16

Path Integrals

16.1 Path Integrals and Classical Physics

Since Richard Feynman invented them over 60 years ago, path integrals have been used with increasing frequency in high-energy and condensed-matter physics, in finance, and in biophysics (Kleinert, 2009). Feynman used them to express matrix elements of the time-evolution operator $\exp(-itH/\hbar)$ in terms of the classical action. Others have used them to compute matrix elements of the Boltzmann operator $\exp(-H/kT)$ which in the limit of zero temperature projects out the ground state $|E_0\rangle$ of the system

$$\lim_{T\to 0} e^{-(H-E_0)/kT} = \lim_{T\to 0} \sum_{n=0}^{\infty} |E_n\rangle e^{-(E_n-E_0)/kT} \langle E_n| = |E_0\rangle\langle E_0| \tag{16.1}$$

a trick used in lattice gauge theory.

Path integrals magically express the quantum-mechanical probability amplitude for a process as a sum of exponentials $\exp(iS/\hbar)$ of the classical action S of the various ways that process might occur.

16.2 Gaussian Integrals

The path integrals we can do are gaussian integrals of infinite order. So we begin by recalling the basic integral formula (5.167)

$$\int_{-\infty}^{\infty} \exp\left[-ia\left(x - \frac{b}{2a}\right)^2\right] dx = \sqrt{\frac{\pi}{ia}} \tag{16.2}$$

which holds for real a and b, and also the one (5.168)

$$\int_{-\infty}^{\infty} \exp\left[-r\left(x-\frac{c}{2r}\right)^2\right]dx = \sqrt{\frac{\pi}{r}} \tag{16.3}$$

which is true for positive r and complex c. Equivalent formulas for real a and b, positive r, and complex c are

$$\int_{-\infty}^{\infty} \exp\left(-iax^2+ibx\right)dx = \sqrt{\frac{\pi}{ia}}\exp\left(i\frac{b^2}{4a}\right) \tag{16.4}$$

$$\int_{-\infty}^{\infty} \exp\left(-rx^2+cx\right)dx = \sqrt{\frac{\pi}{r}}\exp\left(\frac{c^2}{4r}\right). \tag{16.5}$$

This last formula will be useful with $x = p$, $r = \epsilon/(2m)$, and $c = i\epsilon\dot{q}$

$$\int_{-\infty}^{\infty} \exp\left(-\epsilon\frac{p^2}{2m}+i\epsilon\,\dot{q}\,p\right)dp = \sqrt{\frac{2\pi m}{\epsilon}}\exp\left(-\epsilon\frac{1}{2}m\dot{q}^2\right) \tag{16.6}$$

as will (16.4) with $x = p$, $a = \epsilon/(2m)$, and $b = \epsilon\dot{q}$

$$\int_{-\infty}^{\infty} \exp\left(-i\epsilon\frac{p^2}{2m}+i\epsilon\,\dot{q}\,p\right)dp = \sqrt{\frac{2\pi m}{i\epsilon}}\exp\left(i\epsilon\frac{1}{2}m\dot{q}^2\right). \tag{16.7}$$

Doable path integrals are multiple gaussian integrals. One may show (exercise 16.1) that for positive $r_1, \ldots, r_N$ and complex $c_1, \ldots, c_N$, the integral (16.5) leads to

$$\int_{-\infty}^{\infty} \exp\left(\sum_i -r_i x_i^2 + c_i x_i\right)\prod_{i=1}^{N} dx_i = \left(\prod_{i=1}^{N}\sqrt{\frac{\pi}{r_i}}\right)\exp\left(\frac{1}{4}\sum_i \frac{c_i^2}{r_i}\right). \tag{16.8}$$

If R is the $N \times N$ diagonal matrix with positive entries $\{r_1, r_2, \ldots, r_N\}$, and X and C are N-vectors with real $\{x_i\}$ and complex $\{c_i\}$ entries, then this formula (16.8) in matrix notation is

$$\int_{-\infty}^{\infty} \exp\left(-X^{\mathsf{T}}RX + C^{\mathsf{T}}X\right)\prod_{i=1}^{N} dx_i = \sqrt{\frac{\pi^N}{\det(R)}}\exp\left(\frac{1}{4}C^{\mathsf{T}}R^{-1}C\right). \tag{16.9}$$

Now every positive symmetric matrix S is of the form $S = ORO^{\mathsf{T}}$ for some positive diagonal matrix R. So inserting $R = O^{\mathsf{T}}SO$ into the previous equation (16.9) and using the invariance of determinants under orthogonal transformations, we find

$$\int_{-\infty}^{\infty} \exp\left(-X^{\mathsf{T}}O^{\mathsf{T}}SOX + C^{\mathsf{T}}X\right)\prod_{i=1}^{N} dx_i = \sqrt{\frac{\pi^N}{\det(S)}}\exp\left[\frac{1}{4}C^{\mathsf{T}}O^{\mathsf{T}}S^{-1}OC\right]. \tag{16.10}$$

The jacobian of the orthogonal transformations $Y = OX$ and $D = OC$ is unity, and so

$$\int_{-\infty}^{\infty} \exp\big(-Y^{\mathsf{T}} S Y + D^{\mathsf{T}} Y\big) \prod_{i=1}^{N} dy_i = \sqrt{\frac{\pi^N}{\det(S)}}\, \exp\left(\frac{1}{4}\, D^{\mathsf{T}} S^{-1} D\right) \quad (16.11)$$

in which S is a positive symmetric matrix, and D is a complex vector.

The other basic gaussian integral (16.4) leads for real S and D to (exercise 16.2)

$$\int_{-\infty}^{\infty} \exp\big(-iY^{\mathsf{T}} S Y + iD^{\mathsf{T}} Y\big) \prod_{i=1}^{N} dy_i = \sqrt{\frac{\pi^N}{\det(iS)}}\, \exp\left(\frac{i}{4}\, D^{\mathsf{T}} S^{-1} D\right). \quad (16.12)$$

The vector $\overline{Y}$ that makes the argument $-iY^{\mathsf{T}} S Y + iD^{\mathsf{T}} Y$ of the exponential of this multiple gaussian integral (16.12) stationary is (exercise 16.3)

$$\overline{Y} = \frac{1}{2} S^{-1} D. \quad (16.13)$$

The exponential of that integral evaluated at its stationary point $\overline{Y}$ is

$$\exp\big(-i\overline{Y}^{\mathsf{T}} S \overline{Y} + iD^{\mathsf{T}} \overline{Y}\big) = \exp\left(\frac{i}{4}\, D^{\mathsf{T}} S^{-1} D\right). \quad (16.14)$$

Thus, the multiple gaussian integral (16.12) is equal to its exponential evaluated at its stationary point $\overline{Y}$, apart from a prefactor involving the determinant $\det iS$.

Similarly, the vector $\overline{Y}$ that makes the argument $-Y^{\mathsf{T}} S Y + D^{\mathsf{T}} Y$ of the exponential of the multiple gaussian integral (16.11) stationary is $\overline{Y} = S^{-1} D/2$, and that exponential evaluated at $\overline{Y}$ is

$$\exp\big(-Y^{\mathsf{T}} S Y + D^{\mathsf{T}} Y\big) = \exp\left(\frac{1}{4}\, D^{\mathsf{T}} S^{-1} D\right). \quad (16.15)$$

Once again, a multiple gaussian integral is simply its exponential evaluated at its stationary point $\overline{Y}$, apart from a prefactor involving a determinant, $\det S$.

16.3 Path Integrals in Imaginary Time

At the imaginary time $t = -i\beta\hbar$, the time-evolution operator $\exp(-itH/\hbar)$ is $\exp(-\beta H)$ in which the inverse temperature $\beta = 1/kT$ is the reciprocal of Boltzmann's constant $k = 8.617 \times 10^{-5}$ eV/K times the absolute temperature T. In the low-temperature limit, $\exp(-\beta H)$ is a projection operator (16.1) on the ground state of the system. These path integrals in imaginary time are called **euclidian** path integrals.

Let us consider a quantum-mechanical system with hamiltonian

$$H = \frac{p^2}{2m} + V(q) \tag{16.16}$$

in which the commutator of the position q and momentum p operators is $[q, p] = i$ in units in which $\hbar = 1$. For tiny ϵ, the corrections to the approximation

$$\exp\left[-\epsilon \left(\frac{p^2}{2m} + V(q) \right) \right] \approx \exp\left(-\epsilon \, \frac{p^2}{2m} \right) \exp\left(-\epsilon \, V(q) \right) + \mathcal{O}(\epsilon^2) \tag{16.17}$$

are of second order in ϵ.

To evaluate the matrix element $\langle q_1 | \exp(-\epsilon H) | q_0 \rangle$, we insert the identity operator I in the form of an integral over the momentum eigenstates

$$I = \int_{-\infty}^{\infty} |p'\rangle\langle p'| dp' \tag{16.18}$$

and use the inner product $\langle q_1 | p' \rangle = \exp(iq_1 p')/\sqrt{2\pi}$ so as to get as $\epsilon \to 0$

$$\begin{aligned} \langle q_1 | \exp(-\epsilon H) | q_0 \rangle &= \int_{-\infty}^{\infty} \langle q_1 | \exp\left(-\epsilon \, \frac{p^2}{2m} \right) |p'\rangle\langle p'| \exp\left(-\epsilon \, V(q) \right) |q_0\rangle \, dp' \\ &= e^{-\epsilon \, V(q_0)} \int_{-\infty}^{\infty} \exp\left[-\epsilon \, \frac{p'^2}{2m} + i \, p' \, (q_1 - q_0) \right] \frac{dp'}{2\pi}. \end{aligned} \tag{16.19}$$

If we adopt the suggestive notation

$$\frac{q_1 - q_0}{\epsilon} = \dot{q}_0 \tag{16.20}$$

and use the integral formula (16.6), then we find

$$\begin{aligned} \langle q_1 | \exp(- \epsilon \, H) | q_0 \rangle &= \frac{1}{2\pi} \, e^{-\epsilon \, V(q_0)} \int_{-\infty}^{\infty} \exp\left(-\epsilon \, \frac{p'^2}{2m} + i \, \epsilon \, p' \, \dot{q}_0 \right) dp' \\ &= \left(\frac{m}{2\pi\epsilon} \right)^{1/2} \exp\left\{ -\epsilon \left[\tfrac{1}{2} \, m \, \dot{q}_0^2 + V(q_0) \right] \right\} \end{aligned} \tag{16.21}$$

in which q_1 enters through the notation (16.20).

The next step is to link two of these matrix elements together

$$\begin{aligned} \langle q_2 | e^{-2\epsilon H} | q_0 \rangle &= \int_{-\infty}^{\infty} \langle q_2 | e^{-\epsilon H} | q_1 \rangle \langle q_1 | e^{-\epsilon H} | q_0 \rangle dq_1 \qquad (16.22) \\ &= \frac{m}{2\pi\epsilon} \int_{-\infty}^{\infty} \exp\left\{ -\epsilon \left[\tfrac{1}{2} \, m \, \dot{q}_1^2 + V(q_1) + \tfrac{1}{2} \, m \, \dot{q}_0^2 + V(q_0) \right] \right\} dq_1. \end{aligned}$$

Linking three of these matrix elements together and using subscripts instead of primes, we have

$$\begin{aligned}\langle q_3|e^{-3\epsilon H}|q_0\rangle &= \iint_{-\infty}^{\infty} \langle q_3|e^{-\epsilon H}|q_2\rangle\langle q_2|e^{-\epsilon H}|q_1\rangle\langle q_1|e^{-\epsilon H}|q_0\rangle\, dq_1 dq_2 \qquad (16.23)\\ &= \left(\frac{m}{2\pi\epsilon}\right)^{3/2} \iint_{-\infty}^{\infty} \exp\left\{ -\epsilon \sum_{j=0}^{2} \left[\, \tfrac{1}{2}\, m\, \dot{q}_j^2 + V(q_j)\,\right]\right\} dq_2 dq_1.\end{aligned}$$

Boldly passing from 3 to n and suppressing some integral signs, we get

$$\begin{aligned}\langle q_n|e^{-n\epsilon H}|q_0\rangle &= \iiint_{-\infty}^{\infty} \langle q_n|e^{-\epsilon H}|q_{n-1}\rangle \dots \langle q_1|e^{-\epsilon H}|q_0\rangle\, dq_{n-1}\dots dq_1 \qquad (16.24)\\ &= \left(\frac{m}{2\pi\epsilon}\right)^{n/2} \iiint_{-\infty}^{\infty} \exp\left\{ -\epsilon \sum_{j=0}^{n-1} \left[\, \tfrac{1}{2}\, m\, \dot{q}_j^2 + V(q_j)\right]\right\} dq_{n-1}\dots dq_1.\end{aligned}$$

Writing $d\beta$ for ϵ and taking the limits $\epsilon \to 0$ and $n \equiv \beta/\epsilon \to \infty$, we find that the matrix element $\langle q_\beta|e^{-\beta H}|q_0\rangle$ is a path integral of the exponential of the average energy multiplied by $-\beta$

$$\langle q_\beta|e^{-\beta H}|q_0\rangle = \int \exp\left[- \int_0^\beta \tfrac{1}{2} m\dot{q}^2(\beta') + V(q(\beta'))\; d\beta' \right] Dq \qquad (16.25)$$

in which $Dq \equiv (n\, m/2\pi\, \beta\, \hbar^2)^{n/2}\, dq_{n-1}\dots dq_2\, dq_1$ and $\dot{q}$ is the β-derivative of the coordinate $q(\beta)$. We sum over all paths $q(t)$ that go from $q(0) = q_0$ at inverse temperature $\beta = 0$ to $q(\beta) = q_\beta$ at inverse temperature β.

In the limit $\beta \to \infty$, the operator $\exp(-\beta H)$ becomes proportional to a projection operator (16.1) on the ground state of the theory.

In three dimensions with $\dot{\boldsymbol{q}}(\beta) = d\boldsymbol{q}(\beta)/d\beta$, and $\hbar \neq 1$, the analog of equation (16.25) is (exercise 16.28)

$$\langle \boldsymbol{q}_\beta|e^{-\beta H}|\boldsymbol{q}_0\rangle = \int \exp\left[- \int_0^\beta \frac{m}{2\hbar^2}\, \dot{\boldsymbol{q}}^2(\beta') + V(\boldsymbol{q}(\beta'))\; d\beta' \right] D\boldsymbol{q} \qquad (16.26)$$

where $Dq \equiv (n\, m/2\pi\, \beta\, \hbar^2)^{3n/2}\, dq_{n-1}\dots dq_2\, dq_1$.

Path integrals in imaginary time are called *euclidian* mainly to distinguish them from *Minkowski* path integrals, which represent matrix elements of the time-evolution operator $\exp(-itH)$ in real time.

