11

Tensors and Local Symmetries

11.1 Points and Coordinates

A point on a curved surface or in a curved space also is a point in a higher-dimensional flat space called an embedding space. For instance, a point on a sphere also is a point in three-dimensional euclidean space and in four-dimensional spacetime. One always can add extra dimensions, but it's simpler to use as few as possible, three in the case of a sphere.

On a sufficiently small scale, any reasonably smooth space locally looks like n-dimensional euclidean space. Such a space is called a **manifold**. Incidentally, according to Whitney's embedding theorem, every n-dimensional connected, smooth manifold can be embedded in $2n$-dimensional euclidean space $\mathbb{R}^{2n}$. So the embedding space for such spaces in general relativity has no more than eight dimensions.

We use coordinates to label points. For example, we can choose a polar axis and a meridian and label a point on the sphere by its polar and azimuthal angles (θ, ϕ) with respect to that axis and meridian. If we use a different axis and meridian, then the coordinates (θ', ϕ') for the same point will change. **Points are physical, coordinates are metaphysical. When we change our system of coordinates, the points don't change, but their coordinates do.**

Most points p have unique coordinates $x^i(p)$ and $x'^i(p)$ in their coordinate systems. For instance, polar coordinates (θ, ϕ) are unique for all points on a sphere — except the north and south poles which are labeled by $\theta = 0$ and $\theta = \pi$ and all $0 \leq \phi < 2\pi$. By using more than one coordinate system, one usually can arrange to label every point uniquely in some coordinate system. In the flat three-dimensional space in which the sphere is a surface, each point of the sphere has unique coordinates, $\vec{p} = (x, y, z)$.

We will use coordinate systems that represent the points of a manifold uniquely and smoothly at least in local patches, so that the maps

$$x'^i = x'^i(p) = x'^i(p(x)) = x'^i(x) \tag{11.1}$$

and

$$x^i = x^i(p) = x^i(p(x')) = x^i(x') \tag{11.2}$$

are well defined, differentiable, and one to one in the patches. We'll often group the n coordinates x^i together and write them collectively as x without a superscript. Since the coordinates $x(p)$ label the point p, we sometimes will call them "the point x." But p and x are different. The point p is unique with infinitely many coordinates x, x', x'', ... in infinitely many coordinate systems.

11.2 Scalars

A **scalar** is a quantity B that is the same in all coordinate systems

$$B' = B. \tag{11.3}$$

If it also depends upon the coordinates x of the spacetime point p, then it is a **scalar field**, and

$$B'(x') = B(x). \tag{11.4}$$

11.3 Contravariant Vectors

By the chain rule, the change in dx'^i due to changes in the unprimed coordinates is

$$dx'^i = \sum_j \frac{\partial x'^i}{\partial x^j}\, dx^j. \tag{11.5}$$

This transformation defines **contravariant vectors**: a quantity A^i is a component of contravariant vector if it transforms like dx^i

$$A'^i = \sum_j \frac{\partial x'^i}{\partial x^j}\, A^j. \tag{11.6}$$

The coordinate differentials dx^i form a contravariant vector. A contravariant vector $A^i(x)$ that depends on the coordinates x is a **contravariant vector**

field and transforms as

$$A'^i(x') = \sum_j \frac{\partial x'^i}{\partial x^j} A^j(x). \tag{11.7}$$

11.4 Covariant Vectors

The chain rule for partial derivatives

$$\frac{\partial}{\partial x'^i} = \sum_j \frac{\partial x^j}{\partial x'^i} \frac{\partial}{\partial x^j} \tag{11.8}$$

defines **covariant vectors**: a quantity C_i that transforms as

$$C'_i = \sum_j \frac{\partial x^j}{\partial x'^i} C_j \tag{11.9}$$

is a component of a **covariant vector**. A covariant vector $C_i(x)$ that depends on the coordinates x and transforms as

$$C'_i(x') = \sum_j \frac{\partial x^j}{\partial x'^i} C_j(x) \tag{11.10}$$

is a **covariant vector field**.

Example 11.1 (Gradient of a Scalar)**.** The derivatives of a scalar field $B'(x') = B(x)$ form a covariant vector field because

$$\frac{\partial B'(x')}{\partial x'^i} = \frac{\partial B(x)}{\partial x'^i} = \sum_j \frac{\partial x^j}{\partial x'^i} \frac{\partial B(x)}{\partial x^j}, \tag{11.11}$$

which shows that the gradient $\partial B(x)/\partial x^j$ fits the definition (11.10) of a covariant vector field. □

11.5 Euclidean Space in Euclidean Coordinates

If we use euclidean coordinates to describe points in euclidean space, then covariant and contravariant vectors are the same.

Euclidean space has a natural inner product (section 1.6), the usual dot-product, which is real and symmetric. In a euclidean space of n dimensions,

we may choose any n fixed, orthonormal basis vectors e_i

$$(e_i, e_j) \equiv e_i \cdot e_j = \sum_{k=1}^{n} e_i^k \, e_j^k = \delta_{ij} \tag{11.12}$$

and use them to represent any point p as the linear combination

$$p = \sum_{i=1}^{n} e_i \, x^i. \tag{11.13}$$

The **dual vectors** e^i are defined as those vectors whose inner products with the e_j are

$$(e^i, e_j) = \sum_{k=1}^{n} e_k^i e_{jk} = \delta_j^i \tag{11.14}$$

zero or one. Here they are the same as the vectors e_i, and so we don't need to distinguish e^i from $e_i = e^i$, but we will anyway.

The coefficients x^i are the euclidean coordinates in the e_i basis. Since they and the dual vectors e^j are orthonormal, each x^i is an inner product or dot product

$$x^i = e^i \cdot p = \sum_{j=1}^{n} e^i \cdot e_j \, x^j = \sum_{j=1}^{n} \delta_j^i \, x^j. \tag{11.15}$$

If we use different orthonormal vectors e'_i as a basis

$$p = \sum_{i=1}^{n} e'_i \, x'^i \tag{11.16}$$

then we get new euclidean coordinates $x'^i = e'^i \cdot p$ for the same point p. These two sets of coordinates are related by the equations

$$\begin{aligned} x'^i &= e'^i \cdot p = \sum_{j=1}^{n} e'^i \cdot e_j \, x^j \\ x^j &= e^j \cdot p = \sum_{k=1}^{n} e^j \cdot e'_k \, x'^k. \end{aligned} \tag{11.17}$$

Because the dual vectors e^i are the same as the basis vectors e_j and are **independent** of the euclidian coordinates x, the coefficients $\partial x'^i / \partial x^j$ and $\partial x^j / \partial x'^i$ of the transformation laws for contravariant (11.6) and covariant

(11.9) vectors are the same

$$\frac{\partial x'^i}{\partial x^j} = e'^i \cdot e_j = \sum_{k=1}^{n} e'^i_k e_{jk} = \sum_{k=1}^{n} e'_{ik} e^j_k = e^j \cdot e'_i = \frac{\partial x^j}{\partial x'^i}. \tag{11.18}$$

So contravariant and covariant vectors transform the same way in euclidean space with euclidean coordinates.

The relations between x'^i and x^j imply that

$$x'^i = \sum_{j,k=1}^{n} \left(e'^i \cdot e_j\right) \left(e^j \cdot e'_k\right) x'^k. \tag{11.19}$$

Since this holds for all coordinates x'^i, we have

$$\sum_{j=1}^{n} \left(e'^i \cdot e_j\right) \left(e^j \cdot e'_k\right) = \delta_{ik}. \tag{11.20}$$

The coefficients $e'^i \cdot e_j = e'_i \cdot e^j$ form an orthogonal matrix, and the linear operator

$$\sum_{i=1}^{n} e_i e'^{i\mathsf{T}} = \sum_{i=1}^{n} |e_i\rangle\langle e'^i| \tag{11.21}$$

is an orthogonal (real, unitary) transformation. The change $x \to x'$ is a rotation and/or a reflection (exercise 11.2).

Example 11.2 (A Euclidean Space of Two Dimensions)**.** In two-dimensional euclidean space, one can describe the same point by euclidean (x, y) and polar (r, θ) coordinates. The derivatives

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{\partial y}{\partial r} \tag{11.22}$$

respect the symmetry (11.18), but (exercise 11.1) the derivatives

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} \neq \frac{\partial x}{\partial \theta} = -y \quad \text{and} \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} \neq \frac{\partial y}{\partial \theta} = x \tag{11.23}$$

do not because polar coordinates are not euclidean. □

11.6 Summation convention

An index that appears twice in the same monomial, once as a subscript and once as a superscript, is a dummy index that is summed over as in

$$A_i\, B^i \equiv \sum_{i=1}^{n} A_i\, B^i. \tag{11.24}$$

The sum is understood to be over the relevant range of indices, usually from 0 or 1 to 3 or n.

These summation conventions make tensor notation almost as compact as matrix notation. They make equations easier to read and write.

Example 11.3 (The Kronecker Delta)**.** The summation convention and the chain rule imply that

$$\frac{\partial x'^i}{\partial x^k}\,\frac{\partial x^k}{\partial x'^j} = \frac{\partial x'^i}{\partial x'^j} = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{11.25}$$

The repeated index k has disappeared in this **contraction**. □

11.7 Minkowski Space

Minkowski space has one time dimension, labeled by $k = 0$, and n space dimensions. In special relativity, $n = 3$, and the Minkowski metric η is

$$\eta_{k\ell} = \eta^{k\ell} = \begin{cases} -1 & \text{if } k = \ell = 0 \\ 1 & \text{if } 1 \leq k = \ell \leq 3 \\ 0 & \text{if } k \neq \ell \end{cases} . \tag{11.26}$$

It defines an inner product between points p and q with coordinates x^k_p and x^ℓ_q as

$$(p, q) = p \cdot q = p^k\, \eta_{k\ell}\, q^\ell = (q, p). \tag{11.27}$$

If one time component vanishes, the Minkowski inner product reduces to the euclidean dot-product (1.82).

We can use different sets $\{e_i\}$ and $\{e'_i\}$ of $n+1$ **Lorentz-orthonormal** basis vectors

$$(e_i, e_j) = e_i \cdot e_j = e^k_i\, \eta_{k\ell}\, e^\ell_j = \eta_{ij} = e'_i \cdot e'_j = (e'_i, e'_j) \tag{11.28}$$

to represent any point p in the space either as a linear combination of the

vectors e_i with coefficients x^i or as a linear combination of the vectors e'_i with coefficients x'^i

$$p = e_i\, x^i = e'_i\, x'^i. \tag{11.29}$$

The dual vectors, which carry upper indices, are defined as

$$e^i = \eta^{ij}\, e_j \quad \text{and} \quad e'^i = \eta^{ij}\, e'_j. \tag{11.30}$$

They are orthonormal to the vectors e_i and e'_i because

$$(e^i, e_j) = e^i \cdot e_j = \eta^{ik}\, e_k \cdot e_j = \eta^{ik}\, \eta_{kj} = \delta^i_j \tag{11.31}$$

and similarly $(e'^i, e'_j) = e'^i \cdot e'_j = \delta^i_j$. Since the square of the matrix η is the identity matrix $\eta_{\ell i}\eta^{ij} = \delta^j_\ell$, it follows that

$$e_i = \eta_{ij}\, e^j \quad \text{and} \quad e'_i = \eta_{ij}\, e'^j. \tag{11.32}$$

The metric η raises (11.30) and lowers (11.32) the index of a basis vector.

The component x'^i is related to the components x^j by the linear map

$$x'^i = e'^i \cdot p = e'^i \cdot e_j\, x^j. \tag{11.33}$$

Such a map from a 4-vector x to a 4-vector x' is a **Lorentz transformation**

$$x'^i = L^i{}_j\, x^j \quad \text{with matrix} \quad L^i{}_j = e'^i \cdot e_j. \tag{11.34}$$

The inner product (p, q) of two points $p = e_i\, x^i = e'_i\, x'^i$ and $q = e_k\, y^k = e'_k\, y'^k$ is **physical** and so is **invariant under Lorentz transformations**

$$(p, q) = x^i\, y^k\, e_i \cdot e_k = \eta_{ik}\, x^i\, y^k = x'^i\, y'^k\, e'_i \cdot e'_k = \eta_{ik}\, x'^i\, y'^k. \tag{11.35}$$

With $x'^i = L^i{}_r\, x^r$ and $y'^k = L^k{}_s\, y^s$, this invariance is

$$\eta_{rs}\, x^r y^s = \eta_{ik}\, L^i{}_r\, x^r\, L^k{}_s\, y^s = \eta_{ik}\, x'^i y'^k \tag{11.36}$$

or since x^r and y^s are arbitrary

$$\eta_{rs} = \eta_{ik}\, L^i{}_r\, L^k{}_s = L^i{}_r\, \eta_{ik}\, L^k{}_s. \tag{11.37}$$

In matrix notation, a left index labels a row, and a right index labels a column. Transposition interchanges rows and columns $L^i{}_r = L^{\mathsf{T}}{}_r{}^i$, so

$$\eta_{rs} = L^{\mathsf{T}}{}_r{}^i\, \eta_{ik}\, L^k{}_s \quad \text{or} \quad \eta = L^{\mathsf{T}}\, \eta\, L \tag{11.38}$$

in matrix notation. In such matrix products, the height of an index—whether it is up or down—determines whether it is contravariant or covariant but does not affect its place in its matrix.

Example 11.4 (A Boost)**.** The matrix

$$L = \begin{pmatrix} \gamma & \sqrt{\gamma^2-1} & 0 & 0 \\ \sqrt{\gamma^2-1} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11.39}$$

where $\gamma = 1/\sqrt{1-v^2/c^2}$ represents a Lorentz transformation that is a boost in the x-direction. Boosts and rotations are Lorentz transformations. Working with 4×4 matrices can get tedious, so students are advised to think in terms of scalars, like $p \cdot x = p^i \eta_{ij} x^j = \boldsymbol{p} \cdot \boldsymbol{x} - Et$ whenever possible. □

If the basis vectors e and e' are independent of p and of x, then the coefficients of the transformation law (11.6) for contravariant vectors are

$$\frac{\partial x'^i}{\partial x^j} = e'^i \cdot e_j. \tag{11.40}$$

Similarly, the component x^j is $x^j = e^j \cdot p = e^j \cdot e'_i \, x'^i$, so the coefficients of the transformation law (11.9) for covariant vectors are

$$\frac{\partial x^j}{\partial x'^i} = e^j \cdot e'_i. \tag{11.41}$$

Using η to raise and lower the indices in the formula (11.40) for the coefficients of the transformation law (11.6) for contravariant vectors, we find, since the inner product (11.27) is symmetric,

$$\frac{\partial x'^i}{\partial x^j} = e'^i \cdot e_j = \eta^{ik} \, \eta_{j\ell} \, e'_k \cdot e^\ell = \eta^{ik} \, \eta_{j\ell} \, \frac{\partial x^\ell}{\partial x'^k} \tag{11.42}$$

which is $\pm \, \partial x^j / \partial x'^i$. So if we use coordinates associated with fixed basis vectors e_i in Minkowski space, then the coefficients for the two kinds of transformation laws differ only by occasional minus signs.

Thus if A^i is a contravariant vector

$$A'^i = \frac{\partial x'^i}{\partial x^j} \, A^j \tag{11.43}$$

then the relation (11.42) between the two kinds of coefficients implies that

$$\eta_{si} \, A'^i = \eta_{si} \, \frac{\partial x'^i}{\partial x^j} \, A^j = \eta_{si} \, \eta^{ik} \, \eta_{j\ell} \, \frac{\partial x^\ell}{\partial x'^k} \, A^j = \delta^k_s \, \frac{\partial x^\ell}{\partial x'^k} \, \eta_{j\ell} \, A^j = \frac{\partial x^\ell}{\partial x'^s} \, \eta_{\ell j} \, A^j \tag{11.44}$$

which shows that $A_\ell = \eta_{\ell j} \, A^j$ transforms covariantly

$$A'_s = \frac{\partial x^\ell}{\partial x'^s} A_\ell. \tag{11.45}$$

The metric η turns a contravariant vector into a covariant one. It also switches a covariant vector A_ℓ back to its contravariant form A^k

$$\eta^{k\ell} A_\ell = \eta^{k\ell} \eta_{\ell j} A^j = \delta^k_j A^j = A^k. \tag{11.46}$$

In Minkowski space, one uses η to raise and lower indices

$$A_i = \eta_{ij}\, A^j \text{ and } A^i = \eta^{ij}\, A_j. \tag{11.47}$$

In general relativity, the spacetime metric g raises and lowers indices.

