag{17.5}$$

In terms of the sliding scale μ, the scattering amplitude is

$$A(s,t,u) = g_\mu - \frac{g^2}{32\pi^2}\int_0^1 dx \left\{ \ln\left[\frac{1+(\mu/m)^2x(1-x)}{1-s/m^2x(1-x)}\right] \right. \tag{17.6}$$
$$\left. + \ln\left[\frac{1+(\mu/m)^2x(1-x)}{1-t/m^2x(1-x)}\right] + \ln\left[\frac{1+(\mu/m)^2x(1-x)}{1-u/m^2x(1-x)}\right] \right\}$$

plus terms of order g^3 and higher in the perturbative expansion. The key observation of Callan, Symanzik, and others is that this scattering amplitude, like any physical quantity, should be **independent of the sliding scale** μ. Thus its derivative with respect to μ must vanish

$$\begin{aligned} 0 = \frac{\partial A(s,t,u)}{\partial \mu} &= \frac{\partial g_\mu}{\partial \mu} - \frac{3g^2}{32\pi^2}\frac{\partial}{\partial \mu}\int_0^1 dx\, \ln\left[1+(\mu/m)^2x(1-x)\right] \\ &= \frac{\partial g_\mu}{\partial \mu} - \frac{3g^2}{32\pi^2}\int_0^1 dx\, \frac{(2\mu/m^2)x(1-x)}{1+(\mu/m)^2x(1-x)}. \end{aligned} \tag{17.7}$$

For $\mu \gg m$, the integral is $2/\mu$. So at high energies, the running coupling constant obeys the differential equation

$$\mu\frac{\partial g_\mu}{\partial \mu} \equiv \beta(g_\mu) = \frac{3g^2}{16\pi^2}. \tag{17.8}$$

Since the coupling constants g and g_μ are equal to lowest order in g_μ, we may write the **beta function** as

$$\mu\frac{\partial g_\mu}{\partial \mu} \equiv \beta(g_\mu) = \frac{3g_\mu^2}{16\pi^2}. \tag{17.9}$$

Integrating, we get

$$\ln\frac{E}{M} = \int_M^E \frac{d\mu}{\mu} = \int_{g_M}^{g_E} \frac{dg_\mu}{\beta(g_\mu)} = \frac{16\pi^2}{3}\int_{g_M}^{g_E}\frac{dg_\mu}{g_\mu^2} = \frac{16\pi^2}{3}\left(\frac{1}{g_M} - \frac{1}{g_E}\right). \tag{17.10}$$

So the running coupling constant g_μ at energy $\mu = E$ is

$$g_E = \frac{g_M}{1 - 3\,g_M\,\ln(E/M)/16\pi^2}. \tag{17.11}$$

As the energy $E = \sqrt{s}$ rises above M, while staying below the singular value $E = M\exp(16\pi^2/3g_M)$, the running coupling constant g_E slowly increases, as does the scattering amplitude, $A \approx g_E$.

Example 17.1 (Quantum Electrodynamics)**.** Vacuum polarization makes the amplitude for the scattering of two electrons proportional to (Weinberg, 1995, section 11.2)

$$A(q^2) = e^2 \left[1 + \pi(q^2)\right] \tag{17.12}$$

rather than to e^2. Here e is the renormalized charge, $q = p_1' - p_1$ is the four-momentum transferred to the first electron, and

$$\pi(q^2) = \frac{e^2}{2\pi^2} \int_0^1 x(1-x) \ln\left[1 + \frac{q^2 x(1-x)}{m^2}\right] dx \tag{17.13}$$

represents the polarization of the vacuum. We define the square of the running coupling constant e_μ^2 to be the amplitude (17.12) at $q^2 = \mu^2$

$$e_\mu^2 = A(\mu^2) = e^2 \left[1 + \pi(\mu^2)\right]. \tag{17.14}$$

For $q^2 = \mu^2 \gg m^2$, the vacuum-polarization term $\pi(\mu^2)$ is (exercise 17.1)

$$\pi(\mu^2) \approx \frac{e^2}{6\pi^2}\left[\ln\frac{\mu}{m} - \frac{5}{6}\right]. \tag{17.15}$$