16.4 Path Integrals in Real Time

Path integrals in real time represent the time-evolution operator $\exp(-itH)$. Using the integral formula (16.7), we find in the limit $\epsilon \to 0$

$$\begin{aligned}\langle q_1|e^{-i\,\epsilon\,H}|q_0\rangle &= \int_{-\infty}^{\infty}\langle q_1|\exp\left[-\,i\,\epsilon\,\frac{p^2}{2m}\right]|p'\rangle\langle p'|\exp\left[-\,i\,\epsilon\,V(q)\right]|q_0\rangle\,dp' \\ &= \frac{1}{2\pi}\,e^{-i\,\epsilon\,V(q_0)}\int_{-\infty}^{\infty}\exp\left[-\,i\,\epsilon\,\frac{p'^2}{2m}+i\,p'\,(q_1-q_0)\right]dp' \\ &= \frac{1}{2\pi}\,e^{-i\,\epsilon\,V(q_0)}\int_{-\infty}^{\infty}\exp\left[-\,i\,\epsilon\,\frac{p'^2}{2m}+i\,\epsilon\,p'\dot{q}_0\right]dp' \\ &= \left(\frac{m}{2\pi i\epsilon}\right)^{1/2}\exp\left[i\,\epsilon\left(\frac{m\,\dot{q}_0^2}{2}-V(q_0)\right)\right].\end{aligned} \tag{16.27}$$

When we link together n of these matrix elements, we get the real-time version of (16.25)

$$\langle q_n|e^{-in\epsilon H}|q_0\rangle = \left(\frac{m}{2\pi i\epsilon}\right)^{n/2}\iiint_{-\infty}^{\infty}\exp\left\{i\epsilon\sum_{j=0}^{n-1}\left[\tfrac{1}{2}\,m\,\dot{q}_j^2 - V(q_j)\right]\right\}dq_{n-1}\dots dq_1. \tag{16.28}$$

Writing dt for ϵ and taking the limits $\epsilon \to 0$ and $n\epsilon \to t$, we find that the amplitude $\langle q_t|e^{-itH}|q_0\rangle$ is the path integral

$$\langle q_t|e^{-itH/\hbar}|q_0\rangle = \int \exp\left[\frac{i}{\hbar}\int_0^t \tfrac{1}{2}\,m\,\dot{q}^2 - V(q)\,dt'\right]Dq \tag{16.29}$$

in which Dq differs from the one that appears in euclidian path integrals by the substitution $\beta \to it/\hbar$

$$Dq = \lim_{n\to\infty}\left(\frac{nm}{2\pi i\hbar t}\right)^{n/2} dq_1 dq_2 \dots dq_{n-1}. \tag{16.30}$$

The integral in the exponent is the **classical action**

$$S[q] = \int_0^t \tfrac{1}{2}\,m\,\dot{q}^2 - V(q)\,dt' \tag{16.31}$$

for a process $q(t')$ that runs from $q(0) = q_0$ to $q(t) = q_t$. We sum over all such processes.

In three-dimensional space, the analog of (16.29) is

$$\langle \boldsymbol{q}_t|e^{-itH}|\boldsymbol{q}_0\rangle = \int \exp\left[i\int_0^t \tfrac{1}{2}\,m\,\dot{\boldsymbol{q}}^2 - V(\boldsymbol{q})\,dt'\right]D\boldsymbol{q}. \tag{16.32}$$

The units of action are energy × time, and the argument of the exponential must be dimensionless, so in arbitrary units the amplitude (16.29) is

$$\langle q_t | e^{-itH/\hbar} | q_0 \rangle = \int e^{iS[q]/\hbar} Dq. \tag{16.33}$$

When is this amplitude big? When is it tiny? Suppose there is a process $q(t) = q_c(t)$ that goes from $q_c(0) = q_0$ to $q_c(t) = q_t$ in time t and that obeys the classical equation of motion (15.14–15.15)

$$\frac{\delta S[q_c]}{\delta q_c} = m\ddot{q}_c + V'(q_c) = 0. \tag{16.34}$$

The action of such a classical process is stationary. A process $q_c(t) + dq(t)$ that differs from a stationary process $q_c(t)$ by a small deviation $dq(t)$ has an action $S[q_c + dq]$ that differs from $S[q_c]$ only by terms of second order in δq. Thus a stationary process has many neighboring processes that have the same action to within a fraction of $\hbar$. These processes add with nearly the same phase to the path integral (16.33) and make a huge contribution to the amplitude $\langle q_t | e^{-itH/\hbar} | q_0 \rangle$.

But if no classical process goes from q_0 to q_t in time t, then the nonclassical, nonstationary processes that go from q_0 to q_t in time t have actions that differ from each other by large multiples of $\hbar$. These amplitudes cancel each other, and the resulting amplitude is tiny. Thus **the real-time path integral explains the principle of stationary action** (section 11.41).

Example 16.1 (Schrödinger's equation)**.** The amplitude $\langle \vec{x}, t | \psi \rangle = \psi(\vec{x}, t)$ to get to $\vec{x}$ in time t from somewhere $\vec{x}_0$ at time 0 is a sum over all processes

$$\psi(\vec{x}, t) = \sum e^{iS/\hbar} \, \psi(\vec{x}_0, 0). \tag{16.35}$$

So using our formulas (6.213 & 6.226) for the space-time partial derivatives

$\nabla S = \vec{p}$ and $\partial S/\partial t = -H$, we have

$$
\begin{aligned}
i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} &= \sum -\frac{\partial S}{\partial t}\, e^{iS/\hbar}\, \psi(\vec{x}_0,0) = \sum H\, e^{iS/\hbar}\, \psi(\vec{x}_0,0) \\
&= H \sum e^{iS/\hbar}\, \psi(\vec{x}_0,0) = H\psi(\vec{x},t) = \left(\frac{\vec{p}^2}{2m} + V(\vec{x})\right)\psi(\vec{x},t) \\
&= \left(\frac{\vec{p}^2}{2m} + V(\vec{x})\right)\sum e^{iS/\hbar}\, \psi(\vec{x}_0,0) \qquad (16.36)\\
&= \left(\frac{(\nabla S)^2}{2m} + V(\vec{x})\right)\sum e^{iS/\hbar}\, \psi(\vec{x}_0,0) \\
&= \left(-\frac{\hbar^2\nabla^2}{2m} + V(\vec{x})\right)\sum e^{iS/\hbar}\, \psi(\vec{x}_0,0) \qquad (16.37)\\
&= \left(-\frac{\hbar^2\nabla^2}{2m} + V(\vec{x})\right)\, \psi(\vec{x}_0,0)
\end{aligned}
$$

which is Schrödinger's equation. □

16.5 Path Integral for a Free Particle

The amplitude for a free nonrelativistic particle to go from the origin to the point $\boldsymbol{q}$ in time t is the path integral (16.32)

$$\langle \boldsymbol{q}|e^{-itH}|\boldsymbol{q}=\boldsymbol{0}\rangle = \int e^{iS_0[\boldsymbol{q}]}\, D\boldsymbol{q} = \int \exp\left(i\int_0^t \tfrac{1}{2}\, m\, \dot{\boldsymbol{q}}^2(t')\, dt'\right) D\boldsymbol{q}. \quad (16.38)$$

The classical path that goes from $\boldsymbol{0}$ to $\boldsymbol{q}$ in time t is $\boldsymbol{q}_c(t') = (t'/t)\,\boldsymbol{q}$. The general path $\boldsymbol{q}(t')$ over which we integrate is $\boldsymbol{q}(t') = \boldsymbol{q}_c(t') + \delta\boldsymbol{q}(t')$. Since both $\boldsymbol{q}(t')$ and $\boldsymbol{q}_c(t')$ go from $\boldsymbol{0}$ to $\boldsymbol{q}$ in time t, the arbitrary detour $\delta\boldsymbol{q}(t')$ must be a loop that goes from $\delta\boldsymbol{q}(0) = 0$ to $\delta\boldsymbol{q}(t) = 0$ in time t. The velocity $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_c + \delta\dot{\boldsymbol{q}}$ is the sum of the constant classical velocity $\dot{\boldsymbol{q}}_c = \boldsymbol{q}/t$ and the loop's velocity $\delta\dot{\boldsymbol{q}}$. The first-order change vanishes

$$m\int_0^t \dot{\boldsymbol{q}}_c \cdot \frac{d\delta\boldsymbol{q}}{dt}\, dt = m\, \dot{\boldsymbol{q}}_c \cdot \int_0^t \frac{d\delta\boldsymbol{q}}{dt}\, dt = m\, \dot{\boldsymbol{q}}_c \cdot [\delta\boldsymbol{q}(t) - \delta\boldsymbol{q}(0)] = 0 \quad (16.39)$$

and so the action $S_0[\boldsymbol{q}]$ is the classical action plus the loop action

$$S_0[\boldsymbol{q}] = \tfrac{1}{2}\, m\int_0^t \left(\dot{\boldsymbol{q}}_c + \delta\dot{\boldsymbol{q}}\right)^2\, dt' = S_0[\boldsymbol{q}_c] + S_0[\boldsymbol{\delta q}]. \quad (16.40)$$

The path integral therefore factorizes

$$\begin{aligned}\langle \boldsymbol{q}|e^{-i\,t\,H}|\mathbf{0}\rangle &= \int e^{iS_0[\boldsymbol{q}]}\,D\boldsymbol{q} \;=\int e^{iS_0[\boldsymbol{q}_c+\delta\boldsymbol{q}]}\,D\delta\boldsymbol{q}\\ &=\int e^{iS_0[\boldsymbol{q}_c]}\,e^{iS_0[\delta\boldsymbol{q}]}\,D\delta\boldsymbol{q} \;= e^{iS_0[\boldsymbol{q}_c]}\int e^{iS_0[\delta\boldsymbol{q}]}\,D\delta\boldsymbol{q}\end{aligned} \tag{16.41}$$

into the phase of the classical action times a path integral over loops. The loop integral L is independent of the spatial points $\boldsymbol{q}$ and $\mathbf{0}$ and so can only depend upon the time interval, $L = L(t)$. Thus the amplitude is the phase of the classical action times a function $L(t)$ of the time

$$\langle \boldsymbol{q}|e^{-i\,t\,H}|\mathbf{0}\rangle = e^{iS_0[\boldsymbol{q}_c]}\,L(t). \tag{16.42}$$

Since the classical velocity is $\dot{\boldsymbol{q}}_c = \boldsymbol{q}/t$, the classical action is

$$S_0[\boldsymbol{q}_c] = \int_0^t \frac{m}{2}\,\dot{\boldsymbol{q}}_c^2(t)\,dt = \frac{m}{2}\,\frac{\boldsymbol{q}^2}{t}. \tag{16.43}$$

So the amplitude is

$$\langle \boldsymbol{q}|e^{-i(t_2-t_1)H}|\mathbf{0}\rangle = e^{im\boldsymbol{q}^2/2t}\,L(t). \tag{16.44}$$

Because position eigenstates are orthogonal, this amplitude must reduce to a delta function as $t \to 0$

$$\lim_{t\to 0}\langle \boldsymbol{q}|e^{-i\,t\,H}|\mathbf{0}\rangle = \langle \boldsymbol{q}|\mathbf{0}\rangle = \delta^3(\boldsymbol{q}). \tag{16.45}$$

One of the many representations of Dirac's delta function is

$$\delta^3(\boldsymbol{q}) = \lim_{t\to 0}\left(\frac{m}{2\pi i\hbar t}\right)^{3/2} e^{im\boldsymbol{q}^2/2\hbar t}. \tag{16.46}$$

Thus $N\,L(t) = (m/2\pi i t)^{3/2}$ and

$$\langle \boldsymbol{q}|e^{-itH/\hbar}|\mathbf{0}\rangle = \left(\frac{m}{2\pi i\hbar t}\right)^{3/2} e^{im\boldsymbol{q}^2/2\hbar t} \tag{16.47}$$

in arbitrary units. You can verify (exercise 16.6) this result by inserting a complete set of momentum dyadics $|p\rangle\langle p|$ into the left-hand side of the amplitude (16.47) and doing the resulting Fourier transform.

Example 16.2 (The Bohm-Aharonov Effect)**.** From our formula (11.322) for the action of a relativistic particle of mass m and charge q, we infer (exercise 16.7) that the action a nonrelativistic particle in an electromagnetic field with no scalar potential is

$$S = \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2}\left[\tfrac{1}{2}m\boldsymbol{v} + q\,\boldsymbol{A}\right]\cdot d\boldsymbol{x}\,. \tag{16.48}$$

Now imagine that we shoot a beam of such particles past but not through a narrow cylinder in which a magnetic field is confined. The particles can go either way around the cylinder of area S but cannot enter the region of the magnetic field. The difference in the phases of the amplitudes is the loop integral from the source to the detector and back to the source

$$\frac{\Delta S}{\hbar} = \oint \left[\frac{m\boldsymbol{v}}{2} + q\,\boldsymbol{A} \right] \cdot \frac{\boldsymbol{dx}}{\hbar} = \oint \frac{m\boldsymbol{v}\cdot\boldsymbol{dx}}{2\hbar} + \frac{q}{\hbar}\int_S \boldsymbol{B}\cdot\boldsymbol{dS} = \oint \frac{m\boldsymbol{v}\cdot\boldsymbol{dx}}{2\hbar} + \frac{q\Phi}{\hbar} \tag{16.49}$$

in which Φ is the magnetic flux through the cylinder. □

16.6 Free Particle in Imaginary Time

If we mimic the steps of the preceding section (16.5) in which the hamiltonian is $H = \boldsymbol{p}^2/2m$, set $\beta = it/\hbar = 1/kT$, and use Dirac's delta function

$$\delta^3(\boldsymbol{q}) = \lim_{t\to 0} \left(\frac{m}{2\pi\hbar t}\right)^{3/2} e^{-m\boldsymbol{q}^2/2\hbar t} \tag{16.50}$$

then we get

$$\langle \boldsymbol{q}|e^{-\beta H}|\boldsymbol{0}\rangle = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2} \exp\left[-\frac{m\boldsymbol{q}^2}{2\hbar^2\beta}\right] = \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} e^{-mkT\boldsymbol{q}^2/2\hbar^2}. \tag{16.51}$$

For $\beta = t/\hbar$, this is the solution (3.200 & 13.124) of the diffusion equation with diffusion constant $D = \hbar/(2m)$.

16.7 Harmonic Oscillator in Real Time

Biologists have mice; physicists have harmonic oscillators with hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}. \tag{16.52}$$

For this hamiltonian, our formula (16.29) for the coordinate matrix elements of the time-evolution operator $\exp(-itH)$ is

$$\langle q_t|e^{-i\,t\,H}|q_0\rangle = \int e^{iS[q]} Dq \tag{16.53}$$

with action

$$S[q] = \int_0^t \tfrac{1}{2} m\dot{q}^2(t') - \tfrac{1}{2} m\omega^2 q^2(t')\, dt'. \tag{16.54}$$

The classical solution $q_c(t) = q_0 \cos\omega t + \dot{q}_0\, \sin(\omega t)/\omega$ in which $q_0 = q_c(0)$

and $\dot{q}_0 = \dot{q}_c(0)$ are the initial position and velocity satisfies the classical equation of motion $m\ddot{q}_c(t) = -\omega^2 q_c(t)$ and makes the action $S[q]$ stationary

$$\frac{d}{d\epsilon} S[q + \epsilon h]\bigg|_{\epsilon=0} = 0. \tag{16.55}$$

We now apply the trick (16.39–16.41) we used for the free particle. We write an arbitrary process $q(t)$ as the sum of the classical process $q_c(t)$ and a loop $\delta q(t)$ with $\delta q(0) = \delta q(t) = 0$. Since the action $S[q]$ is quadratic in the variables q and $\dot{q}$, the functional Taylor series (15.31) for $S[q_c + \delta q]$ has only two terms

$$S[q_c + \delta q] = S[q_c] + S[\delta q]. \tag{16.56}$$

Thus we can write the path integral (16.53) as

$$\begin{aligned} \langle q_t | e^{-itH} | q_0 \rangle &= \int e^{iS[q]} \, Dq = \int e^{iS[q_c + \delta q]} \, D\delta q \\ &= \int e^{iS[q_c] + iS[\delta q]} \, D\delta q = e^{iS[q_c]} \int e^{iS[\delta q]} \, D\delta q. \end{aligned} \tag{16.57}$$

The remaining path integral over the loops δq does not involve the end points q_0 and q_t and so must be a function $L(t)$ of the time t but not of q_0 or q_t

$$\langle q_t | e^{-itH} | q_0 \rangle = e^{iS[q_c]} \, L(t). \tag{16.58}$$

The action $S[q_c]$ is (exercise 16.8)

$$S[q_c] = \frac{m\omega}{2\sin(\omega t)} \left[\left(q_0^2 + q_t^2 \right) \cos(\omega t) - 2 q_0 q_t \right]. \tag{16.59}$$

The action $S[\delta q]$ of a loop

$$\delta q(t') = \sum_{j=1}^{n-1} a_j \, \sin \frac{j\pi t'}{t} \tag{16.60}$$

is (exercise 16.9)

$$S[\delta q] = \sum_{j=1}^{n-1} \frac{mt}{4} \, a_j^2 \left[\frac{(j\pi)^2}{t^2} - \omega^2 \right]. \tag{16.61}$$

The path integral over the loops is then, apart from a constant jacobian J,

$$\int e^{iS[\delta q]}\, D\delta q = J\left(\frac{nm}{2\pi it}\right)^{n/2} \int \exp\left\{\sum_{j=1}^{n-1} \frac{imt}{4}\, a_j^2 \left[\frac{(j\pi)^2}{t^2} - \omega^2\right]\right\} \prod_{j=1}^{n-1} da_j$$
$$= J\left(\frac{nm}{2\pi it}\right)^{n/2} \prod_{j=1}^{n-1} \int_{-\infty}^{\infty} \exp\left\{\frac{imt}{4}\, a_j^2 \left[\frac{(j\pi)^2}{t^2} - \omega^2\right]\right\} da_j. \tag{16.62}$$

Using the gaussian integral (16.2) and the infinite product (4.147), we get

$$\int e^{iS[\delta q]}\, D\delta q = J n^{n/2} \sqrt{\frac{m}{2\pi it}} \prod_{j=1}^{n-1} \frac{\sqrt{2}}{j\pi}\left(1 - \frac{\omega^2 t^2}{\pi^2 j^2}\right)^{-1/2}$$
$$= \sqrt{\frac{m\omega}{2\pi i \sin\omega t}} \left(\lim_{n\to\infty} J n^{n/2} \prod_{j=1}^{n-1} \frac{\sqrt{2}}{j\pi}\right). \tag{16.63}$$

From (16.58) and (16.59), we see that the number within the parentheses is unity because we then have (Feynman and Hibbs, 1965, ch. 3)

$$\langle q_t|e^{-itH/\hbar}|q_0\rangle = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega t)}} \exp\left[i\frac{m\omega\left[\left(q_0^2 + q_t^2\right)\cos(\omega t) - 2q_0 q_t\right]}{2\hbar\sin(\omega t)}\right] \tag{16.64}$$

which agrees with the amplitude (16.47) in the limit $t \to 0$ (exercise 16.10).