11.8 Lorentz Transformations

In section 11.7, Lorentz transformations arose as linear maps of the coordinates due to a change of basis. They also are linear maps of the basis vectors e_i that preserve the inner products

$$(e_i, e_j) = e_i \cdot e_j = \eta_{ij} = e'_i \cdot e'_j = (e'_i, e'_j). \tag{11.48}$$

The vectors e_i are four linearly independent four-dimensional vectors, and so they span four-dimensional Minkowski space and can represent the vectors e'_i as

$$e'_i = \Lambda_i{}^k\, e_k \tag{11.49}$$

where the coefficients $\Lambda_i{}^k$ are real numbers. The requirement that the new basis vectors e'_i are Lorentz orthonormal gives

$$\eta_{ij} = e'_i \cdot e'_j = \Lambda_i{}^k\, e_k \cdot \Lambda_j{}^\ell\, e_\ell = \Lambda_i{}^k\, e_k \cdot e_\ell\, \Lambda_j{}^\ell = \Lambda_i{}^k\, \eta_{k\ell}\, \Lambda_j{}^\ell \tag{11.50}$$

or in matrix notation

$$\eta = \Lambda\, \eta\, \Lambda^\mathsf{T} \tag{11.51}$$

where Λ^T is the transpose $(\Lambda^\mathsf{T})^\ell{}_j = \Lambda_j{}^\ell$. Evidently Λ^T satisfies the definition (11.38) of a Lorentz transformation. What Lorentz transformation is it? The point p must remain invariant, so by (11.34 & 11.49) one has

$$p = e'_i\, x'^i = \Lambda_i{}^k\, e_k\, L^i{}_j x^j = \delta^k_j\, e_k\, x^j = e_j\, x^j \tag{11.52}$$

whence $\Lambda_i{}^k\, L^i{}_j = \delta^k_j$ or $\Lambda^\mathsf{T} L = I$. So $\Lambda^\mathsf{T} = L^{-1}$.

By multiplying condition (11.51) by the metric η first from the left and then from the right and using the fact that $\eta^2 = I$, we find

$$1 = \eta^2 = \eta\, \Lambda\, \eta\, \Lambda^\mathsf{T} = \Lambda\, \eta\, \Lambda^\mathsf{T}\, \eta \tag{11.53}$$

which gives us the inverse matrices

$$\Lambda^{-1} = \eta\, \Lambda^\mathsf{T}\, \eta = L^\mathsf{T} \quad \text{and} \quad (\Lambda^\mathsf{T})^{-1} = \eta\, \Lambda\, \eta = L. \tag{11.54}$$

In special relativity, contravariant vectors transform as

$$dx'^i = L^i{}_j\, dx^j \tag{11.55}$$

and since $x^j = L^{-1j}{}_i\, x'^i$, the covariant ones transform as

$$\frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j} = L^{-1j}{}_i\,\frac{\partial}{\partial x^j} = \Lambda_i{}^j\,\frac{\partial}{\partial x^j}. \tag{11.56}$$

By taking the determinant of both sides of (11.51) and using the transpose (1.194) and product (1.207) rules for determinants, we find that $\det\Lambda = \pm 1$.

11.9 Special Relativity

The spacetime of special relativity is flat, four-dimensional Minkowski space. The inner product $(p-q)\cdot(p-q)$ of the interval $p-q$ between two points is physical and independent of the coordinates and therefore invariant. Points p and q that have $(p-q)\cdot(p-q) > 0$ are **spacelike**. If p and q are spacelike, then they occur at the same time in some Lorentz frame. Points p and q that have $(p-q)\cdot(p-q) < 0$ are **timelike**. If p and q are timelike, then they occur at the same place in some Lorentz frame.

If the points p and q are close neighbors with coordinates $x^i + dx^i$ for p and x^i for q, then that invariant inner product is

$$(p-q)\cdot(p-q) = e_i\, dx^i \cdot e_j\, dx^j = dx^i\,\eta_{ij}\, dx^j = \boldsymbol{dx}^2 - (dx^0)^2 \tag{11.57}$$

with $dx^0 = c\,dt$. (At some point in what follows, we'll measure distance in light-seconds so that $c = 1$.) If the points p and q are on the trajectory of a massive particle moving at velocity $\boldsymbol{v}$, then this invariant quantity is the square of the **invariant distance**

$$ds^2 = \boldsymbol{dx}^2 - c^2 dt^2 = \left(\boldsymbol{v}^2 - c^2\right) dt^2 \tag{11.58}$$

which always is negative since $v < c$. The time in the rest frame of the particle is the **proper time**. The square of its differential element is

$$d\tau^2 = -\,ds^2/c^2 = \left(1 - v^2/c^2\right) dt^2. \tag{11.59}$$

A particle of mass zero moves at the speed of light, and so its element $d\tau$ of proper time is zero. But for a particle of mass $m > 0$ moving at speed v, the element of proper time $d\tau$ is smaller than the corresponding element of laboratory time dt by the factor $\sqrt{1 - v^2/c^2}$. The proper time is the time in the rest frame of the particle, $d\tau = dt$ when $v = 0$. So if $T(0)$ is the lifetime

of a particle at rest, then the apparent lifetime $T(v)$ when the particle is moving at speed v is

$$T(v) = dt = \frac{d\tau}{\sqrt{1 - v^2/c^2}} = \frac{T(0)}{\sqrt{1 - v^2/c^2}} \tag{11.60}$$

which is longer — an effect known as **time dilation**.

Example 11.5 (Time Dilation in Muon Decay)**.** A muon at rest has a mean life of $T(0) = 2.2 \times 10^{-6}$ seconds. Cosmic rays hitting nitrogen and oxygen nuclei make pions high in the Earth's atmosphere. The pions rapidly decay into muons in 2.6×10^{-8} s. A muon moving at the speed of light from 10 km takes at least $t = 10\,\text{km}/300,000\,(\text{km/sec}) = 3.3 \times 10^{-5}$ s to hit the ground. Were it not for time dilation, the probability P of such a muon reaching the ground as a muon would be

$$P = e^{-t/T(0)} = \exp(-33/2.2) = e^{-15} = 2.6 \times 10^{-7}. \tag{11.61}$$

The (rest) mass of a muon is 105.66 MeV. So a muon of energy $E = 749$ MeV has by (11.68) a time-dilation factor of

$$\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{E}{mc^2} = \frac{749}{105.7} = 7.089 = \frac{1}{\sqrt{1 - (0.99)^2}}. \tag{11.62}$$

So a muon moving at a speed of $v = 0.99\,c$ has an apparent mean life $T(v)$ given by equation (11.60) as

$$T(v) = \frac{E}{mc^2}\,T(0) = \frac{T(0)}{\sqrt{1 - v^2/c^2}} = \frac{2.2 \times 10^{-6}\,\text{s}}{\sqrt{1 - (0.99)^2}} = 1.6 \times 10^{-5}\,\text{s}. \tag{11.63}$$

The probability of survival with time dilation is

$$P = e^{-t/T(v)} = \exp(-33/16) = 0.12 \tag{11.64}$$

so that 12% survive. Time dilation increases the chance of survival by a factor of 460,000 — no small effect. □

11.10 Kinematics

From the scalar $d\tau$, and the contravariant vector dx^i, we can make the 4-vector

$$u^i = \frac{dx^i}{d\tau} = \frac{dt}{d\tau}\left(\frac{dx^0}{dt}, \frac{d\boldsymbol{x}}{dt}\right) = \frac{1}{\sqrt{1 - v^2/c^2}}\,(c, \boldsymbol{v}) \tag{11.65}$$

in which $u^0 = c\,dt/d\tau = c/\sqrt{1 - v^2/c^2}$ and $\boldsymbol{u} = u^0\,\boldsymbol{v}/c$. The product mu^i is the **energy-momentum 4-vector** p^i

$$p^i = m\,u^i = m\,\frac{dx^i}{d\tau} = m\,\frac{dt}{d\tau}\,\frac{dx^i}{dt} = \frac{m}{\sqrt{1 - v^2/c^2}}\,\frac{dx^i}{dt}$$
$$= \frac{m}{\sqrt{1 - v^2/c^2}}\,(c, \boldsymbol{v}) = \left(\frac{E}{c}, \boldsymbol{p}\right). \tag{11.66}$$

Its invariant inner product is a constant characteristic of the particle and proportional to the square of its mass

$$c^2\,p^i\,p_i = mc\,u^i\,mc\,u_i = -E^2 + c^2\,\boldsymbol{p}^2 = -m^2\,c^4. \tag{11.67}$$

Note that the time-dilation factor is the ratio of the energy of a particle to its rest energy

$$\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{E}{mc^2} \tag{11.68}$$

and the velocity of the particle is its momentum divided by its equivalent mass E/c^2

$$\boldsymbol{v} = \frac{\boldsymbol{p}}{E/c^2}. \tag{11.69}$$

The analog of $\boldsymbol{F} = m\,\boldsymbol{a}$ is

$$m\,\frac{d^2x^i}{d\tau^2} = m\,\frac{du^i}{d\tau} = \frac{dp^i}{d\tau} = f^i \tag{11.70}$$

in which $p^0 = E/c$, and f^i is a 4-vector force.

Example 11.6 (Time Dilation and Proper Time)**.** In the frame of a laboratory, a particle of mass m with 4-momentum $p^i_{lab} = (E/c, p, 0, 0)$ travels a distance L in a time t for a 4-vector displacement of $x^i_{lab} = (ct, L, 0, 0)$. In its own rest frame, the particle's 4-momentum and 4-displacement are $p^i_{rest} = (mc, 0, 0, 0)$ and $x^i_{rest} = (c\tau, 0, 0, 0)$. Since the Minkowski inner product of two 4-vectors is Lorentz invariant, we have

$$\left(p^i x_i\right)_{rest} = \left(p^i x_i\right)_{lab} \quad \text{or} \quad Et - pL = mc^2\tau = mc^2 t\sqrt{1 - v^2/c^2} \tag{11.71}$$

so a massive particle's phase $\exp(-ip^i x_i/\hbar)$ is $\exp(imc^2\tau/\hbar)$. □

Example 11.7 ($p + \pi \to \Sigma + K$)**.** What is the minimum energy that a beam of pions must have to produce a sigma hyperon and a kaon by striking a proton at rest? Conservation of the energy-momentum 4-vector gives

$p_p + p_\pi = p_\Sigma + p_K$. We set $c = 1$ and use this equality in the invariant form $(p_p+p_\pi)^2 = (p_\Sigma+p_K)^2$. We compute $(p_p+p_\pi)^2$ in the the $p_p = (m_p, \mathbf{0})$ frame and set it equal to $(p_\Sigma + p_K)^2$ in the frame in which the spatial momenta of the Σ and the K cancel and vanish at minimum pion energy:

$$\begin{aligned}(p_p + p_\pi)^2 &= p_p^2 + p_\pi^2 + 2p_p \cdot p_\pi = -m_p^2 - m_\pi^2 - 2m_p E_\pi \\ &= (p_\Sigma + p_K)^2 = -\left(m_\Sigma + m_K\right)^2 . \end{aligned} \tag{11.72}$$

Thus, since the relevant masses (in MeV) are $m_{\Sigma^+} = 1189.4$, $m_{K^+} = 493.7$, $m_p = 938.3$, and $m_{\pi^+} = 139.6$, the minimum total energy of the pion is

$$E_\pi = \frac{(m_\Sigma + m_K)^2 - m_p^2 - m_\pi^2}{2m_p} \approx 1030 \quad \text{MeV} \tag{11.73}$$

of which 890 MeV is kinetic. □

11.11 Electrodynamics

In electrodynamics and in MKSA (SI) units, the three-dimensional vector potential $\boldsymbol{A}$ and the scalar potential ϕ form a covariant 4-vector potential

$$A_i = \left(\frac{-\phi}{c}, \boldsymbol{A}\right). \tag{11.74}$$

The contravariant 4-vector potential is $A^i = (\phi/c, \boldsymbol{A})$. The **magnetic induction** is

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad \text{or} \quad B_i = \epsilon_{ijk}\partial_j A_k \tag{11.75}$$

in which $\partial_j = \partial/\partial x^j$, the sum over the repeated indices j and k runs from 1 to 3, and ϵ_{ijk} is totally antisymmetric with $\epsilon_{123} = 1$. The electric field is

$$E_i = c\left(\frac{\partial A_0}{\partial x^i} - \frac{\partial A_i}{\partial x^0}\right) = -\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} \tag{11.76}$$

where $x^0 = ct$. In 3-vector notation, $\boldsymbol{E}$ is given by the gradient of ϕ and the time-derivative of $\boldsymbol{A}$

$$\boldsymbol{E} = -\nabla\phi - \dot{\boldsymbol{A}}. \tag{11.77}$$

In terms of the second-rank, antisymmetric Faraday field-strength tensor

$$F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} = -F_{ji} \tag{11.78}$$

the electric field is $E_i = c\,F_{i0}$ and the magnetic field B_i is

$$B_i = \frac{1}{2}\epsilon_{ijk}\,F_{jk} = \frac{1}{2}\epsilon_{ijk}\left(\frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k}\right) = (\nabla \times \boldsymbol{A})_i \tag{11.79}$$

where the sum over repeated indices runs from 1 to 3. The inverse equation $F_{jk} = \epsilon_{jki}B_i$ for spatial j and k follows from the Levi-Civita identity (1.455)

$$\begin{aligned}\epsilon_{jki}B_i &= \frac{1}{2}\epsilon_{jki}\epsilon_{inm}\,F_{nm} = \frac{1}{2}\epsilon_{ijk}\epsilon_{inm}\,F_{nm}\\ &= \frac{1}{2}\left(\delta_{jn}\,\delta_{km} - \delta_{jm}\,\delta_{kn}\right)\,F_{nm} = \frac{1}{2}\left(F_{jk} - F_{kj}\right) = F_{jk}.\end{aligned} \tag{11.80}$$

In 3-vector notation and MKSA = SI units, Maxwell's equations are a ban on magnetic monopoles and **Faraday's law**, both homogeneous,

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \quad \text{and} \quad \boldsymbol{\nabla}\times\boldsymbol{E} + \dot{\boldsymbol{B}} = 0 \tag{11.81}$$

and **Gauss's law** and the **Maxwell-Ampère law**, both inhomogeneous,

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_f \quad \text{and} \quad \boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{j}_f + \dot{\boldsymbol{D}}. \tag{11.82}$$

Here ρ_f is the density of **free** charge and $\boldsymbol{j}_f$ is the **free current density**. By *free*, we understand charges and currents that do not arise from polarization and are not restrained by chemical bonds. The divergence of $\boldsymbol{\nabla}\times\boldsymbol{H}$ vanishes (like that of any curl), and so the Maxwell-Ampère law and Gauss's law imply that free charge is conserved

$$0 = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H}) = \boldsymbol{\nabla}\cdot\boldsymbol{j}_f + \boldsymbol{\nabla}\cdot\dot{\boldsymbol{D}} = \boldsymbol{\nabla}\cdot\boldsymbol{j}_f + \dot{\rho}_f. \tag{11.83}$$

If we use this **continuity equation** to replace $\boldsymbol{\nabla}\cdot\boldsymbol{j}_f$ with $-\dot{\rho}_f$ in its middle form $0 = \boldsymbol{\nabla}\cdot\boldsymbol{j}_f + \boldsymbol{\nabla}\cdot\dot{\boldsymbol{D}}$, then we see that the Maxwell-Ampère law preserves the Gauss-law constraint in time

$$0 = \boldsymbol{\nabla}\cdot\boldsymbol{j}_f + \boldsymbol{\nabla}\cdot\dot{\boldsymbol{D}} = \frac{\partial}{\partial t}\left(-\rho_f + \boldsymbol{\nabla}\cdot\boldsymbol{D}\right). \tag{11.84}$$

Similarly, Faraday's law preserves the constraint $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$

$$0 = -\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) = \frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0. \tag{11.85}$$

In a **linear, isotropic** medium, the **electric displacement** $\boldsymbol{D}$ is related to the electric field $\boldsymbol{E}$ by the **permittivity** ϵ, $\boldsymbol{D} = \epsilon\boldsymbol{E}$, and the **magnetic** or **magnetizing** field $\boldsymbol{H}$ differs from the magnetic induction $\boldsymbol{B}$ by the **permeability** μ, $\boldsymbol{H} = \boldsymbol{B}/\mu$.

On a sub-nanometer scale, the microscopic form of Maxwell's equations

applies. On this scale, the homogeneous equations (11.81) are unchanged, but the inhomogeneous ones are

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=\frac{\rho}{\epsilon_0}\quad\text{and}\quad\boldsymbol{\nabla}\times\boldsymbol{B}=\mu_0\,\boldsymbol{j}+\epsilon_0\,\mu_0\,\dot{\boldsymbol{E}}=\mu_0\,\boldsymbol{j}+\frac{\dot{\boldsymbol{E}}}{c^2}\tag{11.86}$$

in which ρ and $\boldsymbol{j}$ are the total charge and current densities, and $\epsilon_0 = 8.854\times10^{-12}$ F/m and $\mu_0 = 4\pi\times10^{-7}$ N/A^2 are the **electric** and **magnetic constants**, whose product is the inverse of the square of the speed of light, $\epsilon_0\mu_0 = 1/c^2$. Gauss's law and the Maxwell-Ampère law (11.86) imply (exercise 11.6) that the microscopic (total) current 4-vector $j = (c\rho, \boldsymbol{j})$ obeys the continuity equation $\dot{\rho}+\boldsymbol{\nabla}\cdot\boldsymbol{j}=0$. Electric charge is conserved.