The amplitude (17.12) then is

$$A(q^2) = e_\mu^2 \, \frac{1 + \pi(q^2)}{1 + \pi(\mu^2)}, \tag{17.16}$$

and since it must be independent of μ, we have

$$0 = \frac{d}{d\mu}\frac{A(q^2)}{1 + \pi(q^2)} = \frac{d}{d\mu}\,\frac{e_\mu^2}{1 + \pi(\mu^2)} \approx \frac{d}{d\mu}\left\{e_\mu^2 \left[1 - \pi(\mu^2)\right]\right\}. \tag{17.17}$$

So by differentiating e_μ and the vacuum-polarization term (17.15), we find

$$0 = 2e_\mu\left(\frac{de_\mu}{d\mu}\right)\left[1 - \pi(\mu^2)\right] - e_\mu^2 \frac{d\pi(\mu^2)}{d\mu} = 2e_\mu\left(\frac{de_\mu}{d\mu}\right)\left[1 - \pi(\mu^2)\right] - e_\mu^2 \frac{e^2}{6\pi^2\mu}. \tag{17.18}$$

But by (17.13) the vacuum-polarization term $\pi(\mu^2)$ is of order e^2, which is the same as e_μ^2 to lowest order in e_μ. Thus we arrive at the Callan-Symanzik equation

$$\mu \frac{de_\mu}{d\mu} \equiv \beta(e_\mu) = \frac{e_\mu^3}{12\pi^2} \tag{17.19}$$

which we can integrate

$$\ln\frac{E}{M} = \int_M^E \frac{d\mu}{\mu} = \int_{e_M}^{e_E} \frac{de_\mu}{\beta(e_\mu)} = 12\pi^2 \int_{e_M}^{e_E} \frac{de_\mu}{e_\mu^3} = 6\pi^2\left(\frac{1}{e_M^2} - \frac{1}{e_E^2}\right) \tag{17.20}$$

to

$$e_E^2 = \frac{e_M^2}{1 - e_M^2 \ln(E/M)/6\pi^2}. \tag{17.21}$$

The fine-structure constant $e_\mu^2/4\pi$ slowly rises from $\alpha = 1/137.036$ at m_e to

$$\frac{e^2(45.5\text{GeV})}{4\pi} = \frac{\alpha}{1 - 2\alpha \ln(45.5/0.00051)/3\pi} = \frac{1}{134.6} \tag{17.22}$$

at $\sqrt{s} = 91$ GeV. When all light charged particles are included, one finds that the fine-structure constant rises to $\alpha = 1/128.87$ at $E = 91$ GeV. □

Example 17.2 (Quantum Chromodynamics)**.** The beta functions of scalar field theories and of quantum electrodynamics are positive, and so interactions in these theories become stronger at higher energy scales. But Yang-Mills theories have beta functions that can be negative because of the cubic interactions of the gauge fields and the ghost fields (16.250). If the gauge group is $SU(3)$, then the beta function is

$$\mu \frac{dg_\mu}{d\mu} \equiv \beta(g_\mu) = -\frac{11g^3}{16\pi^2} = -\frac{11g_\mu^3}{16\pi^2} \tag{17.23}$$

to lowest order in g_μ. Integrating, we find

$$\ln \frac{E}{M} = \int_M^E \frac{d\mu}{\mu} = \int_{g_M}^{g_E} \frac{dg_\mu}{\beta(g_\mu)} = -\frac{16\pi^2}{11} \int_{g_M}^{g_E} \frac{dg_\mu}{g_\mu^3} = \frac{8\pi^2}{11} \left(\frac{1}{g_M^2} - \frac{1}{g_E^2} \right) \tag{17.24}$$

and

$$g_E^2 = g_M^2 \left[1 + \frac{11 g_M^2}{8\pi^2} \ln \frac{E}{M} \right]^{-1} \tag{17.25}$$

which shows that as the energy E of a scattering process increases, the running coupling slowly **decreases**, going to zero at infinite energy, an effect called **asymptotic freedom**.