16.8 Harmonic Oscillator in Imaginary Time

For the harmonic oscillator with hamiltonian (16.52), our formula (16.25) for euclidian path integrals becomes

$$\langle q_\beta|e^{-\beta H}|q_0\rangle = \int \exp\left\{-\int_0^\beta \tfrac{1}{2}m\dot{q}^2(\beta') + \tfrac{1}{2}m\omega^2 q^2(\beta')\, d\beta'\right\} Dq. \tag{16.65}$$

The euclidian action, which is a β-integral of the energy of the oscillator,

$$S_e[q] = \int_0^\beta \left[\tfrac{1}{2}m\dot{q}^2(\beta') + \tfrac{1}{2}m\omega^2 q^2(\beta')\right] d\beta' \tag{16.66}$$

is purely quadratic, and so we may play a trick (15.32). We look for a path $q_e(\beta)$ that makes the euclidian action (16.66) stationary

$$\begin{aligned}
\delta S_e[q_e][h] &= \frac{d}{d\epsilon}\int_0^\beta \tfrac{1}{2}m(\dot q_e(\beta') + \epsilon \dot h(\beta'))^2 + \tfrac{1}{2}m\omega^2(q_e(\beta') + \epsilon h(\beta'))^2 \, d\beta' \Big|_{\epsilon=0} \\
&= \int_0^\beta m\dot q_e(\beta')\dot h(\beta') + m\omega^2 q_e(\beta') h(\beta')\, d\beta' \\
&= \int_0^\beta \left[-m\ddot q_e(\beta') + m\omega^2 q_e(\beta')\right] h(\beta') d\beta' = 0. \qquad (16.67)
\end{aligned}$$

The path $q_e(\beta)$ must satisfy the euclidian equation of motion

$$\ddot q_e(\beta) = \omega^2 q_e(\beta) \qquad (16.68)$$

whose general solution is

$$q_e(\beta) = A e^{\omega\beta} + B e^{-\omega\beta}. \qquad (16.69)$$

The path from $q_e(0) = q_0$ to $q_e(\beta) = q_\beta$ must have

$$A = \frac{q_\beta e^{-\omega\beta} - q_0 e^{-2\omega\beta}}{1 - e^{-2\omega\beta}} \quad \text{and} \quad B = q_0 - A. \qquad (16.70)$$

Its action $S_e[q_e]$ is (exercise 16.12)

$$S_e[q_e] = \tfrac{1}{2}m\omega\left[A^2\left(e^{2\omega\beta} - 1\right) + B^2\left(1 - e^{-2\omega\beta}\right)\right]. \qquad (16.71)$$

Since the action is purely quadratic, the trick (15.32) tells us that the action $S_e[q]$ of the arbitrary path $q(\beta) = q_e(\beta) + \delta q(\beta)$ is the sum

$$S_e[q] = S_e[q_e] + S_e[\delta q] \qquad (16.72)$$

in which the action $S_e[\delta q]$ of the loop $\delta q(\beta)$ depends upon β but not upon q_β or q_0. It follows then that for some loop function $L(\beta)$ of β alone

$$\begin{aligned}
\langle q_\beta | e^{-\beta H} | q_0\rangle &= \exp\left(-S_e[q_e]\right)\, L(\beta) \qquad (16.73) \\
&= L(\beta)\, \exp\Big\{ -\tfrac{1}{2}m\omega\left[A^2\left(e^{2\omega\beta} - 1\right) + B^2\left(1 - e^{-2\omega\beta}\right)\right]\Big\}.
\end{aligned}$$

To study the ground state of the harmonic oscillator, we let $\beta \to \infty$ in this equation. Inserting a complete set of eigenstates $H|n\rangle = E_n|n\rangle$, we see that the limit of the left-hand side is

$$\lim_{\beta\to\infty} \langle q_\beta | e^{-\beta H} | q_0\rangle = \lim_{\beta\to\infty} \langle q_\beta | n\rangle e^{-\beta E_n} \langle n | q_0\rangle = e^{-\beta E_0}\langle q_\beta|0\rangle\langle 0|q_0\rangle. \qquad (16.74)$$

Our formulas (16.70) for A and B say that $A \to q_\beta e^{-\omega\beta}$ and $B \to q_0$ as $\beta \to \infty$, and so in this limit by (16.73 & 16.74) we have

$$e^{-\beta E_0}\langle q_\beta|0\rangle\langle 0|q_0\rangle = L(\beta)\,\exp\left[-\tfrac{1}{2}m\omega\left(q_\beta^2 + q_0^2\right)\right] \tag{16.75}$$

from which may infer our earlier formula (15.44) for the wave function of the ground state

$$\langle q|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\,\exp\left[-\frac{1}{2}\frac{m\omega q^2}{\hbar}\right] \tag{16.76}$$

in which the prefactor ensures the normalization

$$1 = \int_{-\infty}^{\infty} |\langle q|0\rangle|^2 dq. \tag{16.77}$$

euclidian path integrals help one study ground states.

16.9 Euclidian Correlation Functions

In the Heisenberg picture, the position operator $q(t)$ is

$$q(t) = e^{itH/\hbar} q\, e^{-itH/\hbar} \tag{16.78}$$

in which $q = q(0)$ is the position operator at time $t = 0$ or equivalently the position operator in the Schrödinger picture. The analogous operator in imaginary time $t = -i\hbar\beta = -i\hbar/(kT)$ is the euclidian position operator $q_e(\beta)$ defined as

$$q_e(\beta) = e^{\beta H} q\, e^{-\beta H}. \tag{16.79}$$

The **euclidian-time-ordered** product of two euclidian position operators is

$$\mathcal{T}\left[q_e(\beta_1)\, q_e(\beta_2)\right] = \theta(\beta_1 - \beta_2) q_e(\beta_1)\, q_e(\beta_2) + \theta(\beta_2 - \beta_1) q_e(\beta_2)\, q_e(\beta_1) \tag{16.80}$$

in which $\theta(x) = (x + |x|)/2|x|$ is Heaviside's function. We can use the method of section 16.3 to compute the matrix element of the euclidian-time-ordered product $\mathcal{T}\left[q(\beta_1)q(\beta_2)\right]$ sandwiched between two factors of $\exp(-\beta H)$. For $\beta_1 \ge \beta_2$, this matrix element is

$$\langle q_\beta|e^{-\beta H} q_e(\beta_1) q_e(\beta_2) e^{-\beta H}|q_{-\beta}\rangle = \langle q_\beta|e^{-(\beta-\beta_1)H} q e^{-(\beta_1-\beta_2)H} q e^{-(\beta+\beta_2)H}|q_{-\beta}\rangle. \tag{16.81}$$

Then instead of the path-integral formula (16.25), we get

$$\langle q_\beta|e^{-\beta H}\mathcal{T}\left[q_e(\beta_1) q_e(\beta_2)\right] e^{-\beta H}|q_{-\beta}\rangle = \int q(\beta_1) q(\beta_2) e^{-S_e[q,\beta,-\beta]}\, Dq \tag{16.82}$$

where as in (16.25) $S_e[q, \beta, -\beta]$ is the **euclidian action**

$$S_e[q, \beta, -\beta] = \int_{-\beta}^{\beta} \tfrac{1}{2} m \dot{q}^2(\beta') + V(q(\beta'))\, d\beta' \tag{16.83}$$

or the time integral of the energy. As in the path integral (16.25), the integration is over all paths that go from $q(-\beta) = q_{-\beta}$ to $q(\beta) = q_\beta$. The analog of the single-particle path integral (16.25) is

$$\langle q_\beta | e^{-2\beta H} | q_{-\beta} \rangle = \int e^{-S_e[q,\beta,-\beta]}\, Dq \tag{16.84}$$

and the factors $(nm/2\pi\beta)^{n/2}$ cancel in the ratio of (16.82) to (16.84)

$$\frac{\langle q_\beta | e^{-\beta H} \mathcal{T}\left[q_e(\beta_1) q_e(\beta_2)\right] e^{-\beta H} | q_{-\beta} \rangle}{\langle q_\beta | e^{-2\beta H} | q_{-\beta} \rangle} = \frac{\displaystyle\int q(\beta_1) q(\beta_2) e^{-S_e[q,\beta,-\beta]}\, Dq}{\displaystyle\int e^{-S_e[q,\beta,-\beta]}\, Dq}. \tag{16.85}$$

In the limit $\beta \to \infty$, the operator $\exp(-\beta H)$ projects out the ground state $|0\rangle$ of the system

$$\lim_{\beta\to\infty} e^{-\beta H} | q_{-\beta} \rangle = \lim_{\beta\to\infty} \sum_{n=0}^{\infty} e^{-\beta H} |n\rangle \langle n | q_{-\beta} \rangle = \lim_{\beta\to\infty} e^{-\beta E_0} |0\rangle \langle 0 | q_{-\beta} \rangle \tag{16.86}$$

which we assume to be unique and normalized to unity. In the ratio (16.85), most of these factors cancel, leaving us with

$$\langle 0 | \mathcal{T}\left[q_e(\beta_1) q_e(\beta_2)\right] |0\rangle = \frac{\displaystyle\int q(\beta_1) q(\beta_2) e^{-S_e[q,\infty,-\infty]}\, Dq}{\displaystyle\int e^{-S_e[q,\infty,-\infty]}\, Dq}. \tag{16.87}$$

More generally, the mean value in the ground state $|0\rangle$ of *any* euclidian-time-ordered product of position operators $q(t_i)$ is a ratio of path integrals

$$\langle 0 | \mathcal{T}\left[q(\beta_1) \dots q(\beta_n)\right] |0\rangle = \frac{\displaystyle\int q(\beta_1) \dots q(\beta_n)\, e^{-S_e[q]}\, Dq}{\displaystyle\int e^{-S_e[q]}\, Dq} \tag{16.88}$$

in which $S_e[q]$ stands for $S_e[q, \infty, -\infty]$. Why do we need the time-ordered product $\mathcal{T}$ on the LHS? Because successive factors of $\exp(-(\beta_k - \beta_\ell)H)$ lead to the path integral of $\exp(-S_e[q])$. Why don't we need $\mathcal{T}$ on the RHS? Because the $q(\beta_i)$'s are real numbers which commute with each other.

The result (16.88) is important because it can be generalized to all quantum theories, including field theories.

16.10 Finite-Temperature Field Theory

Matrix elements of the operator $\exp(-\beta H)$ where $\beta = 1/kT$ tell us what a system is like at temperature T. In the low-temperature limit, they describe the ground state of the system.

Quantum mechanics imposes upon n coordinates q_i and conjugate momenta π_k the commutation relations

$$[q_i, p_k] = i\,\delta_{i,k} \quad \text{and} \quad [q_i, q_k] = [p_i, p_k] = 0. \tag{16.89}$$

In quantum field theory, we associate a coordinate $q_{\boldsymbol{x}} \equiv \phi(\boldsymbol{x})$ and a conjugate momentum $p_{\boldsymbol{x}} \equiv \pi(\boldsymbol{x})$ with each point $\boldsymbol{x}$ of space and impose upon them the very similar commutation relations

$$[\phi(\boldsymbol{x}), \pi(\boldsymbol{y})] = i\,\delta(\boldsymbol{x}-\boldsymbol{y}) \quad \text{and} \quad [\phi(\boldsymbol{x}), \phi(\boldsymbol{y})] = [\pi(\boldsymbol{x}), \pi(\boldsymbol{y})] = 0. \tag{16.90}$$

Just as in quantum mechanics the time derivative of a coordinate is its commutator with a hamiltonian $\dot{q}_i = i[H, q_i]$, so too in quantum field theory the time derivative of a field $\dot{\phi}(\boldsymbol{x}, t) \equiv \dot{\phi}(x)$ is $\dot{\phi}(x) = i[H, \phi(x)]$. A typical hamiltonian for a single scalar field ϕ is

$$H = \int \left[\frac{1}{2}\pi^2(\boldsymbol{x}) + \frac{1}{2}\left(\nabla\phi(\boldsymbol{x})\right)^2 + \frac{1}{2}m^2\phi^2(\boldsymbol{x}) + P(\phi(\boldsymbol{x}))\right] d^3x \tag{16.91}$$

in which P is a quartic polynomial.

Since quantum field theory is just the quantum mechanics of many variables, we can use the methods of sections 16.3 & 16.4 to write matrix elements of $\exp(-\beta H)$ as path integrals. We define a potential

$$V(\phi(\boldsymbol{x})) = \frac{1}{2}\left(\nabla\phi(\boldsymbol{x})\right)^2 + \frac{1}{2}m^2\phi^2(\boldsymbol{x}) + P(\phi(\boldsymbol{x})) \tag{16.92}$$

and write the hamiltonian H as

$$H = \int \left[\frac{1}{2}\pi^2(\boldsymbol{x}) + V(\phi(\boldsymbol{x}))\right] d^3x. \tag{16.93}$$

Like $|q'\rangle$ and $|p'\rangle$, the states $|\phi'\rangle$ and $|\pi'\rangle$ are eigenstates of the hermitian operators $\phi(\boldsymbol{x}, 0)$ and $\pi(\boldsymbol{x}, 0)$

$$\phi(\boldsymbol{x}, 0)|\phi'\rangle = \phi'(\boldsymbol{x})|\phi'\rangle \quad \text{and} \quad \pi(\boldsymbol{x}, 0)|\pi'\rangle = \pi'(\boldsymbol{x})|\pi'\rangle. \tag{16.94}$$

The analog of $\langle q'|p'\rangle$ is

$$\langle \phi'|\pi'\rangle = f\ \exp\left[i\int \phi'(\boldsymbol{x})\pi'(\boldsymbol{x})d^3x\right] \tag{16.95}$$

in which f is a factor which eventually will cancel.