In vacuum, $\rho = \boldsymbol{j} = 0$, $\boldsymbol{D} = \epsilon_0\boldsymbol{E}$, and $\boldsymbol{H} = \boldsymbol{B}/\mu_0$, and Maxwell's equations become

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{B}&=0\quad\text{and}\quad\boldsymbol{\nabla}\times\boldsymbol{E}+\dot{\boldsymbol{B}}=0\\ \boldsymbol{\nabla}\cdot\boldsymbol{E}&=0\quad\text{and}\quad\boldsymbol{\nabla}\times\boldsymbol{B}=\frac{1}{c^2}\dot{\boldsymbol{E}}.\end{aligned}\tag{11.87}$$

Two of these equations $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$ and $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0$ are constraints. Taking the curl of the other two equations, we find

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E})=-\frac{1}{c^2}\ddot{\boldsymbol{E}}\quad\text{and}\quad\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B})=-\frac{1}{c^2}\ddot{\boldsymbol{B}}.\tag{11.88}$$

One may use the Levi-Civita identity (1.455) to show (exercise 11.8) that

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{E})-\triangle\boldsymbol{E}\ \text{and}\ \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{B})-\triangle\boldsymbol{B}\tag{11.89}$$

in which $\triangle\equiv\boldsymbol{\nabla}^2$. Since in vacuum the divergence of $\boldsymbol{E}$ vanishes, and since that of $\boldsymbol{B}$ always vanishes, these identities and the curl-curl equations (11.88) tell us that waves of $\boldsymbol{E}$ and $\boldsymbol{B}$ move at the speed of light

$$\frac{1}{c^2}\ddot{\boldsymbol{E}}-\triangle\boldsymbol{E}=0\quad\text{and}\quad\frac{1}{c^2}\ddot{\boldsymbol{B}}-\triangle\boldsymbol{B}=0.\tag{11.90}$$

We may write the two homogeneous Maxwell equations (11.81) as

$$\begin{aligned}\partial_iF_{jk}+\partial_kF_{ij}+\partial_jF_{ki}&=\partial_i\left(\partial_jA_k-\partial_kA_j\right)+\partial_k\left(\partial_iA_j-\partial_jA_i\right)\\&\quad+\partial_j\left(\partial_kA_i-\partial_iA_k\right)=0\end{aligned}\tag{11.91}$$

(exercise 11.9). This relation, known as the **Bianchi identity**, actually is a generally covariant tensor equation

$$\epsilon^{\ell ijk}\partial_iF_{jk}=0\tag{11.92}$$

in which $\epsilon^{\ell ijk}$ is totally antisymmetric, as explained in Sec. 11.37. There are four versions of this identity (corresponding to the four ways of choosing

three different indices i, j, k from among four and leaving out one, ℓ). The $\ell = 0$ case gives the scalar equation $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, and the three that have $\ell \neq 0$ give the vector equation $\boldsymbol{\nabla} \times \boldsymbol{E} + \dot{\boldsymbol{B}} = 0$.

In tensor notation, the microscopic form of the two inhomogeneous equations (11.86)—the laws of Gauss and Ampère—are

$$\partial_i F^{ki} = \mu_0 \, j^k \tag{11.93}$$

in which j^k is the current 4-vector

$$j^k = (c\rho, \boldsymbol{j}) \, . \tag{11.94}$$

The **Lorentz force law** for a particle of charge q is

$$m \, \frac{d^2 x^i}{d\tau^2} = m \, \frac{du^i}{d\tau} = \frac{dp^i}{d\tau} = f^i = q \, F^{ij} \, \frac{dx_j}{d\tau} = q \, F^{ij} \, u_j . \tag{11.95}$$

We may cancel a factor of $dt/d\tau$ from both sides and find for $i = 1, 2, 3$

$$\frac{dp^i}{dt} = q \left(-F^{i0} + \epsilon_{ijk} B_k v_j \right) \quad \text{or} \quad \frac{d\boldsymbol{p}}{dt} = q \, (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{11.96}$$

and for $i = 0$

$$\frac{dE}{dt} = q \, \boldsymbol{E} \cdot \boldsymbol{v} \tag{11.97}$$

which shows that only the electric field does work. The only special-relativistic correction needed in Maxwell's electrodynamics is a factor of $1/\sqrt{1 - v^2/c^2}$ in these equations. That is, we use $\boldsymbol{p} = m\boldsymbol{u} = m\boldsymbol{v}/\sqrt{1 - v^2/c^2}$ not $\boldsymbol{p} = m\boldsymbol{v}$ in (11.96), and we use the total energy E not the kinetic energy in (11.97). The reason why so little of classical electrodynamics was changed by special relativity is that electric and magnetic effects were accessible to measurement during the 1800's. Classical electrodynamics was almost perfect.

Keeping track of factors of the speed of light is a lot of trouble and a distraction; in what follows, we'll often use units with $c = 1$.

11.12 Tensors

Tensors are structures that transform like products of vectors. A first-rank tensor is a covariant or a contravariant vector. Second-rank tensors also are

distiguished by how they transform under changes of coordinates:

$$\begin{aligned} \text{contravariant} \quad M'^{ij} &= \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l}\, M^{kl} \\ \text{mixed} \quad N'^i_j &= \frac{\partial x'^i}{\partial x^k}\frac{\partial x^l}{\partial x'^j}\, N^k_l \\ \text{covariant} \quad F'_{ij} &= \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j}\, F_{kl}. \end{aligned} \tag{11.98}$$

We can define tensors of higher rank by extending the these definitions to quantities with more indices.

Example 11.8 (Some Second-Rank Tensors)**.** If A_k and B_ℓ are covariant vectors, and C^m and D^n are contravariant vectors, then the product $C^m\, D^n$ is a second-rank contravariant tensor, and all four products $A_k\, C^m$, $A_k\, D^n$, $B_k\, C^m$, and $B_k\, D^n$ are second-rank mixed tensors, while $C^m\, D^n$ as well as $C^m\, C^n$ and $D^m\, D^n$ are second-rank contravariant tensors. □

Since the transformation laws that define tensors are linear, any linear combination of tensors of a given rank and kind is a tensor of that rank and kind. Thus if F_{ij} and G_{ij} are both second-rank covariant tensors, then so is their sum

$$H_{ij} = F_{ij} + G_{ij}. \tag{11.99}$$

A covariant tensor is **symmetric** if it is independent of the order of its indices. That is, if $S_{ik} = S_{ki}$, then S is symmetric. Similarly, a contravariant tensor is symmetric if permutations of its indices leave it unchanged. Thus A is symmetric if $A^{ik} = A^{ki}$.

A covariant or contravariant tensor is **antisymmetric** if it changes sign when any two of its indices are interchanged. So A_{ik}, B^{ik}, and C_{ijk} are antisymmetric if

$$\begin{aligned} A_{ik} = -A_{ki} \quad \text{and} \quad B^{ik} = -B^{ki} \quad \text{and} \\ C_{ijk} = C_{jki} = C_{kij} = -C_{jik} = -C_{ikj} = -C_{kji}. \end{aligned} \tag{11.100}$$

Example 11.9 (Three Important Tensors)**.** The Maxwell field strength $F_{kl}(x)$ is a second-rank covariant tensor; so is the metric of spacetime $g_{ij}(x)$. The Kronecker delta δ^i_j is a mixed second-rank tensor; it transforms as

$$\delta'^i_j = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^l}{\partial x'^j}\,\delta^k_l = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^k}{\partial x'^j} = \frac{\partial x'^i}{\partial x^j} = \delta^i_j. \tag{11.101}$$

So it is **invariant** under changes of coordinates. □

Example 11.10 (Contractions)**.** Although the product $A_k\, C^\ell$ is a mixed second-rank tensor, the product $A_k\, C^k$ transforms as a scalar because

$$A'_k\, C'^k = \frac{\partial x^\ell}{\partial x'^k}\frac{\partial x'^k}{\partial x^m} A_\ell\, C^m = \frac{\partial x^\ell}{\partial x^m} A_\ell\, C^m = \delta^\ell_m A_\ell\, C^m = A_\ell\, C^\ell. \tag{11.102}$$

A sum in which an index is repeated once covariantly and once contravariantly is a **contraction** as in the Kronecker-delta equation (11.25). In general, the rank of a tensor is the number of uncontracted indices. □

11.13 Differential Forms

By (11.10 & 11.5), a covariant vector field contracted with contravariant coordinate differentials is invariant under arbitrary coordinate transformations

$$A' = A'_i\, dx'^i = \frac{\partial x^j}{\partial x'^i}\, A_j\, \frac{\partial x'^i}{\partial x^k}\, dx^k = \delta^j_k\, A_j\, dx^k = A_k\, dx^k = A. \tag{11.103}$$

This invariant quantity $A = A_k\, dx^k$ is a called a **1-form** in the language of **differential forms** introduced about a century ago by Élie Cartan (1869–1951, son of a blacksmith).

The **wedge product** $dx \wedge dy$ of two coordinate differentials is the directed area spanned by the two differentials and is defined to be antisymmetric

$$dx \wedge dy = -\, dy \wedge dx \quad \text{and} \quad dx \wedge dx = dy \wedge dy = 0 \tag{11.104}$$

so as to transform correctly under a change of coordinates. In terms of the coordinates $u = u(x, y)$ and $v = v(x, y)$, the new element of area is

$$du \wedge dv = \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) \wedge \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right). \tag{11.105}$$

Labeling partial derivatives by subscripts (6.20) and using the antisymmetry (11.104), we see that the new element of area $du \wedge dv$ is the old area $dx \wedge dy$ multiplied by the Jacobian $J(u, v; x, y)$ of the transformation $x, y \to u, v$

$$\begin{aligned} du \wedge dv &= (u_x dx + u_y dy) \wedge (v_x dx + v_y dy) \\ &= u_x\, v_x\, dx \wedge dx + u_x\, v_y\, dx \wedge dy + u_y\, v_x\, dy \wedge dx + u_y\, v_y\, dy \wedge dy \\ &= (u_x\, v_y - u_y v_x)\; dx \wedge dy \\ &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} dx \wedge dy = J(u, v; x, y)\, dx \wedge dy. \end{aligned} \tag{11.106}$$

A contraction $H = \frac{1}{2} H_{ik}\, dx^i \wedge dx^k$ of a second-rank covariant tensor with a wedge product of two differentials is a 2-form. A **p-form** is a rank-p covariant tensor contracted with a wedge product of p differentials

$$K = \frac{1}{p!}\, K_{i_1 \ldots i_p}\, dx^{i_1} \wedge \ldots dx^{i_p}. \tag{11.107}$$

The **exterior derivative** d differentiates and adds a differential; it turns a p-form into a $(p+1)$-form. It converts a function or a **0-form** f into a 1-form

$$df = \frac{\partial f}{\partial x^i}\, dx^i \tag{11.108}$$

and a 1-form $A = A_j\, dx^j$ into a 2-form $dA = d(A_j\, dx^j) = (\partial_i A_j)\, dx^i \wedge dx^j$.

Example 11.11 (The Curl)**.** The exterior derivative of the 1-form

$$A = A_x\, dx + A_y\, dy + A_z\, dz \tag{11.109}$$

is a 2-form that contains the curl (6.39) of $\boldsymbol{A}$

$$\begin{aligned} dA &= \partial_y A_x\, dy \wedge dx + \partial_z A_x\, dz \wedge dx \\ &\quad + \partial_x A_y\, dx \wedge dy + \partial_z A_y\, dz \wedge dy \\ &\quad + \partial_x A_z\, dx \wedge dz + \partial_y A_z\, dy \wedge dz \\ &= (\partial_y A_z - \partial_z A_y)\, dy \wedge dz \\ &\quad + (\partial_z A_x - \partial_x A_z)\, dz \wedge dx \\ &\quad + (\partial_x A_y - \partial_y A_x)\, dx \wedge dy \\ &= (\nabla \times A)_x\, dy \wedge dz + (\nabla \times A)_y\, dz \wedge dx + (\nabla \times A)_z\, dx \wedge dy. \end{aligned} \tag{11.110}$$

□

The exterior derivative of the 1-form $A = A_j\, dx^j$ is the 2-form

$$dA = d\left(A_j \wedge dx^j\right) = \partial_i A_j\, dx^i \wedge dx^j = \tfrac{1}{2}\, F_{ij}\, dx^i \wedge dx^j = F \tag{11.111}$$

in which $\partial_i = \partial/\partial x^i$. So d turns the electromagnetic 1-form A—the 4-vector potential or gauge field A_j—into the Faraday 2-form—the tensor F_{ij}.

The square dd of the exterior derivative vanishes in the sense that dd applied to any p-form Q is zero

$$\begin{aligned} d\left[d\left(Q_{i\ldots} dx^i \wedge \ldots\right)\right] &= d\left[(\partial_r Q_{i\ldots})\, dx^r \wedge dx^i \wedge \ldots\right] \\ &= (\partial_s \partial_r Q_{i\ldots})\, dx^s \wedge dx^r \wedge dx^i \wedge \ldots = 0 \end{aligned} \tag{11.112}$$

because $\partial_s \partial_r Q$ is symmetric in r and s while $dx^s \wedge dx^r$ is anti-symmetric.

Some writers drop the wedges and write $dx^i \wedge dx^j$ as $dx^i dx^j$ while keeping

the rules of antisymmetry $dx^i dx^j = -dx^j dx^i$ and $\left(dx^i\right)^2 = 0$. But this economy prevents one from using invariant quantities like $S = \frac{1}{2} S_{ik}\, dx^i dx^k$ in which S_{ik} is a second-rank covariant symmetric tensor. If M_{ik} is a covariant second-rank tensor with no particular symmetry, then (exercise 11.7) only its antisymmetric part contributes to the 2-form $M_{ik}\, dx^i \wedge dx^k$ and only its symmetric part contributes to the quantity $M_{ik}\, dx^i dx^k$.

The exterior derivative d applied to the Faraday 2-form $F = dA$ gives

$$dF = ddA = 0 \tag{11.113}$$

which is the Bianchi identity (11.92). A p-form H is **closed** if $dH = 0$. By (11.113), the Faraday 2-form is closed, $dF = 0$.

A p-form H is **exact** if there is a $(p-1)$-form K whose differential is $H = dK$. The identity (11.112) or $dd = 0$ implies that **every exact form is closed**. The lemma of Poincaré shows that **every closed form is locally exact**.

If the A_i in the 1-form $A = A_i dx^i$ commute with each other, then the 2-form $A \wedge A = 0$. But if the A_i don't commute because they are matrices or operators or Grassmann variables, then $A \wedge A = \frac{1}{2}[A_i, A_j]\, dx^i \wedge dx^j$ need not vanish.

Example 11.12 (If $\dot{B} = 0$, the electric field is closed and locally exact)**.** If $\dot{\boldsymbol{B}} = 0$, then by Faraday's law (11.81) the curl of the electric field vanishes, $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$. In terms of the 1-form $E = E_i\, dx^i$ for $i = 1, 2, 3$, the vanishing of its curl $\boldsymbol{\nabla} \times \boldsymbol{E}$ is

$$dE = \partial_j E_i\, dx^j \wedge dx^i = \frac{1}{2}\left(\partial_j E_i - \partial_i E_j\right) dx^j \wedge dx^i = 0. \tag{11.114}$$

So E is closed. We can define a quantity $V_P(\boldsymbol{x})$ as a line integral of the 1-form E along a path P to $\boldsymbol{x}$ from some starting point $\boldsymbol{x}_0$

$$V_P(\boldsymbol{x}) = -\int_{P,\,\boldsymbol{x}_0}^{\boldsymbol{x}} E_i\, dx^i = -\int_P E. \tag{11.115}$$

In general, $V_P(\boldsymbol{x})$ will depend on the path P. But the path dependence $V_{P'}(\boldsymbol{x}) - V_P(\boldsymbol{x})$ is the line integral of E around a closed loop which by Stokes's theorem (6.44) is an integral of the vanishing curl $\boldsymbol{\nabla} \times \boldsymbol{E}$ over any surface S whose boundary ∂S is the closed curve $P' - P$

$$V_{P'}(\boldsymbol{x}) - V_P(\boldsymbol{x}) = \oint_{P'-P} E_i\, dx^i = \int_S (\boldsymbol{\nabla} \times \boldsymbol{E}) \cdot da = 0 \tag{11.116}$$

as long as $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$ on and near the surface S. In the notation of forms,

Stokes's theorem is

$$V_{P'}(\boldsymbol{x}) - V_P(\boldsymbol{x}) = \int_{\partial S} E = \int_S dE = 0 \tag{11.117}$$

(George Stokes, 1819–1903). Thus the potential $V_P(\boldsymbol{x}) = V(\boldsymbol{x})$ is independent of the path, $\boldsymbol{E} = -\boldsymbol{\nabla} V(\boldsymbol{x})$, and the 1-form $E = E_i\, dx^i = -\partial_i V\, dx^i = -dV$ is locally exact. □

The general form of Stokes's theorem is that the integral of any p-form H over the boundary ∂R of any $(p+1)$-dimensional, simply connected, orientable region R is equal to the integral of the $(p+1)$-form dH over R

$$\int_{\partial R} H = \int_R dH \tag{11.118}$$

which for $p = 1$ gives (6.44 & 11.117).