If the gauge group is $SU(N)$, and the theory has n_f flavors of quarks with masses below μ, then the beta function is

$$\beta(g_\mu) = -\frac{g_\mu^3}{4\pi^2} \left(\frac{11N}{12} - \frac{n_f}{6} \right) \tag{17.26}$$

which is negative as long as $n_f < 11N/2$. Using this beta-function with $N = 3$ and again integrating, we get instead of (17.25)

$$g_E^2 = g_M^2 \left[1 + \frac{(11 - 2n_f/3) g_M^2}{16\pi^2} \ln \frac{E^2}{M^2} \right]^{-1}. \tag{17.27}$$

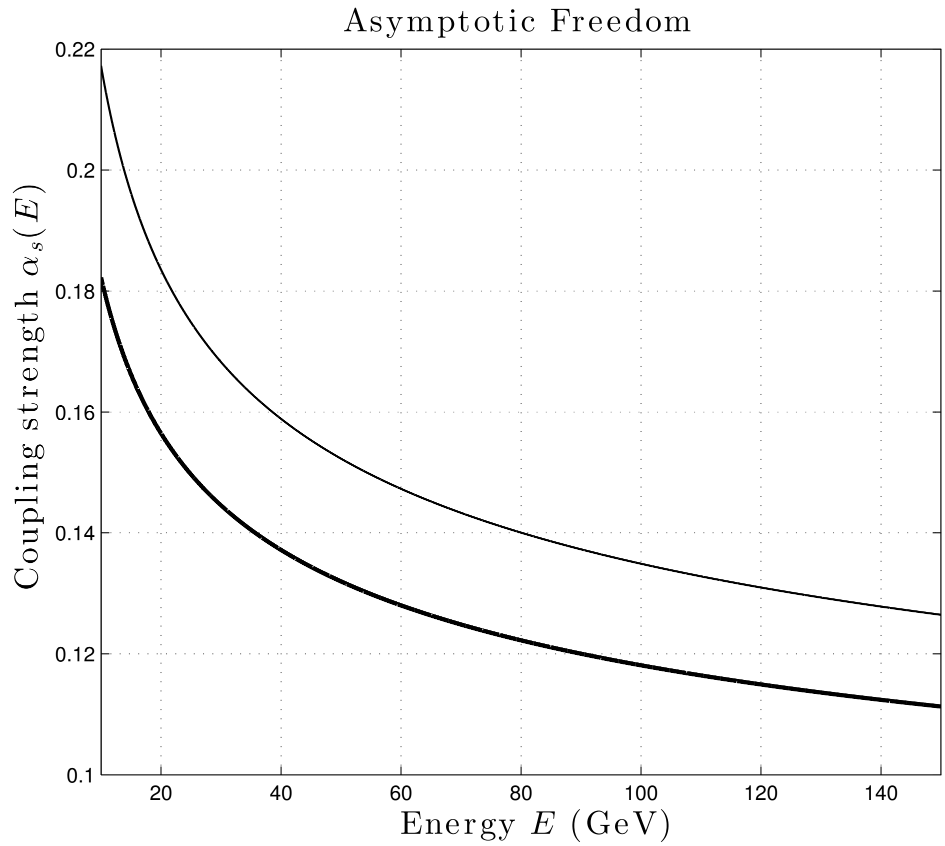

Figure 17.1 The strong-structure constant $\alpha_s(E)$ as given by the one-loop formula (17.29) (thin curve) and by a three-loop formula (thick curve) with $\Lambda = 230$ MeV and $n_f = 5$ is plotted for $m_b \ll E \ll m_t$.

So with

$$M^2 \equiv \Lambda^2 \exp\left(\frac{16\pi^2}{(11 - 2n_f/3)g_M^2}\right) \tag{17.28}$$

we find (exercise 17.2)

$$\alpha_s(E) \equiv \frac{g^2(E)}{4\pi} = \frac{12\pi}{(33 - 2n_f)\ln(E^2/\Lambda^2)} \tag{17.29}$$

which expresses the dimensionless QCD coupling constant $\alpha_s(E)$ appropriate to energy E in terms of a parameter Λ that has the dimension of energy. Sidney Coleman called this **dimensional transmutation**. For $\Lambda = 230$ MeV and $n_f = 5$, Fig. 17.1 displays $\alpha_s(E)$ in the range $4.19 = m_b \ll E \ll$