Repeating our derivation of Eq.(16.21) with

$$D\pi' \equiv \prod_{\boldsymbol{x}} d\pi'(\boldsymbol{x}) \tag{16.96}$$

we find in the limit $\epsilon \to 0$

$$\begin{aligned}
\langle\phi_1|\exp(-\epsilon H)|\phi_0\rangle &= \int \langle\phi_1|\exp\left(-\frac{\epsilon}{2}\int \pi^2(\boldsymbol{x})d^3x\right)|\pi'\rangle \\
&\qquad \times \langle\pi'|\exp\left(-\epsilon\int V(\phi(\boldsymbol{x}))\,d^3x\right)|\phi_0\rangle\, D\pi' \\
&= |f|^2\ \exp\left(-\epsilon\int V(\phi_0(\boldsymbol{x}))\,d^3x\right) \qquad (16.97)\\
&\quad \times\int \exp\left[\int -\tfrac{1}{2}\epsilon\pi'^2(\boldsymbol{x}) + i\pi'(\boldsymbol{x})[\phi_1(\boldsymbol{x})-\phi_0(\boldsymbol{x})]d^3x\right]D\pi'.
\end{aligned}$$

Using the abbreviation

$$\dot\phi_0(\boldsymbol{x}) \equiv \frac{\phi_1(\boldsymbol{x})-\phi_0(\boldsymbol{x})}{\epsilon} \tag{16.98}$$

and the integral formula (16.6), we get

$$\langle\phi_1|\exp(-\epsilon H)|\phi_0\rangle = f'\ \exp\left\{-\epsilon\int\left[\tfrac{1}{2}\,\dot\phi_0^2(\boldsymbol{x}) + V(\phi_0(\boldsymbol{x}))\right]d^3x\right\}.$$

Putting together $n = \beta/\epsilon$ such terms, integrating over the intermediate states $|\phi_j\rangle\langle\phi_j|$ from $j = 1$ to $j = n-1$ and absorbing the normalizing factors into $D\phi$, we get

$$\langle\phi_\beta|e^{-\beta H}|\phi_0\rangle = \int_{\phi_0}^{\phi_\beta}\exp\left[-\int_0^\beta\int \tfrac{1}{2}\dot\phi^2(x) + V(\phi(x))\,d^3x\,d\beta'\right]D\phi. \tag{16.99}$$

Replacing the potential $V(\phi)$ with its definition (16.92), we have

$$\langle\phi_\beta|e^{-\beta H}|\phi_0\rangle = \int_{\phi_0}^{\phi_\beta}\exp\left[-\int_0^\beta\int \tfrac{1}{2}\left[\dot\phi^2 + (\nabla\phi)^2 + m^2\phi^2\right] + P(\phi)\,d^3x\,d\beta'\right]D\phi \tag{16.100}$$

in which the limits ϕ_0 and ϕ_β remind us that we are to integrate over all fields $\phi(\boldsymbol{x},t)$ that run from $\phi(\boldsymbol{x},0) = \phi_0(\boldsymbol{x})$ to $\phi(\boldsymbol{x},\beta) = \phi_\beta(\boldsymbol{x})$.

In terms of the energy density

$$\mathcal{H}(\phi) \equiv \tfrac{1}{2}\left[(\partial_a \phi)^2 + m^2\phi^2\right] + P(\phi) \tag{16.101}$$

in which $\partial_a = (\partial/\partial\beta, \nabla)$, the path integral (16.100) is

$$\langle\phi_\beta|e^{-\beta H}|\phi_0\rangle = \int_{\phi_0}^{\phi_\beta} \exp\left[-\int_0^\beta\!\!\int \mathcal{H}(\phi)\, d^3x\, d\beta'\right] D\phi. \tag{16.102}$$

The partition function $Z(\beta)$—defined as the trace $Z(\beta) \equiv \operatorname{Tr} e^{-\beta H}$ over all the states of the system—is an integral over all periodic ($\phi_\beta = \phi_0$) fields

$$Z(\beta) \equiv \operatorname{Tr} e^{-\beta H} = \int_{\phi_0}^{\phi_0} \langle\phi|e^{-\beta H}|\phi\rangle D\phi = \int_{\phi_0}^{\phi_0} \exp\left[-\int_0^\beta\!\!\int \mathcal{H}(\phi)\, d^3x\, d\beta'\right] D\phi. \tag{16.103}$$

Because the squares of the four derivatives ∂_a occur with the same sign, finite-temperature field theory is called **euclidian** quantum field theory.

Like the definition (16.79) of the euclidian position operator $q_e(\beta)$, the euclidian field operator $\phi_e(x)$ is defined as

$$\phi_e(\boldsymbol{x}, \beta) = e^{\beta H}\phi(\boldsymbol{x}, 0)e^{-\beta H}. \tag{16.104}$$

The **euclidian-time-ordered product** (16.80) of two fields is

$$\begin{aligned}\mathcal{T}\left[\phi_e(\boldsymbol{x}_1,\beta_1)\phi_e(\boldsymbol{x}_2,\beta_2)\right] &= \theta(\beta_1-\beta_2)e^{\beta_1 H}\phi_e(\boldsymbol{x}_1,0)e^{-(\beta_1-\beta_2)H}\phi_e(\boldsymbol{x}_2,0)e^{-\beta_2 H}\\ &\quad + \theta(\beta_2-\beta_1)e^{\beta_2 H}\phi_e(\boldsymbol{x}_2,0)e^{-(\beta_2-\beta_1)H}\phi_e(\boldsymbol{x}_1,0)e^{-\beta_1 H}.\end{aligned}$$

The logic of equations (16.79–16.88) leads us to write its mean value in a system described by a maximum-entropy-at-inverse-temperature-β density operator $\rho = e^{-\beta H}/Z(\beta)$ as the ratio

$$\begin{aligned}\langle\mathcal{T}\left[\phi_e(x_1)\phi_e(x_2)\right]\rangle &= \operatorname{Tr}\left\{\rho\,\mathcal{T}\left[\phi_e(x_1)\phi_e(x_2)\right]\right\}\\ &= \frac{\operatorname{Tr}\left\{e^{-\beta H}\,\mathcal{T}\left[\phi_e(x_1)\phi_e(x_2)\right]\right\}}{\operatorname{Tr}\left[\,e^{-\beta H}\right]}\\ &= \frac{\displaystyle\int_{\phi_0}^{\phi_0} \phi(x_1)\phi(x_2)\exp\left[-\int_0^\beta\!\!\int \mathcal{H}(\phi)\, d^3x\, d\beta'\right] D\phi}{\displaystyle\int_{\phi_0}^{\phi_0} \exp\left[-\int_0^\beta\!\!\int \mathcal{H}(\phi)\, d^3x\, d\beta'\right] D\phi}\end{aligned} \tag{16.105}$$

in which several factors have canceled.

In the zero-temperature ($\beta \to \infty$) limit, the density operator ρ becomes the projection operator $|0\rangle\langle 0|$ on the ground state, and mean-value formulas like (16.105) become

$$\langle 0|\mathcal{T}\left[\phi_e(x_1)\dots\phi_e(x_n)\right]|0\rangle = \frac{\displaystyle\int \phi(x_1)\dots\phi(x_n)\,\exp\left[-\int \mathcal{H}(\phi)\,d^4x\right] D\phi}{\displaystyle\int \exp\left[-\int \mathcal{H}(\phi)\,d^4x\right] D\phi} \tag{16.106}$$

in which Hamilton's density $\mathcal{H}(\phi)$ is integrated over all of euclidian spacetime and over all fields that are periodic on the infinite time interval. Statistical field theory and lattice gauge theory use formulas like (16.105 & 16.106).

16.11 Real-Time Field Theory

We can apply the derivation of section 16.10 to amplitudes in real time. In the amplitude (16.97), we replace $-\epsilon H$ by $-i\epsilon H$ and follow the logic of sections (16.4 & 16.10). We find in the limit $\epsilon \to 0$ with $\dot\phi_0 \equiv (\phi_1 - \phi_0)/\epsilon$

$$\begin{aligned}\langle\phi_1|e^{-i\epsilon H}|\phi_0\rangle &= \int \langle\phi_1|e^{-i\epsilon\int \pi^2/2\,d^3x}|\pi'\rangle\langle\pi'|e^{-i\epsilon\int V(\phi)\,d^3x}|\phi_0\rangle D\pi' \\ &= |f|^2 e^{-i\epsilon\int V(\phi_0)d^3x}\int e^{-i\int \epsilon\pi'^2/2+i\pi'(\phi_1-\phi_0)\,d^3x} D\pi' \\ &= f'\,\exp\left[i\epsilon\int \tfrac{1}{2}\dot\phi_0^2 - V(\phi_0)\,d^3x\right].\end{aligned} \tag{16.107}$$

Putting together $2t/\epsilon$ similar factors and integrating over all the intermediate states $|\phi_j\rangle\langle\phi_j|$, we arrive at the path integral

$$\langle\phi_t|e^{-i2tH}|\phi_{-t}\rangle = \int_{\phi_{-t}}^{\phi_t} \exp\left[i\int \tfrac{1}{2}\dot\phi^2(x) - V(\phi(x))\,d^4x\right] D\phi \tag{16.108}$$

in which we integrate over all fields $\phi(x)$ that run from $\phi_{-t}(\boldsymbol{x},-t)$ to $\phi_t(\boldsymbol{x},t)$. After expanding the definition (16.92) of the potential $V(\phi)$, we have

$$\langle\phi_t|e^{-i2tH}|\phi_{-t}\rangle = \int_{\phi_{-t}}^{\phi_t} \exp\left[i\int \tfrac{1}{2}\left[\dot\phi^2 - (\boldsymbol{\nabla}\phi)^2 - m^2\phi^2\right] - P(\phi)\,d^4x\right] D\phi. \tag{16.109}$$

This amplitude is a path integral

$$\langle\phi_t|e^{-i2tH/\hbar}|\phi_{-t}\rangle = \int_{\phi_{-t}}^{\phi_t} e^{iS[\phi]/\hbar}\,D\phi \tag{16.110}$$

of phases $\exp(iS[\phi]/\hbar)$ that are exponentials of the classical action

$$S[\phi] = \int \tfrac{1}{2}\left[\dot\phi^2 - (\boldsymbol{\nabla}\phi)^2 - m^2\phi^2\right] - P(\phi)\,d^4x. \tag{16.111}$$

The time dependence of the Heisenberg field operator $\phi(\boldsymbol{x}, t)$ is

$$\phi(\boldsymbol{x}, t) = e^{itH/\hbar}\phi(\boldsymbol{x}, 0)e^{-itH/\hbar}. \tag{16.112}$$

The **time-ordered product** of two fields, as in (16.80), is the sum

$$\mathcal{T}\left[\phi(x_1)\phi(x_2)\right] = \theta(x_1^0 - x_2^0)\phi(x_1)\phi(x_2) + \theta(x_2^0 - x_1^0)\phi(x_2)\phi(x_1). \tag{16.113}$$

Between two factors of $\exp(-itH)$, it is for $t_1 > t_2$

$$e^{-itH}\mathcal{T}\left[\phi(x_1)\phi(x_2)\right]e^{-itH} = e^{-i(t-t_1)H}\phi(\boldsymbol{x}_1, 0)e^{-i(t_1-t_2)H}\phi(\boldsymbol{x}_2, 0)e^{-i(t+t_2)H}.$$

So by the logic that led to the path-integral formulas (16.105) and (16.110), we can write a matrix element of the time-ordered product (16.113) as

$$\langle\phi_t|e^{-itH}\mathcal{T}\left[\phi(x_1)\phi(x_2)\right]e^{-itH}|\phi_{-t}\rangle = \int_{\phi_{-t}}^{\phi_t}\phi(x_1)\,\phi(x_2)\,e^{iS[\phi]}\,D\phi \tag{16.114}$$

in which we integrate over fields that go from ϕ_{-t} at time $-t$ to ϕ_t at time t. The time-ordered product of any combination of fields is then

$$\langle\phi_t|e^{-itH}\mathcal{T}\left[\phi(x_1)\ldots\phi(x_n)\right]e^{-itH}|\phi_{-t}\rangle = \int\phi(x_1)\ldots\phi(x_n)\,e^{iS[\phi]}\,D\phi. \tag{16.115}$$

Like the position eigenstates $|q'\rangle$ of quantum mechanics, the eigenstates $|\phi'\rangle$ are states of infinite energy that overlap many states, whereas we often are interested in the ground state $|0\rangle$ or in states of a few particles. To form such matrix elements, we multiply both sides of equations (16.110 & 16.115) by $\langle 0|\phi_t\rangle\langle\phi_{-t}|0\rangle$ and integrate over ϕ_{-t} and ϕ_t. Since the ground state is a normalized eigenstate of the hamiltonian $H|0\rangle = E_0|0\rangle$ with eigenvalue E_0, we find from the path-integral formulas (16.110)

$$\begin{aligned}\int\langle 0|\phi_t\rangle\langle\phi_t|e^{-i2tH}|\phi_{-t}\rangle\langle\phi_{-t}|0\rangle D\phi_t D\phi_{-t} &= \langle 0|e^{-i2tH}|0\rangle \qquad (16.116)\\ &= e^{-i2tE_0} = \int\langle 0|\phi_t\rangle e^{iS[\phi]}\langle\phi_{-t}|0\rangle\,D\phi D\phi_t D\phi_{-t}\end{aligned}$$

and (16.115) (suppressing the differentials $D\phi_t D\phi_{-t}$)

$$e^{-2itE_0}\langle 0|\mathcal{T}\left[\phi(x_1)\ldots\phi(x_n)\right]|0\rangle = \int\langle 0|\phi_t\rangle\phi(x_1)\ldots\phi(x_n)\,e^{iS[\phi]}\langle\phi_{-t}|0\rangle\,D\phi. \tag{16.117}$$

The mean value in the ground state of a time-ordered product of field operators is then a ratio of these path integrals

$$\langle 0|\mathcal{T}\left[\phi(x_1)\ldots\phi(x_n)\right]|0\rangle = \frac{\displaystyle\int \langle 0|\phi_t\rangle\, \phi(x_1)\ldots\phi(x_n)\, e^{iS[\phi]}\langle\phi_{-t}|0\rangle\, D\phi}{\displaystyle\int \langle 0|\phi_t\rangle\, e^{iS[\phi]}\langle\phi_{-t}|0\rangle\, D\phi} \tag{16.118}$$

in which factors involving E_0 have canceled and the integration is over all fields that go from $\phi(\boldsymbol{x},-t) = \phi_{-t}(\boldsymbol{x})$ to $\phi(\boldsymbol{x},t) = \phi_t(\boldsymbol{x})$ and over $\phi_{-t}(\boldsymbol{x})$ and $\phi_t(\boldsymbol{x})$.

16.12 Perturbation Theory

Field theories with hamiltonians that are quadratic in their fields like

$$H_0 = \int \tfrac{1}{2}\left[\pi^2(x) + (\nabla\phi(x))^2 + m^2\phi^2(x)\right] d^3x \tag{16.119}$$

are soluble. Their fields evolve in time as

$$\phi(\boldsymbol{x},t) = e^{itH_0}\phi(\boldsymbol{x},0)e^{-itH_0}. \tag{16.120}$$

The mean value in the ground state of H_0 of a time-ordered product of these fields is by (16.118) a ratio of path integrals

$$\langle 0|\mathcal{T}\left[\phi(x_1)\ldots\phi(x_n)\right]|0\rangle = \frac{\displaystyle\int \langle 0|\phi_t\rangle\, \phi(x_1)\ldots\phi(x_n)\, e^{iS_0[\phi]}\langle\phi'|0\rangle\, D\phi}{\displaystyle\int \langle 0|\phi''\rangle\, e^{iS_0[\phi]}\langle\phi'|0\rangle\, D\phi} \tag{16.121}$$

in which the action $S_0[\phi]$ is quadratic in the fields

$$\begin{aligned} S_0[\phi] &= \int \tfrac{1}{2}\left[\dot\phi^2(x) - (\nabla\phi(x))^2 - m^2\phi^2(x)\right] d^4x \\ &= \int \tfrac{1}{2}\left[-\partial_a\phi(x)\partial^a\phi(x) - m^2\phi^2(x)\right] d^4x. \end{aligned} \tag{16.122}$$

So the path integrals in the ratio (16.121) are gaussian and doable.

The Fourier transforms

$$\tilde\phi(p) = \int e^{-ipx}\phi(x)\, d^4x \quad \text{and} \quad \phi(x) = \int e^{ipx}\tilde\phi(p)\, \frac{d^4p}{(2\pi)^4} \tag{16.123}$$

turn the spacetime derivatives in the action into a quadratic form

$$S_0[\phi] = -\tfrac{1}{2}\int |\tilde\phi(p)|^2\, (p^2+m^2)\, \frac{d^4p}{(2\pi)^4} \tag{16.124}$$

in which $p^2 = \boldsymbol{p}^2 - p^{02}$, and $\tilde\phi(-p) = \tilde\phi^*(p)$ by (3.25) since the field ϕ is real.