Example 11.13 (Stokes's Theorem for 0-forms)**.** Here $p = 0$, the region $R = [a, b]$ is 1-dimensional, H is a 0-form, and Stokes's theorem is

$$H(b) - H(a) = \int_{\partial R} H = \int_R dH = \int_a^b dH(x) = \int_a^b H'(x)\, dx \tag{11.119}$$

familiar from elementary calculus. □

Example 11.14 (Exterior Derivatives Anticommute with Differentials)**.** The exterior derivative acting on two one-forms $A = A_i dx^i$ and $B = B_j dx^j$ is

$$\begin{aligned} d(A \wedge B) &= d(A_i dx^i \wedge B_j dx^j) = \partial_k (A_i B_j)\, dx^k \wedge dx^i \wedge dx^j \qquad (11.120)\\ &= (\partial_k A_i)\, B_j\, dx^k \wedge dx^i \wedge dx^j + A_i\, (\partial_k B_j)\, dx^k \wedge dx^i \wedge dx^j \\ &= (\partial_k A_i)\, B_j\, dx^k \wedge dx^i \wedge dx^j - A_i\, (\partial_k B_j)\, dx^i \wedge dx^k \wedge dx^j \\ &= (\partial_k A_i)\, dx^k \wedge dx^i \wedge B_j dx^j - A_i dx^i \wedge (\partial_k B_j)\, dx^k \wedge dx^j \\ &= dA \wedge B - A \wedge dB. \end{aligned}$$

If A is a p-form, then $d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB$ (exercise 11.10). □

11.14 Tensor Equations

Maxwell's homogeneous equations (11.92) relate the derivatives of the field-strength tensor to each other as

$$0 = \partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki}. \tag{11.121}$$

They are generally covariant **tensor equations** (sections 11.35 & 11.37). They follow from the Bianchi identity (11.113)

$$dF = ddA = 0. \tag{11.122}$$

Maxwell's inhomegneous equations (11.93) relate the derivatives of the field-strength tensor to the current density j^i and to the square root of the modulus g of the determinant of the metric tensor g_{ij} (section 11.18)

$$\frac{\partial(\sqrt{g}\, F^{ik})}{\partial x^k} = \mu_0 \sqrt{g}\, j^i. \tag{11.123}$$

They are generally covariant tensor equations. We'll write them as invariant forms in section 11.29 and derive them from an action principle in section 11.43.

If we can write a physical law in one coordinate system as a tensor equation

$$G^{kl} = 0 \tag{11.124}$$

then in any other coordinate system, the corresponding tensor equation

$$G'^{ij} = 0 \tag{11.125}$$

also is valid since

$$G'^{ij} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} G^{kl} = 0. \tag{11.126}$$

Similarly, physical laws remain the same when expressed in terms of invariant forms. **Thus by writing a theory in terms of tensors or forms, one gets a theory that is true in all coordinate systems if it is true in any**. Only such generally covariant theories have a chance at being right in our coordinate system, which is not special. One way to make a generally covariant theory is to start with an action that is invariant under all coordinate transformations.

11.15 The Quotient Theorem

Suppose that the product $B\,A$ of a quantity B (with unknown transformation properties) with an arbitrary tensor A (of a given rank and kind) is a tensor. Then B is itself a tensor. The simplest example is when $B_i A^i$ is a scalar for all contravariant vectors A^i

$$B'_i A'^i = B_j A^j. \tag{11.127}$$

Then since A^i is a contravariant vector

$$B'_i A'^i = B'_i \frac{\partial x'^i}{\partial x^j} A^j = B_j A^j \tag{11.128}$$

or

$$\left(B'_i \frac{\partial x'^i}{\partial x^j} - B_j \right) A^j = 0. \tag{11.129}$$

Since this equation holds for all vectors A, we may promote it to the level of a vector equation

$$B'_i \frac{\partial x'^i}{\partial x^j} - B_j = 0. \tag{11.130}$$

Multiplying both sides by $\partial x^j / \partial x'^k$ and summing over j, we get

$$B'_i \frac{\partial x'^i}{\partial x^j} \frac{\partial x^j}{\partial x'^k} = B_j \frac{\partial x^j}{\partial x'^k} \tag{11.131}$$

which shows that the unknown quantity B_i transforms as a covariant vector

$$B'_k = \frac{\partial x^j}{\partial x'^k} B_j. \tag{11.132}$$

The quotient rule works for unknowns B and tensors A of arbitrary rank and kind. The proof in each case is similar to the one given here.

11.16 The embedding space

The change in a point $p(x)$ on a smooth manifold due to an infinitesimal change $dx^i(p)$ in the coordinates $x^i(p)$ of the point is

$$dp^\alpha(x) = \frac{\partial p(x)}{\partial x^i}\, dx^i = e_i{}^\alpha(x)\, dx^i \tag{11.133}$$

in which the superscript α labels the n coordinates of the embedding space $\mathbb{R}^n$ and the index i labels the coordinates on the manifold. The embedding space $\mathbb{R}^n$ has its own metric $I_{\alpha\beta}$ which determines the inner products of basis vectors as

$$e_i \cdot e_k = \sum_{\alpha,\beta=1}^{n} e_i{}^\alpha\, I_{\alpha\beta}\, e_j{}^\beta. \tag{11.134}$$

The inner metric $I_{\alpha\beta}$ usually is a diagonal matrix with eigenvalues ± 1.

Example 11.15 (A sphere)**.** For the surface of a 2-sphere in $\mathbb{R}^3$, one might have $\alpha = 1, 2, 3$ and $i = \theta, \phi$. One then could use the 3×3 identity matrix I to form inner products

$$e_i \cdot e_j = e_i{}^\alpha \, I_{\alpha\beta} \, e_j{}^\beta = e_i{}^\alpha \, e_j{}^\alpha. \tag{11.135}$$

□

Example 11.16 (Spacetime)**.** If we use four coordinates to describe ordinary spacetime as a curved surface in a flat embedding space $\mathbb{R}^8$, then we would have $\alpha = 0, 1, \ldots, 7$ and $i = 0, 1, 2, 3$. We then could use an 8×8 diagonal matrix I with entries $I_{\alpha\beta} = \pm\delta_{\alpha\beta} = \pm 1$ to form inner products

$$e_i \cdot e_j = \sum_{\alpha,\beta=0}^{7} e_i{}^\alpha \, I_{\alpha\beta} \, e_j{}^\beta. \tag{11.136}$$

□

11.17 Tangent vectors

Vectors defined by (11.133) as derivatives

$$e_i{}^\alpha(x) \equiv \frac{\partial p^\alpha(x)}{\partial x^i} \tag{11.137}$$

of physical points p with respect to the coordinates x^i are tangent to the manifold at p. They are **local coordinate basis vectors**. These **tangent vectors** span a **tangent space** of as many dimensions as there are coordinates x^i.

In a different system of coordinates x', the same displacement (11.133) is $dp = e'_i(x') \, dx'^i$. The basis vectors $e_i(x)$ and $e'_i(x')$

$$e_i(x) = \frac{\partial p}{\partial x^i} \quad \text{and} \quad e'_i(x') = \frac{\partial p}{\partial x'^i}. \tag{11.138}$$

are linearly related to each other and transform as covariant vectors

$$e'_i(x') = \frac{\partial p}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i} \frac{\partial p}{\partial x^j} = \frac{\partial x^j}{\partial x'^i} \, e_j(x). \tag{11.139}$$

11.18 The metric tensor

Points are physical, coordinate systems metaphysical. So p, q, $p-q$, and $(p-q)\cdot(p-q)$ are all invariant quantities. When p and $q = p + dp$ lie infinitesimally close to each other on the spacetime manifold, the vector $dp = e_i\, dx^i$ is the sum of the basis vectors multiplied by the changes in the coordinates x^i. Both dp and the inner product $dp\cdot dp$ are physical and so are independent of the coordinates. The inner product $e_i \cdot e_j$ of two coordinate basis vectors is defined in terms of the metric the flat embedding space $\mathbb{R}^n$, which is a diagonal matrix I with eigenvalues ± 1; it is a sum over the n values of α and β that label the coordinates of $\mathbb{R}^n$

$$e_i \cdot e_j = \sum_{\alpha,\beta=0}^{n-1} e_i{}^{\alpha}\, I_{\alpha\beta}\, e^{\beta}{}_{j}. \tag{11.140}$$

The squared separation dp^2 is

$$dp^2 \equiv dp\cdot dp = (e_i\, dx^i)\cdot(e_j\, dx^j) = \sum_{\alpha,\beta=0}^{n-1} e_i^{\alpha}\, I_{\alpha\beta}\, e_j^{\beta}\; dx^i dx^j. \tag{11.141}$$

It is the same in any other coordinate system

$$dp^2 \equiv dp\cdot dp = (e'_i\, dx'^i)\cdot(e'_j\, dx'^j) = \sum_{\alpha,\beta=0}^{n-1} e'^{\alpha}_i\, I_{\alpha\beta}\, e'^{\beta}_j\; dx'^i dx'^j. \tag{11.142}$$

This invariant squared separation dp^2 defines a 4×4 metric g_{ij} on spacetime

$$dp^2 = e_i\cdot e_j\, dx^i dx^j = g_{ij}\, dx^i dx^j \tag{11.143}$$

as the inner products

$$g_{ij} = e_i\cdot e_j = \sum_{\alpha,\beta=0}^{n-1} e_i{}^{\alpha}\, I_{\alpha\beta}\, e^{\beta}{}_{j}. \tag{11.144}$$

We normally will apply the summation convention (section 11.6) to these inner indices α and β.

Since the basis vectors e_i transform (11.139) as covariant vectors, the metric g_{ij} transforms as a covariant tensor

$$g'_{ij} = e'^{\alpha}_i\, I_{\alpha\beta}\, e'^{\beta}_j = e^{\alpha}_i\, I_{\alpha\beta}\, e^{\beta}_j\, \frac{\partial x^k}{\partial x'^i}\, \frac{\partial x^\ell}{\partial x'^j} = g_{k\ell}\, \frac{\partial x^k}{\partial x'^i}\, \frac{\partial x^\ell}{\partial x'^j}, \tag{11.145}$$

as it must be since the squared separation dp^2 is independent of our coordinates

$$g'_{ij}\, dx'^i dx'^j = g'_{ij}\, \frac{\partial x'^i}{\partial x^k} dx^k\, \frac{\partial x'^j}{\partial x^\ell} dx^\ell = g_{k\ell}\, dx^k dx^\ell. \tag{11.146}$$

Thus the **metric tensor** g_{ij} is a rank-2 covariant tensor. By its construction (11.144), it also is a 4×4 real, symmetric matrix.

Example 11.17 (Graph paper)**.** Imagine a piece of slightly crumpled graph paper with horizontal and vertical lines. The lines give us a two-dimensional coordinate system (x^1, x^2) that labels each point $p(x)$ on the paper. The vectors $e_1(x) = \partial_1 p(x)$ and $e_2(x) = \partial_2 p(x)$ define how a point moves $dp(x) = e_i(x)\, dx^i$ when we change its coordinates by dx^1 and dx^2. The vectors $e_1(x)$ and $e_2(x)$ span a different tangent space at the intersection of every horizontal line with every vertical line. Each tangent space is like the tiny square of the graph paper at that intersection. We can think of the two vectors $e_i(x)$ as three-component vectors in the three-dimensional embedding space we live in. The squared distance between any two nearby points separated by $dp(x)$ is $ds^2 \equiv dp^2(x) = e_1^2(x)(dx^1)^2 + 2e_1(x) \cdot e_2(x)\, dx^1 dx^2 + e_2^2(x)(dx^2)^2$ in which the inner products $g_{ij} = e_i(x) \cdot e_j(x)$ are defined by the euclidian metric of the embedding euclidian space $\mathbb{R}^3$. □

Example 11.18 (The sphere S^2)**.** Let the point $\boldsymbol{p}$ be a euclidean 3-vector representing a point on the two-dimensional surface of a sphere of radius R. Spherical coordinates label the point $\boldsymbol{p} = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, the coordinate basis vectors are

$$\boldsymbol{e}_\theta = \frac{\partial \boldsymbol{p}}{\partial \theta} = R\,\hat{\theta} \quad \text{and} \quad \boldsymbol{e}_\phi = \frac{\partial \boldsymbol{p}}{\partial \phi} = R\,\sin\theta\,\hat{\phi}, \tag{11.147}$$

and the embedding metric is the 3×3 identity matrix $I_{\alpha\beta} = \delta_{\alpha\beta}$. The inner products of the basis vectors are the components (11.144) of the sphere's metric tensor

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta & \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\phi \\ \boldsymbol{e}_\phi \cdot \boldsymbol{e}_\theta & \boldsymbol{e}_\phi \cdot \boldsymbol{e}_\phi \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\,\sin^2\theta \end{pmatrix} \tag{11.148}$$

which has determinant $R^4 \sin^2\theta$. Since $\boldsymbol{e}_\theta \cdot \boldsymbol{e}_\phi = 0$, the squared distance is

$$ds^2 = \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta\, d\theta^2 + \boldsymbol{e}_\phi \cdot \boldsymbol{e}_\phi\, d\phi^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2. \tag{11.149}$$

If we adopt a dimensionless scale factor a and set $ar = R\sin\theta$, so that $adr = R\cos\theta d\theta$, then

$$ds^2 = \frac{a^2 dr^2}{\cos^2\theta} + a^2 r^2 d\phi^2 = \frac{a^2 dr^2}{1 - (a/R)^2 r^2} + a^2 r^2 d\phi^2, \tag{11.150}$$

and in terms of $k = a^2/R^2$ the r, ϕ metric is

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{\phi r} & g_{\phi\phi} \end{pmatrix} = a^2 \begin{pmatrix} 1/(1-kr^2) & 0 \\ 0 & r^2 \end{pmatrix}. \tag{11.151}$$

□

Example 11.19 (The hyperboloid H^2)**.** The hyperboloid H^2 in $\mathbb{R}^3$ is the surface defined by $R^2 = x^2 + y^2 - z^2$. If we label its points as $\boldsymbol{p} = R(\sinh\theta\cos\phi, \sinh\theta\sin\phi, \cosh\theta)$, then its coordinate basis vectors are

$$\begin{aligned} \boldsymbol{e}_\theta &= \frac{\partial \boldsymbol{p}}{\partial\theta} = R\,(\cosh\theta\cos\phi, \cosh\theta\sin\phi, \sinh\theta) \\ \boldsymbol{e}_\phi &= \frac{\partial \boldsymbol{p}}{\partial\phi} = R\,(-\sinh\theta\sin\phi, \sinh\theta\cos\phi, 0). \end{aligned} \tag{11.152}$$

The embedding metric is $I = \mathrm{diag}(1, 1, -1)$, so z is a time coordinate. Since $\boldsymbol{e}_\theta \cdot \boldsymbol{e}_\phi = 0$, the squared distance between nearby points is

$$ds^2 = \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta\, d\theta^2 + \boldsymbol{e}_\phi \cdot \boldsymbol{e}_\phi\, d\phi^2 = R^2\, d\theta^2 + R^2\, \sinh^2\theta\, d\phi^2. \tag{11.153}$$

In terms of the dimensionless scale factor a, the parameter $k = (a/R)^2$, and the radial variable $r = R\sinh\theta/a$, the squared distance ds^2 is

$$ds^2 = a^2 \frac{dr^2}{\cosh^2\theta} + a^2 r^2 d\phi^2 = a^2 \left(\frac{dr^2}{1+kr^2} + r^2\, d\phi^2 \right). \tag{11.154}$$

□

Example 11.20 (The sphere S^3)**.** The sphere S^3 is a 3-dimensional space in $\mathbb{R}^4$ defined by $R^2 = x^2 + y^2 + z^2 + w^2$. If we label its points as

$$p(\chi, \theta, \phi) = R(\sin\chi\sin\theta\cos\phi, \sin\chi\sin\theta\sin\phi, \sin\chi\cos\theta, \cos\chi), \tag{11.155}$$

then its coordinate basis vectors are

$$\begin{aligned} e_\chi &= \frac{\partial p}{\partial\chi} = R(\cos\chi\sin\theta\cos\phi, \cos\chi\sin\theta\sin\phi, \cos\chi\cos\theta, -\sin\chi) \\ e_\theta &= \frac{\partial p}{\partial\theta} = R(\sin\chi\cos\theta\cos\phi, \sin\chi\cos\theta\sin\phi, -\sin\chi\sin\theta, 0) \\ e_\phi &= \frac{\partial p}{\partial\phi} = R(-\sin\chi\sin\theta\sin\phi, \sin\chi\sin\theta\cos\phi, 0, 0). \end{aligned} \tag{11.156}$$