$m_t = 172$ GeV. The thin curve is the one-loop formula (17.29), and the thick curve is a three-loop formula (Weinberg, 1996, p. 156). □

17.3 The Renormalization Group in Lattice Field Theory

Let us consider a quantum field theory on a lattice (Gattringer and Lang, 2010, chap. 3) in which the strength of the nonlinear interactions depends upon a single dimensionless coupling constant g. The spacing a of the lattice regulates the infinities, which return as $a \to 0$. The value of an observable P computed on this lattice will depend upon the lattice spacing a and on the coupling constant g, and so will be a function $P(a, g)$ of these two parameters. The *right* value of the coupling constant is the value that makes the result of the computation be as close as possible to the physical value P. So the correct coupling constant is not a constant at all, but rather a function $g(a)$ that varies with the lattice spacing or cutoff a. Thus as we vary the lattice spacing and go to the continuum limit $a \to 0$, we must adjust the coupling function $g(a)$ so that what we compute, $P(a, g(a))$, is equal to the physical value P. That is, $g(a)$ must vary with a so as to keep $P(a, g(a))$ constant at $P(a, g(a)) = P$

$$\frac{dP(a, g(a))}{da} = 0. \tag{17.30}$$

Writing this condition as a dimensionless derivative

$$a\,\frac{dP(a, g(a))}{da} = \frac{da}{d\ln a}\,\frac{dP(a, g(a))}{da} = \frac{dP(a, g(a))}{d\ln a} = 0 \tag{17.31}$$

we arrive at the **Callan-Symanzik equation**

$$0 = \frac{dP(a, g(a))}{d\ln a} = \left(\frac{\partial}{\partial \ln a} + \frac{dg}{d\ln a}\,\frac{\partial}{\partial g}\right) P(a, g(a)). \tag{17.32}$$

The coefficient of the second partial derivative with a minus sign is the lattice β-function

$$\beta_L(g) \equiv -\frac{dg}{d\ln a}. \tag{17.33}$$

Since the lattice spacing a and the energy scale μ are inversely related, the lattice β-function differs from the continuum beta-function by a minus sign.

In $SU(N)$ gauge theory, the first two terms of the lattice β-function for small g are

$$\beta_L(g) = -\beta_0\, g^3 - \beta_1\, g^5 \tag{17.34}$$

where for n_f flavors of light quarks

$$\beta_0 = \frac{1}{(4\pi)^2}\left(\frac{11}{3}N - \frac{2}{3}n_f\right)$$
$$\beta_1 = \frac{1}{(4\pi)^4}\left(\frac{34}{3}N^2 - \frac{10}{3}Nn_f - \frac{N^2-1}{N}n_f\right) \tag{17.35}$$

and $N = 3$ in quantum chromodynamics.

Combining the definition (17.33) of the β-function with its expansion (17.34) for small g, one gets the differential equation

$$\frac{dg}{d\ln a} = \beta_0\, g^3 + \beta_1\, g^5 \tag{17.36}$$

which one may integrate

$$\int d\ln a = \ln a\, - \ln c = \int \frac{dg}{\beta_0 g^3 + \beta_1 g^5} = -\frac{1}{2\beta_0 g^2} + \frac{\beta_1}{2\beta_0^2}\,\ln\left(\frac{\beta_0 + \beta_1 g^2}{g^2}\right) \tag{17.37}$$

to find

$$a(g) = c\left(\frac{\beta_0 + \beta_1 g^2}{g^2}\right)^{\beta_1/2\beta_0^2}\, e^{-1/2\beta_0 g^2} \tag{17.38}$$

in which c is a constant of integration. The term $\beta_1 g^2$ is of higher order in g, and if one drops it and absorbs a power of β_0 into a new constant of integration Λ, then one finds

$$a(g) = \frac{1}{\Lambda}\,\left(\beta_0 g^2\right)^{-\beta_1/2\beta_0^2}\, e^{-1/2\beta_0 g^2}. \tag{17.39}$$