The initial $\langle \phi_{-t}|0\rangle$ and final $\langle 0|\phi_t\rangle$ wave functions produce the $i\epsilon$ in the Feynman propagator (5.236). Although its exact form doesn't matter here, the wave function $\langle \phi|0\rangle$ of the ground state of H_0 is the exponential (15.53)

$$\langle \phi|0\rangle = c \exp\left[-\frac{1}{2}\int |\tilde\phi(\boldsymbol{p})|^2 \sqrt{\boldsymbol{p}^2 + m^2}\,\frac{d^3p}{(2\pi)^3}\right] \tag{16.125}$$

in which $\tilde\phi(\boldsymbol{p})$ is the spatial Fourier transform of the eigenvalue $\phi(\boldsymbol{x})$

$$\tilde\phi(\boldsymbol{p}) = \int e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\,\phi(\boldsymbol{x})\,d^3x \tag{16.126}$$

and c is a normalization factor that will cancel in ratios of path integrals.

Apart from $-2i\ln c$ which we will not keep track of, the wave functions $\langle \phi_{-t}|0\rangle$ and $\langle 0|\phi_t\rangle$ add to the action $S_0[\phi]$ the term

$$\Delta S_0[\phi] = \frac{i}{2}\int \sqrt{\boldsymbol{p}^2 + m^2}\left(|\tilde\phi(\boldsymbol{p},t)|^2 + |\tilde\phi(\boldsymbol{p},-t)|^2\right)\frac{d^3p}{(2\pi)^3} \tag{16.127}$$

in which we envision taking the limit $t \to \infty$ with $\phi(\boldsymbol{x},t) = \phi_t(\boldsymbol{x})$ and $\phi(\boldsymbol{x},-t) = \phi_{-t}(\boldsymbol{x})$. The identity (Weinberg, 1995, pp. 386–388)

$$f(+\infty) + f(-\infty) = \lim_{\epsilon\to 0+}\epsilon\int_{-\infty}^{\infty} f(t)\,e^{-\epsilon|t|}\,dt \tag{16.128}$$

allows us to write $\Delta S_0[\phi]$ as

$$\Delta S_0[\phi] = \lim_{\epsilon\to 0+}\frac{i\epsilon}{2}\int \sqrt{\boldsymbol{p}^2 + m^2}\int_{-\infty}^{\infty} |\tilde\phi(\boldsymbol{p},t)|^2\,e^{-\epsilon|t|}\,dt\,\frac{d^3p}{(2\pi)^3}. \tag{16.129}$$

To first order in ϵ, the change in the action is (exercise 16.15)

$$\begin{aligned}\Delta S_0[\phi] &= \lim_{\epsilon\to 0+}\frac{i\epsilon}{2}\int \sqrt{\boldsymbol{p}^2 + m^2}\int_{-\infty}^{\infty} |\tilde\phi(\boldsymbol{p},t)|^2\,dt\,\frac{d^3p}{(2\pi)^3}\\ &= \lim_{\epsilon\to 0+}\frac{i\epsilon}{2}\int \sqrt{\boldsymbol{p}^2 + m^2}\,|\tilde\phi(p)|^2\,\frac{d^4p}{(2\pi)^4}.\end{aligned} \tag{16.130}$$

Thus the modified action is

$$\begin{aligned}S_0[\phi,\epsilon] = S_0[\phi] + \Delta S_0[\phi] &= -\frac{1}{2}\int |\tilde\phi(p)|^2\left(p^2 + m^2 - i\epsilon\sqrt{\boldsymbol{p}^2 + m^2}\right)\frac{d^4p}{(2\pi)^4}\\ &= -\frac{1}{2}\int |\tilde\phi(p)|^2\left(p^2 + m^2 - i\epsilon\right)\frac{d^4p}{(2\pi)^4}\end{aligned} \tag{16.131}$$

since the square root is positive. In terms of the modified action, our formula

(16.121) for the time-ordered product is the ratio

$$\langle 0|\mathcal{T}\left[\phi(x_1)\dots\phi(x_n)\right]|0\rangle = \frac{\displaystyle\int \phi(x_1)\dots\phi(x_n)\, e^{iS_0[\phi,\epsilon]}\, D\phi}{\displaystyle\int e^{iS_0[\phi,\epsilon]}\, D\phi}. \tag{16.132}$$

We can use this formula (16.132) to express the mean value in the vacuum $|0\rangle$ of the time-ordered exponential of a spacetime integral of $j(x)\phi(x)$, in which $j(x)$ is a classical (c-number, external) current, as the ratio

$$\begin{aligned} Z_0[j] &\equiv \langle 0|\,\mathcal{T}\left\{\exp\left[i\int j(x)\,\phi(x)\,d^4x\right]\right\}|0\rangle \\ &= \frac{\displaystyle\int \exp\left[i\int j(x)\,\phi(x)\,d^4x\right] e^{iS_0[\phi,\epsilon]}\, D\phi}{\displaystyle\int e^{iS_0[\phi,\epsilon]}\, D\phi}. \end{aligned} \tag{16.133}$$

Since the state $|0\rangle$ is normalized, the mean value $Z_0[0]$ is unity,

$$Z_0[0] = 1. \tag{16.134}$$

If we absorb the current into the action

$$S_0[\phi,\epsilon,j] = S_0[\phi,\epsilon] + \int j(x)\,\phi(x)\,d^4x \tag{16.135}$$

then in terms of the current's Fourier transform

$$\tilde{j}(p) = \int e^{-ipx}\, j(x)\, d^4x \tag{16.136}$$

the modified action $S_0[\phi,\epsilon,j]$ is (exercise 16.16)

$$S_0[\phi,\epsilon,j] = -\tfrac{1}{2}\int\left[|\tilde{\phi}(p)|^2\left(p^2+m^2-i\epsilon\right) - \tilde{j}^*(p)\tilde{\phi}(p) - \tilde{\phi}^*(p)\tilde{j}(p)\right]\frac{d^4p}{(2\pi)^4}. \tag{16.137}$$

Changing variables to

$$\tilde{\psi}(p) = \tilde{\phi}(p) - \tilde{j}(p)/(p^2+m^2-i\epsilon) \tag{16.138}$$

we write the action $S_0[\phi,\epsilon,j]$ as (exercise 16.17)

$$\begin{aligned} S_0[\phi,\epsilon,j] &= -\tfrac{1}{2}\int\left[|\tilde{\psi}(p)|^2\left(p^2+m^2-i\epsilon\right) - \frac{\tilde{j}^*(p)\tilde{j}(p)}{(p^2+m^2-i\epsilon)}\right]\frac{d^4p}{(2\pi)^4} \\ &= S_0[\psi,\epsilon] + \tfrac{1}{2}\int\left[\frac{\tilde{j}^*(p)\tilde{j}(p)}{(p^2+m^2-i\epsilon)}\right]\frac{d^4p}{(2\pi)^4}. \end{aligned} \tag{16.139}$$

And since $D\phi = D\psi$, our formula (16.133) gives simply (exercise 16.18)

$$Z_0[j] = \exp\left(\frac{i}{2}\int \frac{|\tilde{j}(p)|^2}{p^2+m^2-i\epsilon}\frac{d^4p}{(2\pi)^4}\right). \tag{16.140}$$

Going back to position space, one finds (exercise 16.19)

$$Z_0[j] = \exp\left[\frac{i}{2}\int j(x)\,\Delta(x-x')\,j(x')\,d^4x\,d^4x'\right] \tag{16.141}$$

in which $\Delta(x-x')$ is Feynman's **propagator** (5.236)

$$\Delta(x-x') = \int \frac{e^{ip(x-x')}}{p^2+m^2-i\epsilon}\,\frac{d^4p}{(2\pi)^4}. \tag{16.142}$$

The functional derivative (chapter 15) of $Z_0[j]$, defined by (16.133), is

$$\frac{1}{i}\frac{\delta Z_0[j]}{\delta j(x)} = \langle 0|\,\mathcal{T}\left[\phi(x)\exp\left(i\int j(x')\phi(x')d^4x'\right)\right]|0\rangle \tag{16.143}$$

while that of equation (16.141) is

$$\frac{1}{i}\frac{\delta Z_0[j]}{\delta j(x)} = Z_0[j]\int \Delta(x-x')\,j(x')\,d^4x'. \tag{16.144}$$

Thus the second functional derivative of $Z_0[j]$ evaluated at $j=0$ gives

$$\langle 0|\,\mathcal{T}\left[\phi(x)\phi(x')\right]|0\rangle = \frac{1}{i^2}\frac{\delta^2 Z_0[j]}{\delta j(x)\delta j(x')}\bigg|_{j=0} = -i\,\Delta(x-x'). \tag{16.145}$$

Similarly, one may show (exercise 16.20) that

$$\begin{aligned}\langle 0|\,\mathcal{T}\left[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right]|0\rangle &= \frac{1}{i^4}\frac{\delta^4 Z_0[j]}{\delta j(x_1)\delta j(x_2)\delta j(x_3)\delta j(x_4)}\bigg|_{j=0}\\ &= -\Delta(x_1-x_2)\Delta(x_3-x_4) - \Delta(x_1-x_3)\Delta(x_2-x_4)\\ &\quad - \Delta(x_1-x_4)\Delta(x_2-x_3).\end{aligned} \tag{16.146}$$

Suppose now that we add a potential $V = P(\phi)$ to the free hamiltonian (16.119). Scattering amplitudes are matrix elements of the time-ordered exponential $\mathcal{T}\exp\left[-i\int P(\phi)\,d^4x\right]$. Our formula (16.132) for the mean value in the ground state $|0\rangle$ of the free hamiltonian H_0 of any time-ordered product of fields leads us to

$$\langle 0|\mathcal{T}\left\{\exp\left[-i\int P(\phi)\,d^4x\right]\right\}|0\rangle = \frac{\int \exp\left[-i\int P(\phi)\,d^4x\right]\,e^{iS_0[\phi,\epsilon]}\,D\phi}{\int e^{iS_0[\phi,\epsilon]}\,D\phi}. \tag{16.147}$$

Using (16.145 & 16.146), we can cast this expression into the magical form

$$\langle 0|\mathcal{T}\left\{\exp\left[-i\int P(\phi)\,d^4x\right]\right\}|0\rangle = \exp\left[-i\int P\left(\frac{\delta}{i\delta j(x)}\right)\,d^4x\right] Z_0[j]\Big|_{j=0}. \tag{16.148}$$

The generalization of the path-integral formula (16.132) to the ground state $|\Omega\rangle$ of an interacting theory with action S is

$$\langle\Omega|\mathcal{T}\left[\phi(x_1)\ldots\phi(x_n)\right]|\Omega\rangle = \frac{\displaystyle\int \phi(x_1)\ldots\phi(x_n)\,e^{iS[\phi,\epsilon]}\,D\phi}{\displaystyle\int e^{iS[\phi,\epsilon]}\,D\phi} \tag{16.149}$$

in which a term like $i\epsilon\phi^2$ is added to make the modified action $S[\phi,\epsilon]$.

These are some of the techniques one uses to make states of incoming and outgoing particles and to compute scattering amplitudes (Weinberg, 1995, 1996; Srednicki, 2007; Zee, 2010).

16.13 Application to Quantum Electrodynamics

In the Coulomb gauge $\nabla\cdot\boldsymbol{A}=0$, the QED hamiltonian is

$$H = H_m + \int \left[\tfrac{1}{2}\boldsymbol{\pi}^2 + \tfrac{1}{2}(\nabla\times\boldsymbol{A})^2 - \boldsymbol{A}\cdot\boldsymbol{j}\right]\,d^3x + V_C \tag{16.150}$$

in which H_m is the matter hamiltonian, and V_C is the Coulomb term

$$V_C = \frac{1}{2}\int \frac{j^0(\boldsymbol{x},t)\,j^0(\boldsymbol{y},t)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,d^3x\,d^3y. \tag{16.151}$$

The operators $\boldsymbol{A}$ and $\boldsymbol{\pi}$ are canonically conjugate, but they satisfy the Coulomb-gauge conditions

$$\nabla\cdot\boldsymbol{A}=0 \quad\text{and}\quad \nabla\cdot\boldsymbol{\pi}=0. \tag{16.152}$$

One may show (Weinberg, 1995, pp. 413–418) that in this theory, the analog of equation (16.149) is

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\,e^{iS_C}\,\delta[\nabla\cdot\boldsymbol{A}]\,D\boldsymbol{A}\,D\psi}{\displaystyle\int e^{iS_C}\,\delta[\nabla\cdot\boldsymbol{A}]\,D\boldsymbol{A}\,D\psi} \tag{16.153}$$

in which the Coulomb-gauge action is

$$S_C = \int \tfrac{1}{2}\dot{\boldsymbol{A}}^2 - \tfrac{1}{2}(\nabla\times\boldsymbol{A})^2 + \boldsymbol{A}\cdot\boldsymbol{j} + \mathcal{L}_m\,d^4x - \int V_C\,dt \tag{16.154}$$

and the functional delta function

$$\delta[\boldsymbol{\nabla} \cdot \boldsymbol{A}] = \prod_x \delta(\boldsymbol{\nabla} \cdot \boldsymbol{A}(x)) \tag{16.155}$$

enforces the Coulomb-gauge condition. The term $\mathcal{L}_m$ is the action density of the matter field ψ.

Tricks are available. We introduce a new field $A^0(x)$ and consider the factor

$$F = \int \exp\left[i \int \frac{1}{2} \left(\boldsymbol{\nabla} A^0 + \boldsymbol{\nabla} \triangle^{-1} j^0 \right)^2 \, d^4x \right] DA^0 \tag{16.156}$$

which is just a *number* independent of the charge density j^0 since we can cancel the j^0 term by shifting A^0. By $\triangle^{-1}$, we mean $-1/4\pi|\boldsymbol{x} - \boldsymbol{y}|$. By integrating by parts, we can write the number F as (exercise 16.21)

$$\begin{aligned} F &= \int \exp\left[i \int \tfrac{1}{2} \left(\boldsymbol{\nabla} A^0 \right)^2 - A^0 j^0 - \tfrac{1}{2} j^0 \triangle^{-1} j^0 \, d^4x \right] DA^0 \\ &= \int \exp\left[i \int \tfrac{1}{2} \left(\boldsymbol{\nabla} A^0 \right)^2 - A^0 j^0 \, d^4x + i \int V_C \, dt \right] DA^0. \end{aligned} \tag{16.157}$$

So when we multiply the numerator and denominator of the amplitude (16.153) by F, the awkward Coulomb term cancels, and we get

$$\langle \Omega | \mathcal{T} \left[\mathcal{O}_1 \ldots \mathcal{O}_n \right] | \Omega \rangle = \frac{\displaystyle\int \mathcal{O}_1 \ldots \mathcal{O}_n \, e^{iS'} \, \delta[\boldsymbol{\nabla} \cdot \boldsymbol{A}] \, DA \, D\psi}{\displaystyle\int e^{iS'} \, \delta[\boldsymbol{\nabla} \cdot \boldsymbol{A}] \, DA \, D\psi} \tag{16.158}$$

where now DA includes all four components A^μ and

$$S' = \int \tfrac{1}{2} \dot{\boldsymbol{A}}^2 - \tfrac{1}{2} (\boldsymbol{\nabla} \times \boldsymbol{A})^2 + \tfrac{1}{2} \left(\boldsymbol{\nabla} A^0 \right)^2 + \boldsymbol{A} \cdot \boldsymbol{j} - A^0 j^0 + \mathcal{L}_m \, d^4x. \tag{16.159}$$

Since the delta-function $\delta[\boldsymbol{\nabla} \cdot \boldsymbol{A}]$ enforces the Coulomb-gauge condition, we can add to the action S' the term $(\boldsymbol{\nabla} \cdot \dot{\boldsymbol{A}}) \, A^0$ which is $-\dot{\boldsymbol{A}} \cdot \boldsymbol{\nabla} A^0$ after we integrate by parts and drop the surface term. This extra term makes the action gauge invariant

$$\begin{aligned} S &= \int \tfrac{1}{2} (\dot{\boldsymbol{A}} - \boldsymbol{\nabla} \boldsymbol{A^0})^2 - \tfrac{1}{2} (\boldsymbol{\nabla} \times \boldsymbol{A})^2 + \boldsymbol{A} \cdot \boldsymbol{j} - A^0 j^0 + \mathcal{L}_m \, d^4x \\ &= \int -\tfrac{1}{4} F_{ab} F^{ab} + A^b j_b + \mathcal{L}_m \, d^4x. \end{aligned} \tag{16.160}$$

Thus at this point we have

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\, e^{iS}\,\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}]\,DA\,D\psi}{\displaystyle\int e^{iS}\,\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}]\,DA\,D\psi} \tag{16.161}$$

in which S is the gauge-invariant action (16.160), and the integral is over all fields. The only relic of the Coulomb gauge is the gauge-fixing delta functional $\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}]$.