The inner metric of the embedding space is $I = (1, 1, 1, 1)$, and in this metric the basis vectors are orthogonal. In terms of the dimensionless scale factor

a, the parameter $k = (a/R)^2$, and the radial variable $r = R\sin\chi/a$, the squared distance ds^2 between two nearby points is

$$\begin{aligned} ds^2 &= e_\chi \cdot e_\chi d\chi^2 + e_\theta \cdot e_\theta d\theta^2 + e_\phi \cdot e_\phi d\phi^2 \\ &= R^2\left(d\chi^2 + \sin^2\chi\, d\theta^2 + \sin^2\chi \sin^2\theta\, d\phi^2\right) \\ &= a^2\left(\frac{dr^2}{1-\sin^2\chi} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) \\ &= a^2\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right). \end{aligned} \tag{11.157}$$

□

Example 11.21 (The hyperboloid H^3)**.** The hyperboloid H^3 is a 3-space in $\mathbb{R}^4$ defined by $R^2 = x^2 + y^2 + z^2 - w^2$. If we label its points as

$$p(\chi,\theta,\phi) = R\,(\sinh\chi\sin\theta\cos\phi, \sinh\chi\sin\theta\sin\phi, \sinh\chi\cos\theta, \cosh\chi), \tag{11.158}$$

then its coordinate basis vectors are

$$\begin{aligned} e_\chi &= \frac{\partial p}{\partial\chi} = R(\cosh\chi\sin\theta\cos\phi, \cosh\chi\sin\theta\sin\phi, \cosh\chi\cos\theta, \sinh\chi) \\ e_\theta &= \frac{\partial p}{\partial\theta} = R(\sinh\chi\cos\theta\cos\phi, \sinh\chi\cos\theta\sin\phi, -\sinh\chi\sin\theta, 0) \\ e_\phi &= \frac{\partial p}{\partial\phi} = R(-\sinh\chi\sin\theta\sin\phi, \sinh\chi\sin\theta\cos\phi, 0, 0). \end{aligned} \tag{11.159}$$

The inner metric is $I = (1,1,1,-1)$, and w is a time coordinate. The basis vectors are orthogonal. In terms of the dimensionless scale factor a, the parameter $k = (a/R)^2$, and the radial variable $r = R\sinh\chi/a$, the squared distance ds^2 between two nearby points is

$$\begin{aligned} ds^2 &= e_\chi \cdot e_\chi d\chi^2 + e_\theta \cdot e_\theta d\theta^2 + e_\phi \cdot e_\phi d\phi^2 \\ &= R^2\left(d\chi^2 + \sinh^2\chi\, d\theta^2 + \sinh^2\chi \sin^2\theta\, d\phi^2\right) \\ &= a^2\left(\frac{dr^2}{1+\sinh^2\chi} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) \\ &= a^2\left(\frac{dr^2}{1+kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right). \end{aligned} \tag{11.160}$$

□

The squared distances (11.157 & 11.160) respectively are those of the closed and open Robinson-Walker metrics (11.454) for $dx^0 = 0$.

11.19 The principle of equivalence

Since the metric tensor $g_{ij}(x)$ is real and symmetric, it can be diagonalized at any point $p(x)$ by a 4×4 orthogonal matrix $O(x)$

$$O^{\mathsf{T}}{}_i{}^k \, g_{k\ell} \, O^\ell{}_j = \begin{pmatrix} e_0 & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 \\ 0 & 0 & e_2 & 0 \\ 0 & 0 & 0 & e_3 \end{pmatrix} \tag{11.161}$$

which arranges the four real eigenvalues e_i of the matrix $g_{ij}(x)$ in the order $e_0 \le e_1 \le e_2 \le e_3$. Thus the coordinate transformation

$$\frac{\partial x^k}{\partial x'^i} = \frac{O^{\mathsf{T}}{}_i{}^k}{\sqrt{|e_i|}} \tag{11.162}$$

takes any spacetime metric $g_{k\ell}(x)$ with one negative and three positive eigenvalues into the Minkowski metric η_{ij} of flat spacetime

$$g_{k\ell}(x) \, \frac{\partial x^k}{\partial x'^i} \, \frac{\partial x^\ell}{\partial x'^j} = g'_{ij}(x') = \eta_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11.163}$$

at the point $p(x) = p(x')$. The **principle of equivalence** says that in these free-fall coordinates x', the physical laws of gravity-free special relativity apply in a suitably small region about the point $p(x) = p(x')$. It follows from this principle that the metric g_{ij} of spacetime accounts for all the effects of gravity.

In the x' coordinates, the invariant squared separation dp^2 is that of special relativity

$$\begin{aligned} dp^2 = g'_{ij} \, dx'^i dx'^j &= e'_i(x') \cdot e'_j(x') \, dx'^i dx'^j \\ &= e'^a_i(x') \eta_{ab} e'^b_j(x') \, dx'^i dx'^j = \delta^a_i \eta_{ab} \delta^b_j \, dx'^i dx'^j \\ &= \eta_{ij} \, dx'^i dx'^j = (\boldsymbol{dx'})^2 - (dx'^0)^2 = ds^2. \end{aligned} \tag{11.164}$$

The x' coordinates are not unique because every Lorentz transformation leaves the metric η invariant. Coordinate systems in which $g_{ij}(x') = \eta_{ij}$ are called **inertial coordinate systems**.

The congruency transformation (1.312 & 11.161–11.163) preserves the signs of the eigenvalues e_i which are the **signature** $(-1, 1, 1, 1)$ of the metric tensor.

11.20 Tetrads

We defined metric tensor (11.144) in terms of four n-vectors e_i in the embedding space $\mathbb{R}^n$ as $g_{ij} = e_i \cdot e_j = e_i{}^\mathsf{T} I\, e_j$ in which I is the metric of the embedding space which is a diagonal matrix (11.136) with n eigenvalues ± 1. If we instead invert the equation (11.163) that relates the metric tensor to the flat metric and write

$$g_{ij} = \frac{\partial x'^a}{\partial x^i}\, \eta_{ab}\, \frac{\partial x'^b}{\partial x^j}, \tag{11.165}$$

then we will have expressed the metric in terms of four 4-vectors

$$c_i{}^a(x) = \frac{\partial x'^a}{\partial x^i} = c^a{}_i(x) = c_i^a(x) \tag{11.166}$$

as

$$g_{ij}(x) = c_i{}^a(x)\, \eta_{ab}\, c^b{}_j(x) \tag{11.167}$$

in which η_{ab} is the 4×4 metric (11.163) of flat Minkowski space. Whether the fundamental variables are the four 4-vectors $c_i{}^a(x)$ introduced by Élie Cartan (1869–1951) or the metric tensor g_{ij} is an open question. Cartan's four 4-vectors $c_i^a(x)$ are called a **moving frame**, a **tetrad**, or a **vierbein**.

11.21 The Contravariant Metric Tensor

The **inverse g^{ik} of the covariant metric tensor** g_{kj} satisfies

$$g'^{ik} g'_{kj} = \delta^i{}_j = g^{ik} g_{kj} \tag{11.168}$$

in all coordinate systems. To see how it transforms, we use the transformation law (11.145) of g_{kj}

$$\delta^i{}_j = g'^{ik} g'_{kj} = g'^{ik} \frac{\partial x^t}{\partial x'^k}\, g_{tu}\, \frac{\partial x^u}{\partial x'^j}. \tag{11.169}$$

In matrix notation, this is $I = g'^{-1}\, H\, g\, H^\mathsf{T}$ which implies $g'^{-1} = H^{-1\mathsf{T}}\, g^{-1}\, H^{-1}$ or in tensor notation

$$g'^{i\ell} = \frac{\partial x'^i}{\partial x^v}\, \frac{\partial x'^\ell}{\partial x^w}\, g^{vw}. \tag{11.170}$$

Thus the inverse g^{ik} of the covariant metric tensor is a second-rank contravariant tensor called the **contravariant metric tensor**.

11.22 Dual vectors and the raising and lowering of indices

The contraction of a contravariant vector A^i with any rank-2 covariant tensor gives a covariant vector, but we reserve the symbol A_i for the covariant vector that is the contraction of A^j with the metric tensor

$$A_i = g_{ij} A^j. \tag{11.171}$$

This operation is called **lowering the index** on A^j.

Similarly, the contraction of a covariant vector B_j with any rank-2 contravariant tensor is a contravariant vector, and we reserve the symbol B^i for contravariant vector that is the contraction

$$B^i = g^{ij} B_j \tag{11.172}$$

of B_j with the inverse of the metric tensor. This is called **raising the index** on B_j.

The vectors e^i, for instance, are given by

$$e^i = g^{ij} e_j. \tag{11.173}$$

They are therefore orthonormal or **dual** to the basis vectors e_i

$$e_i \cdot e^j = e_i \cdot g^{jk} e_k = g^{jk} e_i \cdot e_k = g^{jk} g_{ik} = g^{jk} g_{ki} = \delta_i^j \tag{11.174}$$

with respect to the metric η_{ab} of the flat space tangent to the manifold.

11.23 Orthogonal coordinates in $\mathbb{R}^n$

In flat n-dimensional euclidian space, it is convenient to use **orthogonal basis vectors** and **orthogonal coordinates**. A change dx^i in the coordinates moves the point p by (11.133)

$$dp = e_i \, dx^i. \tag{11.175}$$

The metric g_{ij} is the inner product (11.144)

$$g_{ij} = e_i \cdot e_j. \tag{11.176}$$

Since the vectors e_i are orthogonal, the metric is diagonal

$$g_{ij} = e_i \cdot e_j = h_i^2 \delta_{ij}. \tag{11.177}$$

The inverse metric

$$g^{ij} = h_i^{-2} \delta_{ij} \tag{11.178}$$

raises indices. For instance, the dual vectors

$$e^{\,i} = g^{ij}\, e_j = h_i^{-2}\, e_i \quad \text{satisfy} \quad e^{\,i} \cdot e_k = \delta^i_k. \tag{11.179}$$

The invariant squared distance dp^2 between nearby points (11.141) is

$$dp^2 = dp \cdot dp = g_{ij}\, dx^i\, dx^j = h_i^2\, (dx^i)^2 \tag{11.180}$$

and the invariant volume element is

$$dV = d^n p = h_1 \ldots h_n\, dx^1 \wedge \ldots \wedge dx^n = g\, dx^1 \wedge \ldots \wedge dx^n = g\, d^n x \tag{11.181}$$

in which $g = \sqrt{\det g_{ij}}$ is the square root of the positive determinant of g_{ij}.

The important special case in which all the scale factors h_i are unity is cartesian coordinates in euclidean space (section 11.5).

We also can use basis vectors $\hat{e}_i$ that are **orthonormal**. By (11.177 & 11.179), these vectors

$$\hat{e}_i = e_i / h_i = h_i\, e^{\,i} \quad \text{satisfy} \quad \hat{e}_i \cdot \hat{e}_j = \delta_{ij}. \tag{11.182}$$

In terms of them, a physical and invariant vector V takes the form

$$V = e_i\, V^i = h_i\, \hat{e}_i\, V^i = e^{\,i}\, V_i = h_i^{-1}\, \hat{e}_i\, V_i = \hat{e}_i\, \overline{V}_i \tag{11.183}$$

where

$$\overline{V}_i \equiv h_i\, V^i = h_i^{-1}\, V_i \quad \text{(no sum)}. \tag{11.184}$$

The dot-product is then

$$V \cdot U = g_{ij}\, V^i\, U^j = \overline{V}_i\, \overline{U}_i. \tag{11.185}$$

In euclidian n-space, we even can choose coordinates x^i so that the vectors e_i defined by $dp = e_i\, dx^i$ are orthonormal. The metric tensor is then the $n \times n$ identity matrix $g_{ik} = e_i \cdot e_k = I_{ik} = \delta_{ik}$. But since this is euclidian n-space, we also can expand the n fixed orthonormal cartesian unit vectors $\hat{\ell}$ in terms of the $e_i(x)$ which vary with the coordinates as $\hat{\ell} = e_i(x)(e_i(x) \cdot \hat{\ell})$.

11.24 Polar Coordinates

In polar coordinates in flat 2-space, the change dp in a point p due to a change in its coordinates is $dp = \hat{r}\, dr + \hat{\theta}\, r\, d\theta$ so $dp = e_r\, dr + e_\theta\, d\theta$ with $e_r = \hat{e}_r = \hat{r}$ and $e_\theta = r\, \hat{e}_\theta = r\, \hat{\theta}$. The metric tensor for polar coordinates is

$$(g_{ij}) = (e_i \cdot e_j) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{11.186}$$

The contravariant basis vectors are $e^r = \hat{r}$ and $e^\theta = \hat{e}_\theta / r$. A physical vector V is $V = V^i e_i = V_i e^i = \overline{V}_r \hat{r} + \overline{V}_\theta \hat{\theta}$.

11.25 Cylindrical Coordinates

For cylindrical coordinates in flat 3-space, the change $\boldsymbol{dp}$ in a point $\boldsymbol{p}$ due to a change in its coordinates is

$$\boldsymbol{dp} = \hat{\boldsymbol{\rho}}\, d\rho + \hat{\boldsymbol{\phi}}\, \rho\, d\phi + \hat{\boldsymbol{z}}\, dz = \boldsymbol{e}_\rho\, d\rho + \boldsymbol{e}_\phi\, d\phi + \boldsymbol{e}_z\, dz \tag{11.187}$$

with $\boldsymbol{e}_\rho = \hat{\boldsymbol{e}}_\rho = \hat{\boldsymbol{\rho}}$, $\boldsymbol{e}_\phi = \rho\, \hat{\boldsymbol{e}}_\phi = \rho\, \hat{\boldsymbol{\phi}}$, and $\boldsymbol{e}_z = \hat{\boldsymbol{e}}_z = \hat{\boldsymbol{z}}$. The metric tensor for cylindrical coordinates is

$$(g_{ij}) = (\boldsymbol{e_i} \cdot \boldsymbol{e_j}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11.188}$$

with determinant $\det g_{ij} \equiv g = \rho^2$. The invariant volume element is

$$dV = \rho\, dx^1 \wedge dx^2 \wedge dx^3 = \sqrt{g}\, d\rho\, d\phi\, dz = \rho\, d\rho\, d\phi\, dz. \tag{11.189}$$

The contravariant basis vectors are $\boldsymbol{e}^\rho = \hat{\boldsymbol{\rho}}$, $\boldsymbol{e}^\phi = \hat{\boldsymbol{e}}_\phi / \rho$, and $\boldsymbol{e}^z = \hat{\boldsymbol{z}}$. A physical vector V is

$$V = V^i\, \boldsymbol{e}_i = V_i\, \boldsymbol{e}^i = \overline{V}_\rho\, \hat{\boldsymbol{\rho}} + \overline{V}_\phi\, \hat{\boldsymbol{\phi}} + \overline{V}_z\, \hat{\boldsymbol{z}}. \tag{11.190}$$

Incidentally, since

$$\boldsymbol{p} = (\rho \cos\phi, \rho \sin\phi, z)\,, \tag{11.191}$$

the formulas for the basis vectors of cylindrical coordinates in terms of those of rectangular coordinates are (exercise 11.14)

$$\begin{aligned} \hat{\boldsymbol{\rho}} &= \cos\phi\, \hat{\boldsymbol{x}} + \sin\phi\, \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\, \hat{\boldsymbol{x}} + \cos\phi\, \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} &= \hat{\boldsymbol{z}}. \end{aligned} \tag{11.192}$$

11.26 Spherical Coordinates

For spherical coordinates in flat 3-space, the change $\boldsymbol{dp}$ in a point $\boldsymbol{p}$ due to a change in its coordinates is

$$\boldsymbol{dp} = \hat{\boldsymbol{r}}\, dr + \hat{\boldsymbol{\theta}}\, r\, d\theta + \hat{\boldsymbol{\phi}}\, r\, \sin\theta\, d\phi = \boldsymbol{e}_r\, dr + \boldsymbol{e}_\theta\, d\theta + \boldsymbol{e}_\phi\, d\phi \tag{11.193}$$

so $\boldsymbol{e}_r = \hat{\boldsymbol{r}}$, $\boldsymbol{e}_\theta = r\,\hat{\boldsymbol{\theta}}$, and $\boldsymbol{e}_\phi = r\,\sin\theta\,\hat{\boldsymbol{\phi}}$ or

$$\begin{aligned} e_r &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ e_\theta &= r\,(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \\ e_\phi &= r\sin\theta\,(-\sin\phi, \cos\phi, 0)\,. \end{aligned} \tag{11.194}$$

The metric tensor for spherical coordinates is

$$(g_{ij}) = (e_i \cdot e_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\,\sin^2\theta \end{pmatrix} \tag{11.195}$$

with determinant $\det g_{ij} \equiv g = r^4 \sin^2\theta$. The invariant volume element is

$$dV = r^2\,\sin^2\theta\,dx^1 \wedge dx^2 \wedge dx^3 = \sqrt{g}\,dr\,d\theta\,d\phi = r^2 \sin\theta\,dr\,d\theta\,d\phi. \tag{11.196}$$