As $g \to 0$, the lattice spacing $a(g)$ goes to zero *very fast* (as long as $n_f < 17$ for $N = 3$). The inverse of this relation (17.39)

$$g(a) \approx \left[\beta_0 \ln(a^{-2}\Lambda^{-2}) + (\beta_1/\beta_0)\,\ln\left(\ln(a^{-2}\Lambda^{-2})\right)\right]^{-1/2} \tag{17.40}$$

shows that the coupling constant slowly goes to zero with a, which is a lattice version of **asymptotic freedom**. □

17.4 The Renormalization Group in Condensed-Matter Physics

The study of condensed matter is concerned mainly with properties that emerge in the bulk, such as melting and boiling points and conductivity. These properties appear in the physics as we increase the distance scale many orders of magnitude beyond the size a of an individual molecule or the distance to its nearest neighbor.

As a simple example, let's consider a euclidian action in d dimensions

$$S = \int d^d x \left(\frac{1}{2}(\partial\phi)^2 + \sum_n g_{d,n}\,\phi^n \right) \tag{17.41}$$

in which $g_2\,\phi^2 \equiv m^2\,\phi^2/2$ is a mass term and $g_4\,\phi^4 \equiv \lambda\,\phi^4/24$ is a quartic self-interaction. We first define a partition function $Z(\Lambda)$ with an ultraviolet cutoff $\Lambda = 1/a$ as a path integral

$$Z(\Lambda) = \int_\Lambda e^{-S} D\phi \tag{17.42}$$

over fields

$$\phi(x) = \int_\Lambda e^{ikx}\phi(k)\,\frac{d^d k}{(2\pi)^d} \tag{17.43}$$

that only have Fourier coefficients $\phi(k)$ with $k^2 < \Lambda^2$.

Corresponding to each such field $\phi(x)$, we introduce a "stretched" field

$$\phi_L(x) = A(L)\,\phi(x/L) \quad \text{for} \quad L \ge 1 \tag{17.44}$$

in which $A(L)$ is a scale factor that keeps the kinetic part of the action invariant. Since

$$\phi_L(x) = A(L)\,\phi(x/L) = A(L)\int_\Lambda \exp\left(i\frac{kx}{L}\right)\phi(k)\,\frac{d^d k}{(2\pi)^d}, \tag{17.45}$$

the momenta of the stretched field are reduced by the factor $1/L$.

We may define a new partition function in which we integrate over the stretched fields $\phi_L(x)$

$$Z(\Lambda/L) = \int_{\Lambda/L} e^{-S} D\phi \equiv \int_\Lambda e^{-S} D\phi_L. \tag{17.46}$$

The kinetic action of a stretched field is

$$S_k = \int d^d x\,\frac{A^2(L)}{2}\left(\frac{\partial\phi(x/L)}{\partial x}\right)^2 = \int d^d(x/L)\,L^d\,\frac{A^2(L)}{2}\left(\frac{\partial\phi(x/L)}{L\partial x/L}\right)^2 \tag{17.47}$$

and so if we choose

$$A(L) = L^{(2-d)/2} \tag{17.48}$$

then letting $x' = x/L$, we can keep the kinetic action S_k invariant

$$S_k = \int d^d x'\,\frac{1}{2}\left(\frac{\partial\phi(x')}{\partial x'}\right)^2. \tag{17.49}$$

The full action of a stretched field is then

$$S(\phi_L) = \int d^d x \left(\frac{1}{2}(\partial\phi)^2 + \sum_n g_{d,n}(L)\phi^n \right) \tag{17.50}$$

in which

$$g_{d,n}(L) = L^d A^n(L)\, g_n = L^{d+n(2-d)/2}\, g_{d,n}. \tag{17.51}$$

Now the action (17.50) of the stretched field is the same as the original action (17.41) but with the original coupling constants $g_{d,n}$ replaced by the scaled coupling constants $g_{d,n}(L)$. The beta-function

$$\beta(g_{d,n}) \equiv \frac{L}{g_{d,n}(L)} \frac{dg_{d,n}(L)}{dL} = d + n(2-d)/2 \tag{17.52}$$

is just the exponent of the coupling "constant" $g_{d,n}(L)$. If it is positive, then the coupling constant $g_{d,n}(L)$ gets stronger as $L \to \infty$; such couplings are called **relevant**. Couplings with vanishing exponents are insensitive to changes in L and are **marginal**. Those with negative exponents shrink with increasing L; they are **irrelevant**.