We now make the gauge transformation

$$A'_b(x) = A_b(x) + \partial_b\Lambda(x) \quad \text{and} \quad \psi'(x) = e^{iq\Lambda(x)}\psi(x) \tag{16.162}$$

in the numerator and also, using a different gauge transformation Λ', in the denominator of the ratio (16.161) of path integrals. Since we are integrating over all gauge fields, these gauge transformations merely change the order of integration in the numerator and denominator of that ratio. They are like replacing $\int_{-\infty}^{\infty} f(x)\,dx$ by $\int_{-\infty}^{\infty} f(y)\,dy$. They change nothing, and so

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle = \langle\Omega|\mathcal{T}\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle' \tag{16.163}$$

in which the prime refers to the gauge transformation (16.162).

We've seen that the action S is gauge invariant. So is the measure $DA\,D\psi$, and we now restrict ourselves to operators $\mathcal{O}_1\dots\mathcal{O}_n$ that are *gauge invariant*. So in the right-hand side of equation (16.163), the replacement of the fields by their gauge transforms affects only the term $\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}]$ that enforces the Coulomb-gauge condition

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\, e^{iS}\,\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}+\triangle\Lambda]\,DA\,D\psi}{\displaystyle\int e^{iS}\,\delta[\boldsymbol{\nabla}\cdot\boldsymbol{A}+\triangle\Lambda']\,DA\,D\psi}. \tag{16.164}$$

We now have two choices. If we integrate over all gauge functions $\Lambda(x)$ in both the numerator and the denominator of this ratio (16.164), then apart from over-all constants that cancel, the mean value in the vacuum of the time-ordered product is the ratio

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\, e^{iS}\,DA\,D\psi}{\displaystyle\int e^{iS}\,DA\,D\psi} \tag{16.165}$$

in which we integrate over all matter fields, gauge fields, and gauges. That is, **we do not fix the gauge**.

The analogous formula for the euclidian time-ordered product is

$$\langle\Omega|\mathcal{T}_e\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{-S_e}\, DA\, D\psi}{\displaystyle\int e^{-S_e}\, DA\, D\psi} \tag{16.166}$$

in which the euclidian action S_e is the spacetime integral of the energy density. This formula is quite general; it holds in nonabelian gauge theories and is important in lattice gauge theory.

Our second choice is to multiply the numerator and the denominator of the ratio (16.164) by the exponential $\exp[-i\frac{1}{2}\alpha\int(\triangle\Lambda)^2\, d^4x]$ and then integrate over $\Lambda(x)$ separately in the numerator and denominator. This operation just multiplies the numerator and denominator by the same constant factor, which cancels. But if before integrating over all gauge transformations, we shift Λ so that $\triangle\Lambda$ changes to $\triangle\Lambda - \dot{A}^0$, then the exponential factor is $\exp[-i\frac{1}{2}\alpha\int(\dot{A}^0 - \triangle\Lambda)^2\, d^4x]$. Now when we integrate over $\Lambda(x)$, the delta function $\delta(\boldsymbol{\nabla}\cdot\boldsymbol{A}+\triangle\Lambda)$ replaces $\triangle\Lambda$ by $-\boldsymbol{\nabla}\cdot\boldsymbol{A}$ in the inserted exponential, converting it to $\exp[-i\frac{1}{2}\alpha\int(\dot{A}^0+\boldsymbol{\nabla}\cdot\boldsymbol{A})^2\, d^4x]$. This term changes the gauge-invariant action (16.160) to the gauge-fixed action

$$S_\alpha = \int -\frac{1}{4}F_{ab}\,F^{ab} - \frac{\alpha}{2}(\partial_b A^b)^2 + A^b j_b + \mathcal{L}_m\; d^4x. \tag{16.167}$$

This Lorentz-invariant, gauge-fixed action is much easier to use than the Coulomb-gauge action (16.154) with the Coulomb potential (16.151). We can use it to compute scattering amplitudes perturbatively. The mean value of a time-ordered product of operators in the ground state $|0\rangle$ of the free theory is

$$\langle 0|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|0\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS_\alpha}\, DA\, D\psi}{\displaystyle\int e^{iS_\alpha}\, DA\, D\psi}. \tag{16.168}$$

By following steps analogous to those the led to (16.142), one may show (exercise 16.22) that in Feynman's gauge, $\alpha = 1$, the photon propagator is

$$\langle 0|\mathcal{T}\left[A_\mu(x)A_\nu(y)\right]|0\rangle = -\,i\triangle_{\mu\nu}(x-y) = -\,i\int \frac{\eta_{\mu\nu}}{q^2 - i\epsilon}\; e^{iq\cdot(x-y)}\,\frac{d^4q}{(2\pi)^4}. \tag{16.169}$$

16.14 Fermionic Path Integrals

In our brief introduction (1.11–1.12) and (1.43–1.45), to Grassmann variables, we learned that because

$$\theta^2 = 0 \tag{16.170}$$

the most general function $f(\theta)$ of a single Grassmann variable θ is

$$f(\theta) = a + b\,\theta. \tag{16.171}$$

So a complete integral table consists of the integral of this linear function

$$\int f(\theta)\,d\theta = \int a + b\,\theta\,d\theta = a\int d\theta + b\int \theta\,d\theta. \tag{16.172}$$

This equation has two unknowns, the integral $\int d\theta$ of unity and the integral $\int \theta\,d\theta$ of θ. We choose them so that the integral of $f(\theta+\zeta)$

$$\int f(\theta+\zeta)\,d\theta = \int a + b\,(\theta+\zeta)\,d\theta = (a+b\,\zeta)\int d\theta + b\int \theta\,d\theta \tag{16.173}$$

is the same as the integral (16.172) of $f(\theta)$. Thus the integral $\int d\theta$ of unity must vanish, while the integral $\int \theta\,d\theta$ of θ can be any constant, which we choose to be unity. Our complete table of integrals is then

$$\int d\theta = 0 \quad \text{and} \quad \int \theta\,d\theta = 1. \tag{16.174}$$

The anticommutation relations for a fermionic degree of freedom ψ are

$$\{\psi,\psi^\dagger\} \equiv \psi\,\psi^\dagger + \psi^\dagger\psi = 1 \quad \text{and} \quad \{\psi,\psi\} = \{\psi^\dagger,\psi^\dagger\} = 0. \tag{16.175}$$

Because ψ has $\psi^\dagger$, it is conventional to introduce a variable $\theta^* = \theta^\dagger$ that anti-commutes with itself and with θ

$$\{\theta^*,\theta^*\} = \{\theta^*,\theta\} = \{\theta,\theta\} = 0. \tag{16.176}$$

The logic that led to (16.174) now gives

$$\int d\theta^* = 0 \quad \text{and} \quad \int \theta^*\,d\theta^* = 1. \tag{16.177}$$

We define the reference state $|0\rangle$ as $|0\rangle \equiv \psi|s\rangle$ for a state $|s\rangle$ that is not annihilated by ψ. Since $\psi^2 = 0$, the operator ψ annihilates the state $|0\rangle$

$$\psi|0\rangle = \psi^2|s\rangle = 0. \tag{16.178}$$

The effect of the operator ψ on the state

$$|\theta\rangle = \exp\left(\psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta\right)|0\rangle = \left(1 + \psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta\right)|0\rangle \tag{16.179}$$

is

$$\psi|\theta\rangle = \psi(1 + \psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta)|0\rangle = \psi\psi^\dagger\theta|0\rangle = (1 - \psi^\dagger\psi)\theta|0\rangle = \theta|0\rangle \quad (16.180)$$

while that of θ on $|\theta\rangle$ is

$$\theta|\theta\rangle = \theta(1 + \psi^\dagger\theta - \tfrac{1}{2}\theta^*\theta)|0\rangle = \theta|0\rangle. \quad (16.181)$$

The state $|\theta\rangle$ therefore is an eigenstate of ψ with eigenvalue θ

$$\psi|\theta\rangle = \theta|\theta\rangle. \quad (16.182)$$

The bra corresponding to the ket $|\zeta\rangle$ is

$$\langle\zeta| = \langle 0|\left(1 + \zeta^*\psi - \frac{1}{2}\zeta^*\zeta\right) \quad (16.183)$$

and the inner product $\langle\zeta|\theta\rangle$ is (exercise 16.23)

$$\begin{aligned}
\langle\zeta|\theta\rangle &= \langle 0|\left(1 + \zeta^*\psi - \frac{1}{2}\zeta^*\zeta\right)\left(1 + \psi^\dagger\theta - \frac{1}{2}\theta^*\theta\right)|0\rangle \\
&= \langle 0|1 + \zeta^*\psi\psi^\dagger\theta - \frac{1}{2}\zeta^*\zeta - \frac{1}{2}\theta^*\theta + \frac{1}{4}\zeta^*\zeta\theta^*\theta|0\rangle \\
&= \langle 0|1 + \zeta^*\theta - \frac{1}{2}\zeta^*\zeta - \frac{1}{2}\theta^*\theta + \frac{1}{4}\zeta^*\zeta\theta^*\theta|0\rangle \\
&= \exp\left[\zeta^*\theta - \frac{1}{2}\left(\zeta^*\zeta + \theta^*\theta\right)\right]. \quad (16.184)
\end{aligned}$$

Example 16.3 (A gaussian integral)**.** For *any* number c, we can compute the integral of $\exp(c\,\theta^*\theta)$ by expanding the exponential

$$\int e^{c\,\theta^*\theta}\,d\theta^* d\theta = \int (1 + c\,\theta^*\theta)\,d\theta^* d\theta = \int (1 - c\,\theta\,\theta^*)\,d\theta^* d\theta = -c. \quad (16.185)$$

□

The identity operator for the space of states

$$c|0\rangle + d|1\rangle \equiv c|0\rangle + d\psi^\dagger|0\rangle \quad (16.186)$$

is (exercise 16.24) the integral

$$I = \int |\theta\rangle\langle\theta|\,d\theta^* d\theta = |0\rangle\langle 0| + |1\rangle\langle 1| \quad (16.187)$$

in which the differentials anti-commute with each other and with other fermionic variables: $\{d\theta, d\theta^*\} = 0$, $\{d\theta, \theta\} = 0$, $\{d\theta, \psi\} = 0$, and so forth.

The case of several Grassmann variables $\theta_1, \theta_2, \ldots, \theta_n$ and several Fermi operators $\psi_1, \psi_2, \ldots, \psi_n$ is similar. The θ_k anticommute among themselves

$$\{\theta_i, \theta_j\} = \{\theta_i, \theta_j^*\} = \{\theta_i^*, \theta_j^*\} = 0 \quad (16.188)$$

while the ψ_k satisfy

$$\{\psi_k, \psi_\ell^\dagger\} = \delta_{k\ell} \quad \text{and} \quad \{\psi_k, \psi_l\} = \{\psi_k^\dagger, \psi_\ell^\dagger\} = 0. \tag{16.189}$$

The reference state $|0\rangle$ is

$$|0\rangle = \left(\prod_{k=1}^{n} \psi_k\right)|s\rangle \tag{16.190}$$

in which $|s\rangle$ is any state not annihilated by any ψ_k (so the resulting $|0\rangle$ isn't zero). The direct-product state

$$|\theta\rangle \equiv \exp\left(\sum_{k=1}^{n} \psi_k^\dagger \theta_k - \frac{1}{2}\theta_k^* \theta_k\right)|0\rangle = \left[\prod_{k=1}^{n}\left(1 + \psi_k^\dagger \theta_k - \frac{1}{2}\theta_k^*\theta_k\right)\right]|0\rangle \tag{16.191}$$

is (exercise 16.25) a simultaneous eigenstate of each ψ_k

$$\psi_k|\theta\rangle = \theta_k|\theta\rangle. \tag{16.192}$$

It follows that

$$\psi_\ell\psi_k|\theta\rangle = \psi_\ell\theta_k|\theta\rangle = -\theta_k\psi_\ell|\theta\rangle = -\theta_k\theta_\ell|\theta\rangle = \theta_\ell\theta_k|\theta\rangle \tag{16.193}$$

and so too $\psi_k\psi_\ell|\theta\rangle = \theta_k\theta_\ell|\theta\rangle$. Since the ψ's anticommute, their eigenvalues must also

$$\theta_\ell\theta_k|\theta\rangle = \psi_\ell\psi_k|\theta\rangle = -\psi_k\psi_\ell|\theta\rangle = -\theta_k\theta_\ell|\theta\rangle \tag{16.194}$$

(even if they commuted with the ψ's in which case we'd have $\psi_\ell\psi_k|\theta\rangle = \theta_k\theta_\ell|\theta\rangle = -\psi_k\psi_\ell|\theta\rangle = -\theta_\ell\theta_k|\theta\rangle$).