The orthonormal basis vectors are $\hat{\boldsymbol{e}}_r = \hat{\boldsymbol{r}}$, $\hat{\boldsymbol{e}}_\theta = \hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{e}}_\phi = \hat{\boldsymbol{\phi}}$. The contravariant basis vectors are $\boldsymbol{e}^{\,r} = \hat{\boldsymbol{r}}$, $\boldsymbol{e}^{\,\theta} = \hat{\boldsymbol{\theta}}/r$, $\boldsymbol{e}^{\,\phi} = \hat{\boldsymbol{\phi}}/r\sin\theta$. A physical vector V is

$$V = V^i\,\boldsymbol{e}_i = V_i\,\boldsymbol{e}^{\,i} = \overline{V}_r\,\hat{\boldsymbol{r}} + \overline{V}_\theta\,\hat{\boldsymbol{\theta}} + \overline{V}_\phi\,\hat{\boldsymbol{\phi}}. \tag{11.197}$$

Incidentally, since

$$\boldsymbol{p} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\,, \tag{11.198}$$

the formulas for the basis vectors of spherical coordinates in terms of those of rectangular coordinates are (exercise 11.15)

$$\begin{aligned} \boldsymbol{e_r} &= \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}} \\ \boldsymbol{e_\theta} &= r\,(\cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}}) \\ \boldsymbol{e_\phi} &= r\sin\theta\,(-\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}})\,. \end{aligned} \tag{11.199}$$

11.27 The Gradient of a Scalar Field

If $f(x)$ is a scalar field, then the difference between it and $f(x+dx)$ defines the **gradient** ∇f as (6.26)

$$df(x) = f(x+dx) - f(x) = \frac{\partial f(x)}{\partial x^i}\,dx^i = \nabla f(x)\cdot dp. \tag{11.200}$$

Since $dp = e_j\,dx^j$, the **invariant** form

$$\nabla f = e^i\,\frac{\partial f}{\partial x^i} = \frac{\hat{e}_i}{h_i}\,\frac{\partial f}{\partial x^i} \tag{11.201}$$

satisfies this definition (11.200) of the gradient

$$\nabla f \cdot dp = \frac{\partial f}{\partial x^i} e^i \cdot e_j dx^j = \frac{\partial f}{\partial x^i}\, \delta^i_j\, dx^j = \frac{\partial f}{\partial x^i}\, dx^i = df. \tag{11.202}$$

In two polar coordinates, the gradient is

$$\nabla f = e^i\, \frac{\partial f}{\partial x^i} = \frac{\hat{e}_i}{h_i}\, \frac{\partial f}{\partial x^i} = \hat{r}\, \frac{\partial f}{\partial r} + \frac{\hat{\theta}}{r}\, \frac{\partial f}{\partial \theta}. \tag{11.203}$$

In three cylindrical coordinates, it is (6.27)

$$\nabla f = e^i\, \frac{\partial f}{\partial x^i} = \frac{\hat{e}_i}{h_i}\, \frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial \rho}\, \hat{\rho} + \frac{1}{\rho} \frac{\partial f}{\partial \phi}\, \hat{\phi} + \frac{\partial f}{\partial z}\, \hat{z} \tag{11.204}$$

and in three spherical coordinates it is (6.28)

$$\nabla f = \frac{\partial f}{\partial x^i}\, e^i = \frac{\hat{e}_i}{h_i}\, \frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial r}\, \hat{r} + \frac{1}{r}\, \frac{\partial f}{\partial \theta}\, \hat{\boldsymbol{\theta}} + \frac{1}{r \sin\theta}\, \frac{\partial f}{\partial \phi}\, \hat{\phi}. \tag{11.205}$$

11.28 Levi-Civita's Tensor

In 3 dimensions, Levi-Civita's **symbol** $\epsilon_{ijk} \equiv \epsilon^{ijk}$ is totally antisymmetric with $\epsilon_{123} = 1$ in all coordinate systems.

We can turn his symbol into something that transforms as a tensor by multiplying it by the square root of the determinant of a rank-2 covariant tensor. A natural choice is the metric tensor. Thus the Levi-Civita **tensor** η_{ijk} is the totally antisymmetric rank-3 covariant (pseudo)tensor

$$\eta_{ijk} = \sqrt{g}\, \epsilon_{ijk} \tag{11.206}$$

in which $g = |\det g_{mn}|$ is the absolute value of the determinant of the metric tensor g_{mn}. The determinant's definition (1.184) and product rule (1.207) imply that Levi-Civita's tensor η_{ijk} transforms as

$$\begin{aligned}
\eta'_{ijk} = \sqrt{g'}\, \epsilon'_{ijk} = \sqrt{g'}\, \epsilon_{ijk} &= \sqrt{\left|\det\left(\frac{\partial x^t}{\partial x'^m}\, \frac{\partial x^u}{\partial x'^n}\, g_{tu}\right)\right|}\, \epsilon_{ijk} \\
&= \sqrt{\left|\det\left(\frac{\partial x^t}{\partial x'^m}\right) \det\left(\frac{\partial x^u}{\partial x'^n}\right) \det\left(g_{tu}\right)\right|}\, \epsilon_{ijk} \\
&= \left|\det\left(\frac{\partial x}{\partial x'}\right)\right| \sqrt{g}\, \epsilon_{ijk} = \sigma\, \det\left(\frac{\partial x}{\partial x'}\right) \sqrt{g}\, \epsilon_{ijk} \\
&= \sigma\, \frac{\partial x^t}{\partial x'^i}\, \frac{\partial x^u}{\partial x'^j}\, \frac{\partial x^v}{\partial x'^k}\, \sqrt{g}\, \epsilon_{tuv} = \sigma\, \frac{\partial x^t}{\partial x'^i}\, \frac{\partial x^u}{\partial x'^j}\, \frac{\partial x^v}{\partial x'^k}\, \eta_{tuv}
\end{aligned} \tag{11.207}$$

in which σ is the sign of the Jacobian $\det(\partial x/\partial x')$. Levi-Civita's tensor is a **pseudotensor** because it doesn't change sign under the parity transformation $x'^i = -x^i$.

We get η with upper indices by using the inverse g^{nm} of the metric tensor

$$\begin{aligned}\eta^{ijk} &= g^{it}\, g^{ju}\, g^{kv}\, \eta_{tuv} = g^{it}\, g^{ju}\, g^{kv}\, \sqrt{g}\, \epsilon_{tuv} = \sqrt{g}\, \epsilon_{ijk}\, \det(g^{mn}) \\ &= \sqrt{g}\, \epsilon_{ijk}/\det(g_{mn}) = s\epsilon_{ijk}/\sqrt{g} = s\epsilon^{ijk}/\sqrt{g}\end{aligned} \tag{11.208}$$

in which s is the sign of the determinant $\det g_{ij} = sg$.

Similarly in 4 dimensions, Levi-Civita's **symbol** $\epsilon_{ijk\ell} \equiv \epsilon^{ijk\ell}$ is totally antisymmetric with $\epsilon_{0123} = 1$ in all coordinate systems. No meaning attaches to whether the indices of the Levi-Civita symbol are up or down; some authors even use the notation $\epsilon(ijk\ell)$ or $\epsilon[ijk\ell]$ to emphasize this fact.

In 4 dimensions, the Levi-Civita **pseudotensor** is

$$\eta_{ijk\ell} = \sqrt{g}\, \epsilon_{ijk\ell} \tag{11.209}$$

where again $g = |\det g_{ij}|$. The determinant's definition (1.184) and product rule (1.207) imply that it transforms as

$$\begin{aligned}\eta'_{ijk\ell} &= \sqrt{g'}\, \epsilon_{ijk\ell} = \left|\det\left(\frac{\partial x}{\partial x'}\right)\right| \sqrt{g}\, \epsilon_{ijk\ell} = \sigma \det\left(\frac{\partial x}{\partial x'}\right) \sqrt{g}\, \epsilon_{ijk\ell} \\ &= \sigma\, \frac{\partial x^t}{\partial x'^i}\, \frac{\partial x^u}{\partial x'^j}\, \frac{\partial x^v}{\partial x'^k}\, \frac{\partial x^w}{\partial x'^\ell}\, \sqrt{g}\, \epsilon_{tuvw} = \sigma\, \frac{\partial x^t}{\partial x'^i}\, \frac{\partial x^u}{\partial x'^j}\, \frac{\partial x^v}{\partial x'^k}\, \frac{\partial x^w}{\partial x'^\ell}\, \eta_{tuvw}\end{aligned} \tag{11.210}$$

where σ is the sign of the Jacobian $\det(\partial x/\partial x')$.

Raising the indices on η with $\det g_{ij} = sg$ we have

$$\begin{aligned}\eta^{ijk\ell} &= g^{it}\, g^{ju}\, g^{kv}\, g^{\ell w}\, \eta_{tuvw} = g^{it}\, g^{ju}\, g^{kv}\, g^{\ell w}\, \sqrt{g}\, \epsilon_{tuvw} = \sqrt{g}\, \epsilon_{ijk\ell} \det(g^{mn}) \\ &= \sqrt{g}\, \epsilon_{ijk\ell}/\det(g_{mn}) = s\, \epsilon_{ijk\ell}/\sqrt{g} \equiv s\, \epsilon^{ijk\ell}/\sqrt{g}.\end{aligned} \tag{11.211}$$

In n dimensions, one may define Levi-Civita's symbol $\epsilon(i_1 \ldots i_n)$ as totally antisymmetric with $\epsilon(1 \ldots n) = 1$ and his pseudotensor as $\eta_{i_1 \ldots i_n} = \sqrt{g}\, \epsilon(i_1 \ldots i_n)$.

11.29 The Hodge Star

This section is optional on a first reading.

In 3 cartesian coordinates, the Hodge dual turns 1-forms into 2-forms

$$*\, dx = dy \wedge dz \qquad *\, dy = dz \wedge dx \qquad *\, dz = dx \wedge dy \tag{11.212}$$

and 2-forms into 1-forms

$$*(dx \wedge dy) = dz \qquad *(dy \wedge dz) = dx \qquad *(dz \wedge dx) = dy. \tag{11.213}$$

It also maps the 0-form 1 and the volume 3-form into each other

$$*1 = dx \wedge dy \wedge dz \qquad *(dx \wedge dy \wedge dz) = 1 \tag{11.214}$$

(William Vallance Douglas Hodge, 1903–1975). More generally in 3-space, we define the Hodge dual, also called the Hodge star, as

$$\begin{aligned} *1 &= \frac{1}{3!}\eta_{\ell jk} dx^\ell \wedge dx^j \wedge dx^k \quad *(dx^\ell \wedge dx^j \wedge dx^k) = g^{\ell t} g^{ju} g^{kv} \eta_{tuv} \\ *dx^i &= \frac{1}{2}\, g^{i\ell}\, \eta_{\ell jk}\, dx^j \wedge dx^k \quad *(dx^i \wedge dx^j) = g^{ik}\, g^{j\ell}\, \eta_{k\ell m}\, dx^m \end{aligned} \tag{11.215}$$

and so if the sign of $\det g_{ij}$ is $s = +1$, then $**1 = 1$, $**dx^i = dx^i$, $**(dx^i \wedge dx^k) = dx^i \wedge dx^k$, and $**(dx^i \wedge dx^j \wedge dx^k) = dx^i \wedge dx^j \wedge dx^k$.

Example 11.22 (Divergence and Laplacian)**.** The dual of the 1-form

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz \tag{11.216}$$

is the 2-form

$$*df = \frac{\partial f}{\partial x}\, dy \wedge dz + \frac{\partial f}{\partial y}\, dz \wedge dx + \frac{\partial f}{\partial z}\, dx \wedge dy \tag{11.217}$$

and its exterior derivative is the laplacian

$$d*df = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) dx \wedge dy \wedge dz \tag{11.218}$$

multiplied by the volume 3-form.

Similarly, the dual of the one form

$$A = A_x\, dx + A_y\, dy + A_z\, dz \tag{11.219}$$

is the 2-form

$$*A = A_x\, dy \wedge dz + A_y\, dz \wedge dx + A_z\, dx \wedge dy \tag{11.220}$$

and its exterior derivative is the divergence

$$d*A = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dx \wedge dy \wedge dz \tag{11.221}$$

times $dx \wedge dy \wedge dz$. □

In flat Minkowski 4-space with $c = 1$, the Hodge dual turns 1-forms into 3-forms

$$\begin{aligned} *\,dt &= -\,dx \wedge dy \wedge dz \qquad & *\,dx &= -\,dy \wedge dz \wedge dt \\ *\,dy &= -\,dz \wedge dx \wedge dt \qquad & *\,dz &= -\,dx \wedge dy \wedge dt \end{aligned} \tag{11.222}$$

2-forms into 2-forms

$$\begin{aligned} *(dx \wedge dt) &= dy \wedge dz \qquad & *\,(dx \wedge dy) &= -\,dz \wedge dt \\ *(dy \wedge dt) &= dz \wedge dx \qquad & *\,(dy \wedge dz) &= -\,dx \wedge dt \\ *(dz \wedge dt) &= dx \wedge dy \qquad & *\,(dz \wedge dx) &= -\,dy \wedge dt \end{aligned} \tag{11.223}$$

3-forms into 1-forms

$$\begin{aligned} *\,(dx \wedge dy \wedge dz) &= -\,dt \qquad & *\,(dy \wedge dz \wedge dt) &= -\,dx \\ *\,(dz \wedge dx \wedge dt) &= -\,dy \qquad & *\,(dx \wedge dy \wedge dt) &= -\,dz \end{aligned} \tag{11.224}$$

and interchanges 0-forms and 4-forms

$$*1 = dt \wedge dx \wedge dy \wedge dz \qquad *\,(dt \wedge dx \wedge dy \wedge dz) = -\,1. \tag{11.225}$$

More generally in 4 dimensions, we define the Hodge star as

$$\begin{aligned} *1 &= \frac{1}{4!}\,\eta_{k\ell mn}\,dx^k \wedge dx^\ell \wedge dx^m \wedge dx^n \\ *\,dx^i &= \frac{1}{3!}\,g^{ik}\,\eta_{k\ell mn}\,dx^\ell \wedge dx^m \wedge dx^n \\ *\,(dx^i \wedge dx^j) &= \frac{1}{2}\,g^{ik}\,g^{j\ell}\,\eta_{k\ell mn}\,dx^m \wedge dx^n \\ *\,(dx^i \wedge dx^j \wedge dx^k) &= g^{it}\,g^{ju}\,g^{kv}\,\eta_{tuvw}\,dx^w \\ *\left(dx^i \wedge dx^j \wedge dx^k \wedge dx^\ell\right) &= g^{it}\,g^{ju}\,g^{kv}\,g^{\ell w}\eta_{tuvw} = \eta^{ijk\ell}. \end{aligned} \tag{11.226}$$

Thus (exercise 11.17) if the determinant $\det g_{ij}$ of the metric is negative, then

$$\begin{aligned} &*\,*\,dx^i = dx^i \qquad *\,*\,(dx^i \wedge dx^j) = -\,dx^i \wedge dx^j \\ &*\,*\,(dx^i \wedge dx^j \wedge dx^k) = dx^i \wedge dx^j \wedge dx^k \qquad *\,*1 = -\,1. \end{aligned} \tag{11.227}$$

In n dimensions, the Hodge star turns p-forms into $n-p$-forms

$$*\left(dx^{i_1} \wedge \ldots \wedge dx^{i_p}\right) = g^{i_1 k_1} \ldots g^{i_p k_p}\,\frac{\eta_{k_1 \ldots k_p \ell_1 \ldots \ell_{n-p}}}{(n-p)!}\,dx^{\ell_1} \wedge \ldots \wedge dx^{\ell_{n-p}}. \tag{11.228}$$

Example 11.23 (The Inhomogeneous Maxwell Equations)**.** The homogeneous Maxwell equations are

$$dF = ddA = 0. \tag{11.229}$$

To get the inhomogeneous Maxwell equations, we first form the dual $*F = *dA$

$$*F = \tfrac{1}{2} F_{ij} * \big(dx^i \wedge dx^j\big) = \tfrac{1}{4} F_{ij} g^{ik} g^{j\ell} \eta_{k\ell mn} dx^m \wedge dx^n = \tfrac{1}{4} F^{k\ell} \eta_{k\ell mn} dx^m \wedge dx^n,$$

and then apply the exterior derivative

$$d * F = \tfrac{1}{4} d \left(F^{k\ell} \eta_{k\ell mn} dx^m \wedge dx^n \right) = \tfrac{1}{4} \partial_p \left(F^{k\ell} \eta_{k\ell mn} \right) dx^p \wedge dx^m \wedge dx^n.$$