The coupling constant $g_{d,n,p}$ of a term with p derivatives and n powers of the field ϕ in a space of d dimensions varies as

$$g_{d,n,p}(L) = L^d\, A^n(L)\, L^{-p}\, g_{d,n,p} = L^{d+n(2-d)/2-p}\, g_{d,n,p}. \tag{17.53}$$

Example 17.3 (QCD)**.** In quantum chromodynamics, there is a cubic term $g\, f_{abc}\, A_0^a A_i^b \partial_0 A_i^c$ which we can treat like $g\, f_{abc}\, \phi_a \phi_b \dot{\phi}_c$. Is it relevant? Well, if we stretch space but not time, then the time derivative has no effect, so $d = 3$. Then the scaling formula (17.53) says that the cubic coefficient $g_{3,3,0}(L)$, which has $n = 3, d = 3$, and $p = 0$, grows as $L^{3/2}$

$$g_{3,3,0}(L) = L^{d+n(2-d)/2}\, g_{3,3,0} = L^{3/2}\, g_{3,3,0}. \tag{17.54}$$

It is strongly relevant. The related term $g\, f_{abc}\, A_0^a A_i^b \partial_i A_0^c$ has $d = 3, n = 3$, and one spatial derivative, $p = 1$. So the scaling formula (17.53) says that it rises as $L^{1/2}$

$$g_{3,3,1}(L) = L^{d+n(2-d)/2-1}\, g_{3,3,1}(1) = L^{1/2}\, g_{3,3,1}. \tag{17.55}$$

It also is relevant. Similarly, the term $g\, f_{abc}\, A_i^a A_j^b \partial_i A_j^c$ has $n = 3, d = 3$, and $p = 1$. By (17.53), it also rises as $L^{1/2}$. It too is relevant.

Since these cubic terms make the attractive force that drives asymptotic freedom, their strengthening as space is stretched by the dimensionless factor L may point to a qualitative explanation of confinement. For if $g_{3,3,0}(L)$

grows with distance as $L^{3/2}$, then the strength of the $g_{3,3,0}$ part of the QCD-structure constant grows as the cube of the separation

$$\alpha_{s,3,3,0}(L) = \frac{g^2_{3,3,0}(L)}{4\pi} = L^3\,\alpha_{s,3,3,0}, \tag{17.56}$$

and the force due to it between two quarks separated by a distance Lr grows linearly with L

$$F(Lr) = \frac{\alpha_s(Lr)}{(Lr)^2} = \frac{L^3\,\alpha_s(r)}{(Lr)^2} = L\,\frac{\alpha_s(r)}{r^2}. \tag{17.57}$$

The $g_{3,3,1}$ part of α_s grows linearly with the separation

$$\alpha_{s,3,3,1}(L) = \frac{g^2_{3,3,1}(L)}{4\pi} = L\,\alpha_{s,3,3,1}. \tag{17.58}$$

The coupling constant of the quartic terms $g^2 f_{abc} f_{ade} A^b_i A^c_j A^d_i A^e_j$ is g^2, and it varies as $g^2(L) = g^2 L^{3-2} = g^2 L$. So the quartic coupling constant is relevant. □

Further Reading

Quantum Field Theory in a Nutshell (Zee, 2010, chapters III & VI), *An Introduction to Quantum Field Theory* (Peskin and Schroeder, 1995, chapter 12), and *The Quantum Theory of Fields* (Weinberg, 1995, 1996, sections 12.2 & 18.1–2).

Exercises

17.1 Show that for $\mu^2 \gg m^2$, the vacuum polarization term (17.13) reduces to (17.15). Hint: Use $\ln a\,b = \ln a + \ln b$ when integrating.

17.2 Show that by choosing the energy scale Λ according to (17.28), one can derive (17.29) from (17.27).

17.3 Show that if we stretch both space and time, then in the notation of (17.53), the cubic $g_{4,3,1}(L)$ and quartic $g_{4,4,0}(L)$ couplings are marginal, that is, are independent of L.