The inner product $\langle\zeta|\theta\rangle$ is

$$\begin{aligned}\langle\zeta|\theta\rangle &= \langle 0|\left[\prod_{k=1}^{n}(1 + \zeta_k^*\psi_k - \frac{1}{2}\zeta_k^*\zeta_k)\right]\left[\prod_{\ell=1}^{n}(1 + \psi_\ell^\dagger\theta_\ell - \frac{1}{2}\theta_\ell^*\theta_\ell)\right]|0\rangle \\ &= \exp\left[\sum_{k=1}^{n}\zeta_k^*\theta_k - \frac{1}{2}\left(\zeta_k^*\zeta_k + \theta_k^*\theta_k\right)\right] = e^{\zeta^\dagger\theta - (\zeta^\dagger\zeta + \theta^\dagger\theta)/2}.\end{aligned} \tag{16.195}$$

The identity operator is

$$I = \int |\theta\rangle\langle\theta| \prod_{k=1}^{n} d\theta_k^* d\theta_k. \tag{16.196}$$

Example 16.4 (Gaussian Grassmann Integral)**.** For *any* 2×2 matrix A, we may compute the gaussian integral

$$g(A) = \int e^{-\theta^\dagger A\theta}\, d\theta_1^* d\theta_1 d\theta_2^* d\theta_2 \tag{16.197}$$

by expanding the exponential. The only terms that survive are the ones that have exactly one of each of the four variables θ_1, θ_2, θ_1^*, and θ_2^*. Thus, the integral is the determinant of the matrix A

$$\begin{aligned} g(A) &= \int \frac{1}{2} \left(\theta^*{}_k A_{k\ell} \theta_\ell \right)^2 \, d\theta_1^* d\theta_1 d\theta_2^* d\theta_2 \\ &= \int \left(\theta_1^* A_{11} \theta_1 \, \theta_2^* A_{22} \theta_2 + \theta_1^* A_{12} \theta_2 \, \theta_2^* A_{21} \theta_1 \right) \, d\theta_1^* d\theta_1 d\theta_2^* d\theta_2 \\ &= A_{11} A_{22} - A_{12} A_{21} = \det A. \end{aligned} \tag{16.198}$$

The natural generalization to n dimensions

$$\int e^{-\theta^\dagger A \theta} \prod_{k=1}^{n} d\theta_k^* d\theta_k = \det A \tag{16.199}$$

is true for *any* $n \times n$ matrix A. If A is invertible, then the invariance of Grassmann integrals under translations implies that

$$\begin{aligned} \int e^{-\theta^\dagger A \theta + \theta^\dagger \zeta + \zeta^\dagger \theta} \prod_{k=1}^{n} d\theta_k^* d\theta_k &= \int e^{-\theta^\dagger A (\theta + A^{-1}\zeta) + \theta^\dagger \zeta + \zeta^\dagger (\theta + A^{-1}\zeta)} \prod_{k=1}^{n} d\theta_k^* d\theta_k \\ &= \int e^{-\theta^\dagger A \theta + \zeta^\dagger \theta + \zeta^\dagger A^{-1} \zeta} \prod_{k=1}^{n} d\theta_k^* d\theta_k \\ &= \int e^{-(\theta^\dagger + \zeta^\dagger A^{-1}) A \theta + \zeta^\dagger \theta + \zeta^\dagger A^{-1} \zeta} \prod_{k=1}^{n} d\theta_k^* d\theta_k \\ &= \int e^{-\theta^\dagger A \theta + \zeta^\dagger A^{-1} \zeta} \prod_{k=1}^{n} d\theta_k^* d\theta_k \\ &= \det A \; e^{\zeta^\dagger A^{-1} \zeta}. \end{aligned} \tag{16.200}$$

The values of θ and $\theta^\dagger$ that make the argument $-\theta^\dagger A\theta + \theta^\dagger \zeta + \zeta^\dagger \theta$ of the exponential stationary are $\overline{\theta} = A^{-1}\zeta$ and $\overline{\theta^\dagger} = \zeta^\dagger A^{-1}$. So a gaussian Grassmann integral is equal to its exponential evaluated at its stationary point, apart from a prefactor involving the determinant $\det A$. This result is a fermionic echo of the bosonic results (16.13–16.15). □

One may further extend these definitions to a Grassmann field $\chi_m(x)$ and an associated Dirac field $\psi_m(x)$. The $\chi_m(x)$'s anticommute among themselves and with all fermionic variables at all points of spacetime

$$\{\chi_m(x), \chi_n(x')\} = \{\chi_m^*(x), \chi_n(x')\} = \{\chi_m^*(x), \chi_n^*(x')\} = 0 \tag{16.201}$$

and the Dirac field $\psi_m(x)$ obeys the equal-time anticommutation relations

$$\begin{aligned} \{\psi_m(\boldsymbol{x},t), \psi_n^\dagger(\boldsymbol{x'},t)\} &= \delta_{mn}\, \delta(\boldsymbol{x}-\boldsymbol{x'}) \quad (n,m=1,\ldots,4) \\ \{\psi_m(\boldsymbol{x},t), \psi_n(\boldsymbol{x'},t)\} &= \{\psi_m^\dagger(\boldsymbol{x},t), \psi_n^\dagger(\boldsymbol{x'},t)\} = 0. \end{aligned} \tag{16.202}$$

As in (16.94 & 16.190), we use eigenstates of the field ψ at $t=0$. If $|0\rangle$ is defined in terms of a state $|s\rangle$ that is not annihilated by any $\psi_m(\boldsymbol{x},0)$ as

$$|0\rangle = \left[\prod_{m,\boldsymbol{x}} \psi_m(\boldsymbol{x},0)\right] |s\rangle \tag{16.203}$$

then (exercise 16.26) the state

$$\begin{aligned} |\chi\rangle &= \exp\left(\int \sum_m \psi_m^\dagger(\boldsymbol{x},0)\, \chi_m(\boldsymbol{x}) - \frac{1}{2}\chi_m^*(\boldsymbol{x})\chi_m(\boldsymbol{x})\, d^3x\right) |0\rangle \\ &= \exp\left(\int \psi^\dagger\chi - \tfrac{1}{2}\chi^\dagger\chi\, d^3x\right) |0\rangle \end{aligned} \tag{16.204}$$

is an eigenstate of the operator $\psi_m(\boldsymbol{x},0)$ with eigenvalue $\chi_m(\boldsymbol{x})$

$$\psi_m(\boldsymbol{x},0)|\chi\rangle = \chi_m(\boldsymbol{x})|\chi\rangle. \tag{16.205}$$

The inner product of two such states is (exercise 16.27)

$$\langle\chi'|\chi\rangle = \exp\left[\int \chi'^\dagger\chi - \frac{1}{2}\chi'^\dagger\chi' - \frac{1}{2}\chi^\dagger\chi\ d^3x\right]. \tag{16.206}$$

The identity operator is the integral

$$I = \int |\chi\rangle\langle\chi|\, D\chi^* D\chi \tag{16.207}$$

in which

$$D\chi^* D\chi \equiv \prod_{m,\boldsymbol{x}} d\chi_m^*(\boldsymbol{x}) d\chi_m(\boldsymbol{x}). \tag{16.208}$$

The hamiltonian for a free Dirac field ψ of mass m is the spatial integral

$$H_0 = \int \overline{\psi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\psi\ d^3x \tag{16.209}$$

in which $\overline{\psi} \equiv i\psi^\dagger\gamma^0$ and the gamma matrices (10.304) satisfy

$$\{\gamma^a, \gamma^b\} = 2\,\eta^{ab} \tag{16.210}$$

where η is the 4×4 diagonal matrix with entries $(-1,1,1,1)$. Since $\psi|\chi\rangle =$

$\chi|\chi\rangle$ and $\langle\chi'|\psi^\dagger = \langle\chi'|\chi'^\dagger$, the quantity $\langle\chi'|\exp(-i\epsilon H_0)|\chi\rangle$ is by (16.206)

$$\begin{aligned}\langle\chi'|e^{-i\epsilon H_0}|\chi\rangle &= \langle\chi'|\chi\rangle \exp\left[-i\epsilon\int \overline{\chi}'\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\chi\; d^3x\right] \qquad (16.211)\\ &= \exp\left[\int \tfrac{1}{2}(\chi'^\dagger-\chi^\dagger)\chi - \tfrac{1}{2}\chi'^\dagger(\chi'-\chi) - i\epsilon\overline{\chi}'(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\chi d^3x\right]\\ &= \exp\left\{\epsilon\int\left[\tfrac{1}{2}\dot{\chi}^\dagger\chi - \tfrac{1}{2}\chi'^\dagger\dot{\chi} - i\overline{\chi}'\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\chi\right]\; d^3x\right\}\end{aligned}$$

in which $\chi'^\dagger - \chi^\dagger = \epsilon\dot{\chi}^\dagger$ and $\chi' - \chi = \epsilon\dot{\chi}$. Everything within the square brackets is multiplied by ϵ, so we may replace $\chi'^\dagger$ by $\chi^\dagger$ and $\overline{\chi}'$ by $\overline{\chi}$ so as to write to first order in ϵ

$$\langle\chi'|e^{-i\epsilon H_0}|\chi\rangle = \exp\left[\epsilon\int \tfrac{1}{2}\dot{\chi}^\dagger\chi - \tfrac{1}{2}\chi^\dagger\dot{\chi} - i\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\chi\; d^3x\right] \qquad (16.212)$$

in which the dependence upon χ' is through the time derivatives.

Putting together $n = 2t/\epsilon$ such matrix elements, integrating over all intermediate-state dyadics $|\chi\rangle\langle\chi|$, and using our formula (16.207), we find

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int \exp\left[\int \tfrac{1}{2}\dot{\chi}^\dagger\chi - \tfrac{1}{2}\chi^\dagger\dot{\chi} - i\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\chi d^4x\right] D\chi^* D\chi. \qquad (16.213)$$

Integrating $\dot{\chi}^\dagger\chi$ by parts and dropping the surface term, we get

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int \exp\left[\int -\chi^\dagger\dot{\chi} - i\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\chi\, d^4x\right] D\chi^* D\chi. \qquad (16.214)$$

Since $-\chi^\dagger\dot{\chi} = -i\overline{\chi}\gamma^0\dot{\chi}$, the argument of the exponential is

$$i\int -\overline{\chi}\gamma^0\dot{\chi} - \overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\,\chi\; d^4x = i\int -\overline{\chi}\,(\gamma^\mu\partial_\mu+m)\,\chi\; d^4x. \qquad (16.215)$$

We then have

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int \exp\left(i\int \mathcal{L}_0(\chi)\, d^4x\right) D\chi^* D\chi \qquad (16.216)$$

in which $\mathcal{L}_0(\chi) = -\overline{\chi}\,(\gamma^\mu\partial_\mu+m)\,\chi$ is the action density (10.306) for a free Dirac field. Thus the amplitude is a path integral with phases given by the classical action $S_0[\chi]$

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int e^{i\int\mathcal{L}_0(\chi)\,d^4x} D\chi^* D\chi = \int e^{iS_0[\chi]} D\chi^* D\chi \qquad (16.217)$$

and the integral is over all fields that go from $\chi(\boldsymbol{x},-t) = \chi_{-t}(\boldsymbol{x})$ to $\chi(\boldsymbol{x},t) = \chi_t(\boldsymbol{x})$. Any normalization factor will cancel in ratios of such integrals.

Since Fermi fields anticommute, their time-ordered product has an extra minus sign

$$\mathcal{T}\left[\overline{\psi}(x_1)\psi(x_2)\right] = \theta(x_1^0 - x_2^0)\,\overline{\psi}(x_1)\,\psi(x_2) - \theta(x_2^0 - x_1^0)\,\psi(x_2)\,\overline{\psi}(x_1). \quad (16.218)$$

The logic behind our formulas (16.115) and (16.121) for the time-ordered product of bosonic fields now leads to an expression for the time-ordered product of $2n$ Dirac fields (with $D\chi''$ and $D\chi'$ and so forth suppressed)

$$\langle 0|\mathcal{T}\left[\overline{\psi}(x_1)\ldots\psi(x_{2n})\right]|0\rangle = \frac{\int \langle 0|\chi''\rangle\,\overline{\chi}(x_1)\ldots\chi(x_{2n})\,e^{iS_0[\chi]}\langle\chi'|0\rangle\,D\chi^* D\chi}{\int \langle 0|\chi''\rangle\,e^{iS_0[\chi]}\langle\chi'|0\rangle\,D\chi^* D\chi}. \quad (16.219)$$

As in (16.132), the effect of the inner products $\langle 0|\chi''\rangle$ and $\langle\chi'|0\rangle$ is to insert ϵ-terms which modify the Dirac propagators

$$\langle 0|\mathcal{T}\left[\overline{\psi}(x_1)\ldots\psi(x_{2n})\right]|0\rangle = \frac{\int \overline{\chi}(x_1)\ldots\chi(x_{2n})\,e^{iS_0[\chi,\epsilon]}\,D\chi^* D\chi}{\int e^{iS_0[\chi,\epsilon]}\,D\chi^* D\chi}. \quad (16.220)$$

Imitating (16.133), we introduce a Grassmann external current $\zeta(x)$ and define a fermionic analog of $Z_0[j]$

$$Z_0[\zeta] \equiv \langle 0|\,\mathcal{T}\left[e^{\int \overline{\zeta}\psi + \overline{\psi}\zeta\, d^4x}\right]|0\rangle = \frac{\int e^{\int \overline{\zeta}\chi + \overline{\chi}\zeta\, d^4x} e^{iS_0[\chi,\epsilon]} D\chi^* D\chi}{\int e^{iS_0[\chi,\epsilon]} D\chi^* D\chi}. \quad (16.221)$$

Example 16.5 (Feynman's fermion propagator)**.** Since

$$\begin{aligned} i\left(\gamma^\mu\partial_\mu + m\right)\Delta(x-y) &\equiv i\left(\gamma^\mu\partial_\mu + m\right)\int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)}\,\frac{-i\left(-i\gamma^\nu p_\nu + m\right)}{p^2+m^2-i\epsilon} \\ &= \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)}\left(i\gamma^\mu p_\mu + m\right)\frac{\left(-i\gamma^\nu p_\nu + m\right)}{p^2+m^2-i\epsilon} \\ &= \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)}\,\frac{p^2+m^2}{p^2+m^2-i\epsilon} = \delta^4(x-y), \end{aligned} \quad (16.222)$$

the function $\Delta(x-y)$ is the inverse of the differential operator $i(\gamma^\mu\partial_\mu + m)$.

Thus the Grassmann identity (16.200) implies that $Z_0[\zeta]$ is

$$\langle 0|\, \mathcal{T}\left[e^{\int \overline{\zeta}\psi+\overline{\psi}\zeta\, d^4x}\right]|0\rangle = \frac{\displaystyle\int e^{\int[\overline{\zeta}\chi+\overline{\chi}\zeta - i\,\overline{\chi}(\gamma^\mu\partial_\mu+m)\chi]d^4x} D\chi^* D\chi}{\displaystyle\int e^{iS_0[\chi,\epsilon]} D\chi^* D\chi} \tag{16.223}$$
$$= \exp\left[\int \overline{\zeta}(x)\Delta(x-y)\zeta(y)\, d^4x d^4y\right].$$

Differentiating we get

$$\langle 0|\mathcal{T}\left[\psi(x)\overline{\psi}(y)\right]|0\rangle = \Delta(x-y) = -i\int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)}\,\frac{-i\gamma^\nu p_\nu + m}{p^2+m^2-i\epsilon}. \tag{16.224}$$

□

16.15 Application to nonabelian gauge theories

The action of a generic non-abelian gauge theory is

$$S = \int -\tfrac{1}{4}F_{a\mu\nu}F_a^{\mu\nu} - \overline{\psi}\left(\gamma^\mu D_\mu + m\right)\psi\ d^4x \tag{16.225}$$

in which the Maxwell field is

$$F_{a\mu\nu} \equiv \partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} + g\, f_{abc}\, A_{b\mu}\, A_{c\nu} \tag{16.226}$$

and the covariant derivative is

$$D_\mu\psi \equiv \partial_\mu\psi - ig\, t_a\, A_{a\mu}\,\psi. \tag{16.227}$$

Here g is a coupling constant, f_{abc} is a structure constant (10.65), and t_a is a generator (10.56) of the Lie algebra (section 10.15) of the gauge group.

One may show (Weinberg, 1996, pp. 14–18) that the analog of equation (16.161) for quantum electrodynamics is

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\ldots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\ldots\mathcal{O}_n\, e^{iS}\,\delta[A_{a3}]\, DA\, D\psi}{\displaystyle\int e^{iS}\,\delta[A_{a3}]\, DA\, D\psi} \tag{16.228}$$

in which the functional delta function

$$\delta[A_{a3}] \equiv \prod_{x,b}\delta(A_{a3}(x)) \tag{16.229}$$

enforces the axial-gauge condition, and $D\psi$ stands for $D\psi^* D\psi$.

Initially, physicists had trouble computing nonabelian amplitudes beyond the lowest order of perturbation theory. Then DeWitt showed how to compute to second order (DeWitt, 1967), and Faddeev and Popov, using path integrals, showed how to compute to all orders (Faddeev and Popov, 1967).

16.16 The Faddeev-Popov trick

The path-integral tricks of Faddeev and Popov are described in (Weinberg, 1996, pp. 19–27). We will use gauge-fixing functions $G_a(x)$ to impose a gauge condition on our non-abelian gauge fields $A^a_\mu(x)$. For instance, we can use $G_a(x) = A^3_a(x)$ to impose an axial gauge or $G_a(x) = i\partial_\mu A^\mu_a(x)$ to impose a Lorentz-invariant gauge.

Under an infinitesimal gauge transformation (11.544)

$$A^\lambda_{a\mu} = A_{a\mu} - \partial_\mu \lambda_a - g\, f_{abc}\, A_{b\mu}\, \lambda_c \tag{16.230}$$

the gauge fields change, and so the gauge-fixing functions $G_b(x)$, which depend upon them, also change. The jacobian J of that change at $\lambda = 0$ is

$$J = \det\left(\frac{\delta G^\lambda_a(x)}{\delta \lambda_b(y)}\right)\bigg|_{\lambda=0} \equiv \frac{DG^\lambda}{D\lambda}\bigg|_{\lambda=0} \tag{16.231}$$

and it typically involves the delta function $\delta^4(x-y)$.