To get back to a 1-form like $j = j_k \, dx^k$, we apply a second Hodge star

$$\begin{aligned}
*d * F &= \tfrac{1}{4} \, \partial_p \left(F^{k\ell} \, \eta_{k\ell mn} \right) * \left(dx^p \wedge dx^m \wedge dx^n \right) \\
&= \tfrac{1}{4} \, \partial_p \left(F^{k\ell} \, \eta_{k\ell mn} \right) g^{ps} g^{mt} g^{nu} \eta_{stuv} \, dx^v \\
&= \tfrac{1}{4} \, \partial_p \left(\sqrt{g} \, F^{k\ell} \right) \epsilon_{k\ell mn} \, g^{ps} g^{mt} g^{nu} \sqrt{g} \, \epsilon_{stuv} \, dx^v \\
&= \tfrac{1}{4} \, \partial_p \left(\sqrt{g} \, F^{k\ell} \right) \epsilon_{k\ell mn} \, g^{ps} g^{mt} g^{nu} g^{wv} \, \epsilon_{stuv} \, \sqrt{g} \, dx_w \\
&= \tfrac{1}{4} \, \partial_p \left(\sqrt{g} \, F^{k\ell} \right) \epsilon_{k\ell mn} \, \epsilon_{pmnw} \, \frac{\sqrt{g}}{\det g_{ij}} \, dx_w \\
&= \frac{s}{4\sqrt{g}} \, \partial_p \left(\sqrt{g} \, F^{k\ell} \right) \epsilon_{k\ell mn} \, \epsilon_{pmnw} \, dx_w
\end{aligned} \tag{11.230}$$

in which we used the definition (1.184) of the determinant. Levi-Civita's 4-symbol obeys the identity (exercise 11.18)

$$\epsilon_{k\ell mn} \, \epsilon^{pwmn} = 2 \left(\delta^p_k \, \delta^w_\ell - \delta^w_k \, \delta^p_\ell \right). \tag{11.231}$$

Applying it to $* \, d * F$, we get

$$* \, d * F = \frac{s}{2\sqrt{g}} \partial_p \left(\sqrt{g} \, F^{k\ell} \right) \left(\delta^p_k \, \delta^w_\ell - \delta^w_k \, \delta^p_\ell \right) dx_w = - \frac{s}{\sqrt{g}} \, \partial_p \left(\sqrt{g} \, F^{kp} \right) \, dx_k.$$

In our spacetime $s = -1$. Setting $* \, d * F$ equal to $j = j_k \, dx^k = j^k \, dx_k$ multiplied by the permeability μ_0 of the vacuum, we arrive at expressions for the microscopic inhomogeneous Maxwell equations in terms of both tensors and forms

$$\frac{1}{\sqrt{g}} \, \partial_p \left(\sqrt{g} \, F^{kp} \right) = \mu_0 \, j^k \quad \text{and} \quad * \, d * F = \mu_0 \, j. \tag{11.232}$$

They and the homogeneous Bianchi identity (11.92, 11.113, & 11.294)

$$\epsilon^{ijk\ell} \, \partial_\ell F_{jk} = dF = d \, dA = 0 \tag{11.233}$$

are invariant under general coordinate transformations. □

11.30 Notations for Derivatives

We have various notations for derivatives. We can use the variables x, y, and so forth as subscripts to label derivatives

$$f_x = \partial_x f = \frac{\partial f}{\partial x} \quad \text{and} \quad f_y = \partial_y f = \frac{\partial f}{\partial y}. \tag{11.234}$$

If we use indices to label variables, then we can use commas

$$f_{,i} = \partial_i f = \frac{\partial f}{\partial x^i} \quad \text{and} \quad f_{,ik} = \partial_k \partial_i f = \frac{\partial^2 f}{\partial x^k \partial x^i} \tag{11.235}$$

and $f_{,k'} = \partial f / \partial x'^k$. In the next section, we will use a semicolon to mean a covariant derivative.

11.31 Covariant derivatives and affine connections

If $F(x)$ is a vector field, then its invariant description in terms of spacetime-dependent basis vectors $e_i(x)$ is

$$F(x) = F^i(x)\, e_i(x). \tag{11.236}$$

Since the basis vectors $e_i(x)$ vary with x, the derivative of $F(x)$ contains two terms

$$\frac{\partial F}{\partial x^\ell} = \frac{\partial F^i}{\partial x^\ell}\, e_i + F^i\, \frac{\partial e_i}{\partial x^\ell}. \tag{11.237}$$

In general, the derivative of a vector e_i is not a linear combination of the basis vectors e_k. For instance, on the 2-dimensional surface of a sphere in 3-dimensions, the derivative

$$\frac{\partial e_\theta}{\partial \theta} = -\hat{r} \tag{11.238}$$

points to the sphere's center and isn't a linear combination of e_θ and e_ϕ.

The inner product of a derivative $\partial e_i / \partial x^\ell$ with a dual basis vector e^k is the **Levi-Civita affine connection**

$$\Gamma^k_{i\ell} = e^k \cdot e_{i,\ell} = e^k \cdot \frac{\partial e_i}{\partial x^\ell} \tag{11.239}$$

which relates spaces that are tangent to the manifold at infinitesimally separated points. It is called an *affine* connection because the different tangent spaces lack a common origin.

In terms of the affine connection (11.239), the inner product of the derivative (11.237) with e^k is

$$e^k \cdot \frac{\partial F}{\partial x^\ell} = e^k \cdot \frac{\partial F^i}{\partial x^\ell}\, e_i + F^i\, e^k \cdot \frac{\partial e_i}{\partial x^\ell} = \frac{\partial F^k}{\partial x^\ell} + F^i\, \Gamma^k_{i\ell} \tag{11.240}$$

a combination that is called a **covariant derivative** (section 11.34)

$$D_\ell F^k \equiv \nabla_\ell F^k \equiv \frac{\partial F^k}{\partial x^\ell} + F^i\, \Gamma^k_{i\ell} \equiv F^k_{;\ell} = e^k \cdot \frac{\partial F}{\partial x^\ell}. \tag{11.241}$$

It is a second-rank mixed tensor. The covariant derivative of the scalar F is

$$D_\ell F = e_k\, e^k \cdot \partial_\ell F = \left(\frac{\partial F^k}{\partial x^\ell} + F^i\, \Gamma^k_{i\ell} \right) e_k = F^k_{;\ell}\; e_k. \tag{11.242}$$

Some physicists write the affine connection $\Gamma^k_{i\ell}$ as

$$\left\{ \begin{matrix} k \\ i\ell \end{matrix} \right\} = \Gamma^k_{i\ell} \tag{11.243}$$

and call it a **Christoffel symbol of the second kind**.

The coordinate basis vectors e_i are the spacetime derivatives (11.138) of the point p, and so the affine connection (11.239) is a double derivative of p

$$\Gamma^k_{i\ell} = e^k \cdot \frac{\partial e_i}{\partial x^\ell} = e^k \cdot \frac{\partial^2 p}{\partial x^\ell \partial x^i} = e^k \cdot \frac{\partial^2 p}{\partial x^i \partial x^\ell} = e^k \cdot \frac{\partial e_\ell}{\partial x^i} = \Gamma^k_{\ell i} \tag{11.244}$$

and thus is symmetric in its two lower indices

$$\Gamma^k_{i\ell} = \Gamma^k_{\ell i}. \tag{11.245}$$

Affine connections are *not* tensors. Tensors transform homogeneously; connections transform inhomogeneously. The connection $\Gamma^k_{i\ell}$ transforms as

$$\begin{aligned} \Gamma'^k_{i\ell} &= e'^k \cdot \frac{\partial e'_i}{\partial x'^\ell} = \frac{\partial x'^k}{\partial x^p}\, e^p \cdot \frac{\partial x^m}{\partial x'^\ell}\, \frac{\partial}{\partial x^m} \left(\frac{\partial x^n}{\partial x'^i}\, e_n \right) \\ &= \frac{\partial x'^k}{\partial x^p}\, \frac{\partial x^m}{\partial x'^\ell}\, \frac{\partial x^n}{\partial x'^i}\, e^p \cdot \frac{\partial e_n}{\partial x^m} + \frac{\partial x'^k}{\partial x^n}\, \frac{\partial^2 x^n}{\partial x'^\ell \partial x'^i} \\ &= \frac{\partial x'^k}{\partial x^p}\, \frac{\partial x^m}{\partial x'^\ell}\, \frac{\partial x^n}{\partial x'^i}\, \Gamma^p_{nm} + \frac{\partial x'^k}{\partial x^n}\, \frac{\partial^2 x^n}{\partial x'^\ell \partial x'^i}. \end{aligned} \tag{11.246}$$

Although the connection $\Gamma^k_{i\ell}$ is not a tensor, its variation

$$\delta\Gamma^k_{i\ell} = \Gamma^k_{i\ell}(g_{nm} + \delta g_{nm}) - \Gamma^k_{i\ell}(g_{nm}) \tag{11.247}$$

is a tensor because the inhomogeneous term in this equation (11.246) cancels in the difference $\delta\Gamma^k_{i\ell}$. The electromagnetic field $A_i(x)$ and other gauge fields are connections.

Since the Levi-Civita connection $\Gamma^k_{i\ell}$ is symmetric in i and ℓ, in four-dimensional spacetime, there are 10 of them for each k, or 40 in all. The 10 correspond to 3 rotations, 3 boosts, and 4 translations.

11.32 Torsion

We can use Cartan's tetrads to define a more general covariant derivative. The quantity $F = F^k c_k$ is a scalar under general coordinate transformations. It has one flat-space index $F^a = F^k c^a_k$ which participates under Lorentz transformations, but we'll ignore that for the moment. Its k-derivative is

$$F^a_{,i} = F^k_{,i}\, c^a_k + F^k\, c^a_{k,i}. \tag{11.248}$$

The four 4-vectors c^a_k have four **dual** vectors c^k_a that obey the rules

$$c^a_k\, c^\ell_a = \delta^\ell_k \quad \text{and} \quad c^a_k\, c^k_b = \delta^a_b. \tag{11.249}$$

The inner product of this equation with the dual vector c^ℓ

$$\begin{aligned} F^a_{,i}\, c^\ell_a &= F^k_{,i}\, c^a_k\, c^\ell_a + F^k\, c^\ell_a\, c^a_{k,i} = F^k_{,i}\, \delta^\ell_k + F^k\, c^\ell_a\, c^a_{k,i} \\ &= F^\ell_{,i} + F^k\, c^\ell_a\, c^a_{k,i} \equiv F^\ell_{,i} + F^k\, \omega^\ell{}_{ki} \equiv F^\ell_{;i} \end{aligned} \tag{11.250}$$

is Cartan's covariant derivative $F^\ell_{;i}$ of the contravariant vector F^k. It is a mixed tensor $F^\ell_{;i}$ defined in terms of a more general connection

$$\omega^\ell{}_{ki} = c^\ell_a\, c^a_{k,i} \tag{11.251}$$

that, unlike the Levi-Civita connection $\Gamma^\ell{}_{ki} = \Gamma^\ell{}_{ik}$, is not neccessarily symmetric in its two lower indices k and i.

The antisymmetric part of the Cartan connection is the **torsion tensor**

$$T^\ell{}_{ki} \equiv \frac{1}{2}\left(\omega^\ell{}_{ki} - \omega^\ell{}_{ik}\right) \tag{11.252}$$

which *is* a tensor.

11.33 Parallel Transport

The movement of a vector along a curve on a manifold so that its length and direction in successive tangent spaces do not change is called **parallel transport**. If the vector is $F = F^i e_i$, then we want the inner product $e^k \cdot dF$ of dF with all dual tangent vectors e^k to vanish along the curve. But this is just the condition that the covariant derivative (11.241) of F should vanish along the curve

$$e^k \cdot \frac{\partial F}{\partial x^\ell} = \frac{\partial F^k}{\partial x^\ell} + F^i \, \Gamma^k_{i\ell} = D_\ell F^k = 0. \tag{11.253}$$

Example 11.24 (Parallel Transport on a Sphere)**.** We parallel-transport the vector $\boldsymbol{v} = (0, 1, 0)$ up from the equator along the line of longitude $\phi = 0$. Along this path, the vector $\boldsymbol{v} = (0, 1, 0)$ is constant, so $\partial_\theta \boldsymbol{v} = 0$ and $D_\theta v^k = 0$. Then we parallel-transport it down from the north pole along the line of longitude $\phi = \pi/2$ to the equator. Along this path, $\phi = \pi/2$, and the vector $\boldsymbol{v} = \boldsymbol{e}_\theta / r = (0, \cos\theta, -\sin\theta)$ obeys the parallel-transport condition (11.253) because its θ-derivative

$$\frac{\partial \boldsymbol{v}}{\partial \theta} = \frac{1}{r} \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = \frac{\partial}{\partial \theta} (0, \cos\theta, -\sin\theta) = (0, -\sin\theta, -\cos\theta) = \left. -\hat{\boldsymbol{r}} \right|_{\phi = \pi/2} \tag{11.254}$$

is perpendicular to the tangent vectors $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$ along the curve $\phi = \pi/2$. We then parallel-transport $\boldsymbol{v}$ along the equator back to the starting point $\phi = 0$. Along this path, the vector $\boldsymbol{v} = (0, 0, -1)$ is constant, so $\partial_\phi \boldsymbol{v} = 0$ and $D_\phi v^k = 0$. The change from $\boldsymbol{v} = (0, 1, 0)$ to $\boldsymbol{v} = (0, 0, -1)$ is due to the curvature of the sphere. □

11.34 Covariant Derivatives

In comma notation, the derivative of a contravariant vector field $F = F^i \, e_i$ is

$$F_{,\ell} = F^i_{,\ell} \, e_i + F^i \, e_{i,\ell} \tag{11.255}$$

which in general lies outside the space spanned by the basis vectors e_i. So we use the affine connections (11.239) to form the inner product

$$e^k \cdot F_{,\ell} = e^k \cdot \left(F^i_{,\ell} e_i + F^i e_{i,\ell} \right) = F^i_{,\ell} \, \delta^k_i + F^i \, \Gamma^k_{i\ell} = F^k_{,\ell} + F^i \, \Gamma^k_{i\ell} \, . \tag{11.256}$$

This **covariant derivative of a contravariant vector field** often is written with a semicolon

$$F^k_{;\ell} = e^k \cdot F_{,\ell} = F^k_{,\ell} + F^i \, \Gamma^k_{i\ell}. \tag{11.257}$$

It transforms as a mixed second-rank tensor. The invariant change dF projected onto e^k is

$$e^k \cdot dF = e^k \cdot F_{,\ell}\, dx^\ell = F^k_{;\ell}\, dx^\ell. \tag{11.258}$$

In terms of its covariant components, the derivative of a vector V is

$$V_{,\ell} = (V_k\, e^k)_{,\ell} = V_{k,\ell}\, e^k + V_k\, e^k_{,\ell}. \tag{11.259}$$

To relate the derivatives of the vectors e^i to the affine connections $\Gamma^k_{i\ell}$, we differentiate the orthonormality relation

$$\delta^k_i = e^k \cdot e_i \tag{11.260}$$

which gives us

$$0 = e^k_{,\ell} \cdot e_i + e^k \cdot e_{i,\ell} \quad \text{or} \quad e^k_{,\ell} \cdot e_i = -\, e^k \cdot e_{i,\ell} = -\,\Gamma^k_{i\ell}. \tag{11.261}$$

Since $e_i \cdot e^k_{,\ell} = -\Gamma^k_{i\ell}$, the inner product of e_i with the derivative of V is

$$e_i \cdot V_{,\ell} = e_i \cdot \Big(V_{k,\ell} e^k + V_k\, e^k_{,\ell}\Big) = V_{i,\ell} - V_k \Gamma^k_{i\ell}. \tag{11.262}$$

This **covariant derivative of a covariant vector field** often is written with a semicolon

$$V_{i;\ell} = e_i \cdot V_{,\ell} = V_{i,\ell} - V_k \Gamma^k_{i\ell}. \tag{11.263}$$

It transforms as a rank-2 covariant tensor. Note the minus sign in $V_{i;\ell}$ and the plus sign in $F^k_{;\ell}$. The change $e_i \cdot dV$ is

$$e_i \cdot dV = e_i \cdot V_{,\ell}\, dx^\ell = V_{i;\ell}\, dx^\ell. \tag{11.264}$$

Since dV is invariant, e_i covariant, and dx^ℓ contravariant, the quotient rule (section 11.15) confirms that the covariant derivative $V_{i;\ell}$ of a covariant vector V_i is a rank-2 covariant tensor.

We used similar logic to derive Cartan's covariant derivative (11.250) of a contravariant vector

$$F^k_{;\ell} = F^k_{,\ell} + F^i\, \omega^k{}_{i\ell}. \tag{11.265}$$

To find his formula for the covariant derivative of a covariant vector, we first differentiate the scalar $V = V_k\, c^k$

$$V_{a,\ell} = V_{k,\ell}\, c^k_a + V_k\, c^k{}_{a,\ell} \tag{11.266}$$

and then use (11.249) to contract it with the dual vector c^a_i

$$\begin{aligned} c^a_i\, V_{a,\ell} &= V_{k,\ell}\, c^k_a\, c^a_i + V_k\, c^a_i\, c^k{}_{a,\ell} = V_{k,\ell}\, \delta^k_i + V_k\, c^a_i\, c^k{}_{a,\ell} \\ &= V_{i,\ell} + V_k\, c^a_i\, c^k{}_{a,\ell}. \end{aligned} \tag{11.267}$$

To evaluate $c^a_i\, c^k{}_{a,\ell}$, we differentiate the first of the dual-vector relations (11.249) to show that

$$c^a{}_i\, c^k{}_{a,\ell} = -\, c^k{}_a\, c^a{}_{i,\ell}. \tag{11.268}$$

Cartan's covariant derivative of a covariant vector then is

$$V_{i;\ell} = V_{i,\ell} - V_k\; c^k{}_a\, c^a{}_{i,\ell} = V_{i,\ell} - V_k\; \omega^k{}_{i\ell}. \tag{11.269}$$

11.35 The Covariant Curl

Because the connection $\Gamma^k_{i\ell}$ is symmetric (11.245) in its lower indices, the covariant curl of a covariant vector V_i is simply its ordinary curl

$$V_{\ell;i} - V_{i;\ell} = V_{\ell,i} - V_k\, \Gamma^k_{\ell i} - V_{i,\ell} + V_k\, \Gamma^k_{i\ell} = V_{\ell,i} - V_{i,\ell}. \tag{11.270}$$

Thus the Faraday field-strength tensor $F_{i\ell}$ which is defined as the curl of the covariant vector field A_i

$$F_{i\ell} = A_{\ell,i} - A_{i,\ell} \tag{11.271}$$

is a generally covariant second-rank tensor.