Let $B[G]$ be any functional of the gauge-fixing functions $G_b(x)$ such as

$$B[G] = \prod_{x,a} \delta(G_a(x)) = \prod_{x,a} \delta(A^3_a(x)) \tag{16.232}$$

in an axial gauge or

$$B[G] = \exp\left[\frac{i}{2}\int (G_a(x))^2\, d^4x\right] = \exp\left[-\frac{i}{2}\int (\partial_\mu A^\mu_a(x))^2\, d^4x\right] \tag{16.233}$$

in a Lorentz-invariant gauge.

We want to understand functional integrals like (16.228)

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1 \dots \mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1 \dots \mathcal{O}_n\, e^{iS}\, B[G]\, J\, DA\, D\psi}{\displaystyle\int e^{iS}\, B[G]\, J\, DA\, D\psi} \tag{16.234}$$

in which the operators $\mathcal{O}_k$, the action functional $S[A]$, and the differentials $DAD\psi$ (but not the gauge-fixing functional $B[G]$ or the Jacobian J) are gauge invariant. The axial-gauge formula (16.228) is a simple example in which $B[G] = \delta[A_{a3}]$ enforces the axial-gauge condition $A_{a3}(x) = 0$ and the determinant $J = \det(\delta_{ab}\partial_3\delta(x-y))$ is a constant that cancels.

If we translate the gauge fields by gauge transformations Λ and Λ', then the ratio (16.234) does not change

$$\langle\Omega|\mathcal{T}[\mathcal{O}_1 \dots \mathcal{O}_n]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1^\Lambda \dots \mathcal{O}_n^\Lambda \, e^{iS^\Lambda} \, B[G^\Lambda] \, J^\Lambda \, DA^\Lambda \, D\psi^\Lambda}{\displaystyle\int e^{iS^{\Lambda'}} \, B[G^{\Lambda'}] \, J^{\Lambda'} \, DA^{\Lambda'} \, D\psi^{\Lambda'}} \tag{16.235}$$

any more than $\int f(y)\,dy$ is different from $\int f(x)\,dx$. Since the operators $\mathcal{O}_k$, the action functional $S[A]$, and the differentials $DAD\psi$ are gauge invariant, most of the Λ-dependence goes away

$$\langle\Omega|\mathcal{T}[\mathcal{O}_1 \dots \mathcal{O}_n]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1 \dots \mathcal{O}_n \, e^{iS} \, B[G^\Lambda] \, J^\Lambda \, DA \, D\psi}{\displaystyle\int e^{iS} \, B[G^{\Lambda'}] \, J^{\Lambda'} \, DA \, D\psi}. \tag{16.236}$$

Let $\Lambda\lambda$ be a gauge transformation Λ followed by an infinitesimal gauge transformation λ. The jacobian J^Λ is a determinant of a product of matrices which is a product of their determinants

$$\begin{aligned} J^\Lambda &= \det\left(\frac{\delta G_a^{\Lambda\lambda}(x)}{\delta\lambda_b(y)}\right)\bigg|_{\lambda=0} = \det\left(\int \frac{\delta G_a^{\Lambda\lambda}(x)}{\delta\Lambda\lambda_c(z)} \frac{\delta\Lambda\lambda_c(z)}{\delta\lambda_b(y)} \, d^4z\right)\bigg|_{\lambda=0} \\ &= \det\left(\frac{\delta G_a^{\Lambda\lambda}(x)}{\delta\Lambda\lambda_c(z)}\right)\bigg|_{\lambda=0} \det\left(\frac{\delta\Lambda\lambda_c(z)}{\delta\lambda_b(y)}\right)\bigg|_{\lambda=0} \\ &= \det\left(\frac{\delta G_a^{\Lambda}(x)}{\delta\Lambda_c(z)}\right) \det\left(\frac{\delta\Lambda\lambda_c(z)}{\delta\lambda_b(y)}\right)\bigg|_{\lambda=0} \equiv \frac{DG^\Lambda}{D\Lambda} \frac{D\Lambda\lambda}{D\lambda}\bigg|_{\lambda=0}. \end{aligned} \tag{16.237}$$

Now we integrate over the gauge transformations Λ (and Λ') with weight function $\rho(\Lambda) = (D\Lambda\lambda/D\lambda|_{\lambda=0})^{-1}$ and find, since the ratio (16.236) is Λ-

independent

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\, e^{iS}\, B[G^\Lambda]\, \frac{DG^\Lambda}{D\Lambda}\, D\Lambda\, DA\, D\psi}{\displaystyle\int e^{iS}\, B[G^{\Lambda'}]\, \frac{DG^{\Lambda'}}{D\Lambda'}\, D\Lambda'\, DA\, D\psi}$$

$$= \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\, e^{iS}\, B[G^\Lambda]\, DG^\Lambda\, DA\, D\psi}{\displaystyle\int e^{iS}\, B[G^\Lambda]\, DG^\Lambda\, DA\, D\psi}$$

$$= \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\, e^{iS}\; DA\, D\psi}{\displaystyle\int e^{iS}\; DA\, D\psi}. \tag{16.238}$$

Thus the mean-value in the vacuum of a time-ordered product of gauge-invariant operators is a ratio of path integrals over all gauge fields without any gauge fixing. No matter what gauge condition G or gauge-fixing functional $B[G]$ we use, the resulting gauge-fixed ratio (16.234) is equal to the ratio (16.238) of path integrals over all gauge fields without any gauge fixing. All gauge-fixed ratios (16.234) give the same time-ordered products, and so we can use whatever gauge condition G or gauge-fixing functional $B[G]$ is most convenient.

The analogous formula for the euclidian time-ordered product is

$$\langle\Omega|\mathcal{T}_e\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\, e^{-S_e}\, DA\, D\psi}{\displaystyle\int e^{-S_e}\, DA\, D\psi} \tag{16.239}$$

where the euclidian action S_e is the spacetime integral of the energy density. This formula is the basis for lattice gauge theory.

The path-integral formulas (16.165 & 16.166) derived for quantum electrodynamics therefore also apply to nonabelian gauge theories.

16.17 Ghosts

Faddeev and Popov showed how to do perturbative calculations in which one does fix the gauge. To continue our description of their tricks, we return to the gauge-fixed expression (16.234) for the time-ordered product

$$\langle\Omega|\mathcal{T}\left[\mathcal{O}_1\dots\mathcal{O}_n\right]|\Omega\rangle = \frac{\displaystyle\int \mathcal{O}_1\dots\mathcal{O}_n\, e^{iS}\, B[G]\, J\, DA\, D\psi}{\displaystyle\int e^{iS}\, B[G]\, J\, DA\, D\psi} \tag{16.240}$$

set $G_b(x) = -i\partial_\mu A^\mu_b(x)$ and use (16.233) as the gauge-fixing functional $B[G]$

$$B[G] = \exp\left[\frac{i}{2}\int (G_a(x))^2\; d^4x\right] = \exp\left[-\frac{i}{2}\int \left(\partial_\mu A^\mu_a(x)\right)^2\; d^4x\right]. \tag{16.241}$$

This functional adds to the action density the term $-(\partial_\mu A^\mu_a)^2/2$ which leads to a gauge-field propagator like the photon's (16.169)

$$\langle 0|\mathcal{T}\left[A^a_\mu(x)A^b_\nu(y)\right]|0\rangle = -i\delta_{ab}\triangle_{\mu\nu}(x-y) = -i\int \frac{\eta_{\mu\nu}\delta_{ab}}{q^2-i\epsilon}\; e^{iq\cdot(x-y)}\;\frac{d^4q}{(2\pi)^4}. \tag{16.242}$$

What about the determinant J? Under an infinitesimal gauge transformation (16.230), the gauge field becomes

$$A^\lambda_{a\mu} = A_{a\mu} - \partial_\mu\lambda_a - g\, f_{abc}\, A_{b\mu}\,\lambda_c \tag{16.243}$$

and so $G^\lambda_a(x) = i\partial^\mu A^\lambda_{a\mu}(x)$ is

$$G^\lambda_a(x) = i\partial^\mu A_{a\mu}(x) + i\partial^\mu \int\left[-\delta_{ac}\partial_\mu - g\, f_{abc}A_{b\mu}(x)\right]\delta^4(x-y)\lambda_c(y)\, d^4y. \tag{16.244}$$

The jacobian J then is the determinant (16.231) of the matrix

$$\left(\frac{\delta G^\lambda_a(x)}{\delta\lambda_c(y)}\right)\Bigg|_{\lambda=0} = -i\delta_{ac}\,\Box\,\delta^4(x-y) - ig\, f_{abc}\,\frac{\partial}{\partial x^\mu}\left[A^\mu_b(x)\delta^4(x-y)\right] \tag{16.245}$$

that is

$$J = \det\left(-i\delta_{ac}\,\Box\,\delta^4(x-y) - ig\, f_{abc}\,\frac{\partial}{\partial x^\mu}\left[A^\mu_b(x)\delta^4(x-y)\right]\right). \tag{16.246}$$

But we've seen (16.199) that a determinant can be written as a fermionic path integral

$$\det A = \int e^{-\theta^\dagger A\,\theta}\,\prod_{k=1}^{n} d\theta^*_k d\theta_k. \tag{16.247}$$

So we can write the jacobian J as

$$J = \int \exp\left[\int i\omega^*_a\Box\omega_a + igf_{abc}\omega^*_a\partial_\mu(A^\mu_b\omega_c)\, d^4x\right] D\omega^* D\omega \tag{16.248}$$

which contributes the terms $-\partial_\mu \omega_a^* \partial^\mu \omega_a$ and

$$-\partial_\mu \omega_a^* \, g \, f_{abc} \, A_b^\mu \omega_c = \partial_\mu \omega_a^* \, g \, f_{abc} \, A_c^\mu \omega_b \qquad (16.249)$$

to the action density.

Thus we can do perturbation theory by using the modified action density

$$\mathcal{L}' = -\tfrac{1}{4} F_{a\mu\nu} F_a^{\mu\nu} - \tfrac{1}{2} \left(\partial_\mu A_a^\mu\right)^2 - \partial_\mu \omega_a^* \partial^\mu \omega_a + \partial_\mu \omega_a^* \, g \, f_{abc} \, A_c^\mu \omega_b - \overline{\psi} \left(\not{D} + m\right) \psi \qquad (16.250)$$

in which $\not{D} \equiv \gamma^\mu D_\mu = \gamma^\mu (\partial_\mu - i g t^a A_{a\mu})$. The **ghost** field ω is a mathematical device, not a physical field describing real particles, which would be spinless fermions violating the spin-statistics theorem (example 10.21).

16.18 Integrating over the momenta

When the hamiltonian is quadratic in the momenta like (16.16) and (16.91), one easily integrates over the momenta and converts the hamiltonian into the lagrangian. It may happen, however, that the hamiltonian is so complicated a function of the momenta that one can't integrate over the momenta. In such cases, the partition function for a scalar field $\phi(x)$ is a path integral over both ϕ and π

$$Z(\beta) = \int \exp\left\{ \int_0^\beta \int \left[i\dot{\phi}(x)\pi(x) - H(\phi, \pi) \right] dt d^3x \right\} D\phi \, D\pi. \qquad (16.251)$$

For some values of $\dot{\phi}\,\pi$, the exponential is not positive, and so is not a probability distribution for ϕ and π. The Monte Carlo methods of chapter 14 are designed to work with probability distributions, not with weight functions that assume values that are negative or complex. This is one aspect of the **sign problem**.

The integral over the momenta

$$P[\phi] = \int \exp\left\{ \int_0^\beta \int \left[i\dot{\phi}(x)\pi(x) - H(\phi, \pi) \right] dt d^3x \right\} D\pi \qquad (16.252)$$

is a probability distribution. So one can numerically integrate over the momenta, make a look-up table for $P[\phi]$, and then apply the usual Monte Carlo method to the probability functional $P[\phi]$ (Amdahl and Cahill, 2016).

Further Reading

Quantum Field Theory (Srednicki, 2007), *The Quantum Theory of Fields*

I, II, & III (Weinberg, 1995, 1996, 2005), and *Quantum Field Theory in a Nutshell* (Zee, 2010) all provide excellent treatments of path integrals.

Exercises

16.1 Derive the multiple gaussian integral (16.8) from (5.168).

16.2 Derive the multiple gaussian integral (16.12) from (5.167).

16.3 Show that the vector $\overline{Y}$ that makes the argument of the multiple gaussian integral (16.12) stationary is given by (16.13), and that the multiple gaussian integral (16.12) is equal to its exponential evaluated at its stationary point $\overline{Y}$ apart from a prefactor involving $\det iS$.

16.4 Repeat the previous exercise for the multiple gaussian integral (16.11).

16.5 Compute the double integral (16.23) for the case $V(q_j) = 0$.

16.6 Insert into the LHS of (16.47) a complete set of momentum dyadics $|p\rangle\langle p|$, use the inner product $\langle q|p\rangle = \exp(iqp\hbar)/\sqrt{2\pi\hbar}$, do the resulting Fourier transform, and so verify the free-particle path integral (16.47).

16.7 By taking the nonrelativistic limit of the formula (11.322) for the action of a relativistic particle of mass m and charge q, derive the expression (16.48) for the action a nonrelativistic particle in an electromagnetic field with no scalar potential.

16.8 Show that for the hamiltonian (16.52) of the simple harmonic oscillator the action $S[q_c]$ of the classical path is (16.59).

16.9 Show that the harmonic-oscillator action of the loop (16.60) is (16.61).

16.10 Show that the harmonic-oscillator amplitude (16.64) for $q' = 0$ and $q'' = q$ reduces as $t \to 0$ to the one-dimensional version of the free-particle amplitude (16.47).

16.11 Work out the path-integral formula for the amplitude for a mass m initially at rest to fall to the ground from height h in a gravitational field of local acceleration g to lowest order and then including loops up to an overall constant. Hint: use the technique of section 16.7.

16.12 Show that the action (16.66) of the stationary solution (16.69) is (16.71).

16.13 Derive formula (16.124) for the action $S_0[\phi]$ from (16.122 & 16.123).

16.14 Derive identity (16.128). Split the time integral at $t = 0$ into two halves, use

$$\epsilon\, e^{\pm\epsilon t} = \pm \frac{d}{dt}\, e^{\pm\epsilon t} \tag{16.253}$$

and then integrate each half by parts.

16.15 Derive the third term in equation (16.130) from the second term.

16.16 Use (16.135) and the Fourier transform (16.136) of the external current j to derive the formula (16.137) for the modified action $S_0[\phi, \epsilon, j]$.

16.17 Derive equation (16.139) from equations (16.137) and (16.138).

16.18 Derive the formula (16.140) for $Z_0[j]$ from the expression (16.139) for $S_0[\phi, \epsilon, j]$.

16.19 Derive equations (16.141 & 16.142) from formula (16.140).

16.20 Derive equation (16.146) from the formula (16.141) for $Z_0[j]$.

16.21 Show that the time integral of the Coulomb term (16.151) is the term that is quadratic in j^0 in the number F defined by (16.156).

16.22 By following steps analogous to those the led to (16.142), derive the formula (16.169) for the photon propagator in Feynman's gauge.

16.23 Derive expression (16.184) for the inner product $\langle \zeta | \theta \rangle$.

16.24 Derive the representation (16.187) of the identity operator I for a single fermionic degree of freedom from the rules (16.174 & 16.177) for Grassmann integration and the anticommutation relations (16.170 & 16.176).

16.25 Derive the eigenvalue equation (16.192) from the definition (16.190 & 16.191) of the eigenstate $|\theta\rangle$ and the anticommutation relations (16.188 & 16.189).

16.26 Derive the eigenvalue relation (16.205) for the Fermi field $\psi_m(\boldsymbol{x}, t)$ from the anticommutation relations (16.201 & 16.202) and the definitions (16.203 & 16.204).

16.27 Derive the formula (16.206) for the inner product $\langle \chi' | \chi \rangle$ from the definition (16.204) of the ket $|\chi\rangle$.

16.28 Without setting $\hbar = 1$, imitate the derivation of the path-integral formula (16.25) and derive its three-dimensional version (16.26).