If however we must use Cartan's covariant derivative (11.250), then we would define the Faraday field-strength tensor $F_{i\ell}$ as

$$F_{i\ell} = A_{\ell;i} - A_{i;\ell} = A_{\ell,i} - A_{i,\ell} + A_k \left(\omega^k{}_{i\ell} - \omega^k{}_{\ell i}\right) = A_{\ell,i} - A_{i,\ell} + 2A_k T^k{}_{i\ell} \tag{11.272}$$

in which $T^k{}_{i\ell}$ is the torsion tensor (11.252).

Example 11.25 (Orthogonal coordinates)**.** In orthogonal coordinates, the curl is defined (6.39, 11.110) in terms of the totally antisymmetric Levi-Civita symbol ϵ^{ijk} (with $\epsilon_{123} = \epsilon^{123} = 1$), as

$$\nabla \times \overline{V} = \sum_{i=1}^{3} (\nabla \times \overline{V})_i\, \hat{e}_i = \frac{1}{h_1 h_2 h_3} \sum_{ijk=1}^{3} \boldsymbol{e}_i\, \epsilon^{ijk}\, V_{k;j} \tag{11.273}$$

which, in view of (11.270) and the antisymmetry of ϵ^{ijk}, is

$$\nabla \times \overline{V} = \sum_{i=1}^{3} (\nabla \times \overline{V})_i\, \hat{e}_i = \sum_{ijk=1}^{3} \frac{1}{h_i h_j h_k}\, \boldsymbol{e}_i\, \epsilon^{ijk}\, V_{k,j} \tag{11.274}$$

or by (11.182 & 11.184)

$$\nabla \times \overline{V} = \sum_{ijk=1}^{3} \frac{1}{h_i h_j h_k}\, h_i \hat{e}_i\, \epsilon^{ijk}\, V_{k,j} = \sum_{ijk=1}^{3} \frac{1}{h_i h_j h_k}\, h_i \hat{e}_i\, \epsilon^{ijk}\, (h_k \overline{V}_k)_{,j}\,. \tag{11.275}$$

Often one writes this as a determinant

$$\nabla \times \overline{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ V_1 & V_2 & V_3 \end{vmatrix} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ h_1 \overline{V}_1 & h_2 \overline{V}_2 & h_3 \overline{V}_3 \end{vmatrix}. \tag{11.276}$$

In cylindrical coordinates, the curl is

$$\nabla \times \overline{V} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\,\hat{\phi} & \hat{z} \\ \partial_\rho & \partial_\phi & \partial_z \\ \overline{V}_\rho & \rho \overline{V}_\phi & \overline{V}_z \end{vmatrix}. \tag{11.277}$$

In spherical coordinates, it is

$$\nabla \times \overline{V} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ \overline{V}_r & r\overline{V}_\theta & r\sin\theta\,\overline{V}_\phi \end{vmatrix}. \tag{11.278}$$

In more formal language, the curl is

$$dV = d\left(V_k dx^k\right) = V_{k,i}\, dx^i \wedge dx^k = \frac{1}{2}\left(V_{k,i} - V_{i,k}\right)\, dx^i \wedge dx^k. \tag{11.279}$$

□

11.36 Covariant derivative of a tensor

Let's write a second-rank contravariant tensor in a coordinate-invariant way as a sum of direct products of tangent vectors e_i

$$T = T^{jm}\, e_j \otimes e_m. \tag{11.280}$$

Its derivative is

$$T_{,k} = T^{jm}{}_{,k}\; e_j \otimes e_m + T^{jm}\, e_{j,k} \otimes e_m + T^{jm}\, e_j \otimes e_{m,k}. \tag{11.281}$$

The covariant derivative $T^{i\ell}{}_{;k}$ of the tensor is the inner product of this derivative with the contravariant or dual tangent vectors $e^i \otimes e^\ell$

$$\begin{aligned} T^{i\ell}{}_{;k} &= e^i \otimes e^\ell \cdot T_{,k} \\ &= T^{jm}{}_{,k}\; e^i \cdot e_j\, e^\ell \cdot e_m + T^{jm}\, e^i \cdot e_{j,k}\, e^\ell \cdot e_m + T^{jm}\, e^i \cdot e_j\, e^\ell \cdot e_{m,k} \\ &= T^{jm}{}_{,k}\; \delta^i_j\, \delta^\ell_m + T^{jm}\, e^i \cdot e_{j,k}\, \delta^\ell_m + T^{jm}\, \delta^i_j\, e^\ell \cdot e_{m,k} \\ &= T^{i\ell}{}_{,k} + T^{j\ell}\, e^i \cdot e_{j,k} + T^{im}\, e^\ell \cdot e_{m,k}. \end{aligned} \tag{11.282}$$

In it we recognize two instances of our formula $\Gamma^i_{jk} = e^i \cdot e_{j,k}$ for the affine connection (11.239). Replacing these inner products with affine connections, we may write the covariant derivative $T^{i\ell}_{;k}$ of the tensor as

$$T^{i\ell}_{;k} = T^{i\ell}_{,k} + T^{j\ell}\,\Gamma^i_{jk} + T^{im}\,\Gamma^\ell_{mk}. \tag{11.283}$$

In particular, when $T^{i\ell} = F^i\,G^\ell$, this formula says that covariant derivatives, like ordinary derivatives, obey the Leibniz rule

$$(F^i\,G^\ell)_{;k} = F^i_{;k}\,G^\ell + F^i\,G^\ell_{;k}. \tag{11.284}$$

Covariant derivatives, like ordinary derivatives, are **derivations**

$$(A\,B)_{;k} = A_{;k}\,B + A\,B_{;k}. \tag{11.285}$$

A similar argument leads to a similar formula for the covariant derivative of a covariant tensor. A coordinate-independent form of a covariant tensor T_{jm} is

$$T = T_{jm}\,e^j \otimes e^m. \tag{11.286}$$

Its derivative is

$$T_{,k} = T_{jm,k}\;e^j \otimes e^m + T_{jm}\,e^j{}_{,k} \otimes e^m + T_{jm}\,e^j \otimes e^m{}_{,k}\,. \tag{11.287}$$

The covariant derivative $T_{i\ell;k}$ of the tensor is the inner product of this derivative with the covariant tangent vectors $e_i \otimes e_\ell$

$$\begin{aligned} T_{i\ell;k} &= e_i \otimes e_\ell \cdot T_{,k} \\ &= T_{i\ell,k} + T_{j\ell}\,e_i \cdot e^j{}_{,k} + T_{im}\,e_\ell \cdot e^m{}_{,k}\,. \end{aligned} \tag{11.288}$$

In it we see two instances of our other formula $e_i \cdot e^k_{,\ell} = -\,\Gamma^k_{i\ell}$ for the affine connection (11.261). Replacing them, we may write we may write the covariant derivative $T_{i\ell;k}$ of the tensor as

$$T_{i\ell;k} = T_{i\ell,k} - T_{j\ell}\,\Gamma^j_{ik} - T_{im}\,\Gamma^m_{\ell k}. \tag{11.289}$$

The rule for a general tensor is to treat every contravariant index as in (11.283) and every covariant index as in (11.289). The covariant derivative of a mixed rank-4 tensor, for instance, is

$$T^{ab}_{xy;k} = T^{ab}_{xy,k} + T^{jb}_{xy}\Gamma^a_{jk} + T^{am}_{xy}\Gamma^b_{mk} - T^{ab}_{jy}\Gamma^j_{xk} - T^{ab}_{xm}\Gamma^m_{yk}. \tag{11.290}$$

11.37 Covariant Derivatives and Antisymmetry

Let us apply our rule (11.289) for the covariant derivative of a second-rank tensor $A_{i\ell}$

$$A_{i\ell;k} = A_{i\ell,k} - A_{m\ell}\,\Gamma^m_{ik} - A_{im}\,\Gamma^m_{\ell k} \tag{11.291}$$

to an antisymmetric tensor

$$A_{i\ell} = -A_{\ell i}. \tag{11.292}$$

Then by adding together the three cyclic permutations of the indices $i\ell k$ we find that the antisymmetry of the tensor and the symmetry (11.245) of the affine connection $\Gamma^m_{ik} = \Gamma^m_{ki}$ conspire to cancel the quadratic terms leaving us with

$$\begin{aligned} A_{i\ell;k} + A_{ki;\ell} + A_{\ell k;i} = {} & A_{i\ell,k} - A_{m\ell}\,\Gamma^m_{ik} - A_{im}\,\Gamma^m_{\ell k} \\ & + A_{ki,\ell} - A_{mi}\,\Gamma^m_{k\ell} - A_{km}\,\Gamma^m_{i\ell} \\ & + A_{\ell k,i} - A_{mk}\,\Gamma^m_{\ell i} - A_{\ell m}\,\Gamma^m_{ki} \\ = {} & A_{i\ell,k} + A_{ki,\ell} + A_{\ell k,i} \end{aligned} \tag{11.293}$$

an identity named after Luigi Bianchi (1856–1928).

The Maxwell field-strength tensor $F_{i\ell}$ is antisymmetric by construction ($F_{i\ell} = A_{\ell,i} - A_{i,\ell}$), and so the Maxwell's homogeneous equations

$$\begin{aligned} \tfrac{1}{2}\,\epsilon^{ijk\ell}\,F_{jk,\ell} &= F_{jk,\ell} + F_{k\ell,j} + F_{\ell j,k} \\ &= A_{k,j\ell} - A_{j,k\ell} + A_{\ell,kj} - A_{k,\ell j} + A_{j,\ell k} - A_{\ell,jk} = 0 \end{aligned} \tag{11.294}$$

are tensor equations valid in all coordinate systems. This remains true even if we use Cartan's covariant derivative (11.250) to define the Cartan-Faraday tensor (11.272). It is amazing how right Maxwell was in the middle of the nineteenth century.

11.38 Affine Connection and Metric Tensor

To relate the affine connection $\Gamma^m_{\ell i}$ to the derivatives of the metric tensor $g_{k\ell}$, we lower the contravariant index m to get

$$\Gamma_{k\ell i} = g_{km}\,\Gamma^m_{\ell i} = g_{km}\,\Gamma^m_{i\ell} = \Gamma_{ki\ell} \tag{11.295}$$

which is symmetric in its last two indices and which some call a **Christoffel symbol of the first kind**, written $[\ell i, k]$. One can raise the index k back up by using the inverse of the metric tensor

$$g^{mk}\,\Gamma_{k\ell i} = g^{mk}\,g_{kn}\,\Gamma^n_{\ell i} = \delta^m_n\,\Gamma^n_{\ell i} = \Gamma^m_{\ell i}. \tag{11.296}$$

Although we can raise and lower these indices, the connections $\Gamma^m_{\ell i}$ and $\Gamma_{k\ell i}$ are not tensors.

The definition (11.239) of the affine connection tells us that

$$\Gamma_{k\ell i} = g_{km}\,\Gamma^m_{\ell i} = g_{km}\,e^m \cdot e_{\ell,i} = e_k \cdot e_{\ell,i} = \Gamma_{ki\ell} = e_k \cdot e_{i,\ell}. \tag{11.297}$$

By differentiating the definition $g_{i\ell} = e_i \cdot e_\ell$ of the metric tensor, we find

$$g_{i\ell,k} = e_{i,k} \cdot e_\ell + e_i \cdot e_{\ell,k} = e_\ell \cdot e_{i,k} + e_i \cdot e_{\ell,k} = \Gamma_{\ell ik} + \Gamma_{i\ell k}. \tag{11.298}$$

Permuting the indices cyclicly, we have

$$\begin{aligned} g_{ki,\ell} &= \Gamma_{ik\ell} + \Gamma_{ki\ell} \\ g_{\ell k,i} &= \Gamma_{k\ell i} + \Gamma_{\ell ki}. \end{aligned} \tag{11.299}$$

If we now subtract relation (11.298) from the sum of the two formulas (11.299) keeping in mind the symmetry $\Gamma_{abc} = \Gamma_{acb}$, then we find that four of the six terms cancel

$$g_{ki,\ell} + g_{\ell k,i} - g_{i\ell,k} = \Gamma_{ik\ell} + \Gamma_{ki\ell} + \Gamma_{k\ell i} + \Gamma_{\ell ki} - \Gamma_{\ell ik} - \Gamma_{i\ell k} = 2\Gamma_{k\ell i} \tag{11.300}$$

leaving a formula for $\Gamma_{k\ell i}$

$$\Gamma_{k\ell i} = \tfrac{1}{2}\left(g_{ki,\ell} + g_{\ell k,i} - g_{i\ell,k}\right). \tag{11.301}$$

Thus the connection is three derivatives of the metric tensor

$$\Gamma^s_{i\ell} = g^{sk}\Gamma_{k\ell i} = \tfrac{1}{2}g^{sk}\left(g_{ki,\ell} + g_{\ell k,i} - g_{i\ell,k}\right). \tag{11.302}$$

11.39 Covariant Derivative of the Metric Tensor

Let us apply our formula (11.289) for the covariant derivative of a covariant tensor to the metric tensor $g_{i\ell}$

$$g_{i\ell;k} \equiv g_{i\ell,k} - g_{m\ell}\,\Gamma^m_{ik} - g_{in}\,\Gamma^n_{\ell k}. \tag{11.303}$$

If we now substitute our formula (11.302) for the connections Γ^l_{ik} and $\Gamma^n_{\ell k}$

$$g_{i\ell;k} = g_{i\ell,k} - g_{m\ell}\,\tfrac{1}{2}g^{ms}\left(g_{is,k} + g_{sk,i} - g_{ik,s}\right) - g_{in}\,\tfrac{1}{2}g^{ns}\left(g_{s\ell,k} + g_{ks,\ell} - g_{\ell k,s}\right) \tag{11.304}$$

and use the fact (11.168) that the metric tensors $g_{i\ell}$ and $g^{\ell k}$ are mutually inverse and symmetric, then we find

$$\begin{aligned} g_{i\ell;k} &= g_{i\ell,k} - \tfrac{1}{2}\delta^s_\ell\left(g_{is,k} + g_{sk,i} - g_{ik,s}\right) - \tfrac{1}{2}\delta^s_i\left(g_{s\ell,k} + g_{ks,\ell} - g_{\ell k,s}\right) \\ &= g_{i\ell,k} - \tfrac{1}{2}\left(g_{i\ell,k} + g_{\ell k,i} - g_{ik,\ell}\right) - \tfrac{1}{2}\left(g_{i\ell,k} + g_{ki,\ell} - g_{\ell k,i}\right) = 0. \end{aligned} \tag{11.305}$$

The covariant derivative of the metric tensor vanishes. This result follows from our choice of the Levi-Civita connection (11.239); it is not true for some other connections.

Covariant derivatives obey the Leibniz rule (11.284), so

$$\begin{aligned}\delta^a_{b;\ell} &= (g^{ac}\, g_{cb})_{;\ell} = g^{ac}_{;\ell}\, g_{cb} + g^{ac}\, g_{cb;\ell} = g^{ac}_{;\ell}\, g_{cb} \\ &= \Gamma^a_{i\ell}\delta^i_b\, g_{cb} - \Gamma^k_{b\ell}\delta^a_k\, g_{cb} = \Gamma^a_{b\ell}\, g_{cb} - \Gamma^a_{b\ell}\, g_{cb} = 0.\end{aligned} \tag{11.306}$$

The covariant derivative of the inverse metric tensor vanishes, $g^{ac}_{;\ell} = 0$.

11.40 Divergence of a contravariant vector

The contracted covariant derivative of a contravariant vector is a **scalar** known as the **divergence**,

$$\nabla \cdot V = V^i_{;i} = V^i_{,i} + V^k\, \Gamma^i_{ki}. \tag{11.307}$$

Because $g_{ik} = g_{ki}$, in the sum (11.302) over i

$$\Gamma^i_{ki} = \tfrac{1}{2} g^{i\ell}\, (g_{i\ell,k} + g_{\ell k,i} - g_{ki,\ell}) \tag{11.308}$$

the last two terms cancel because they differ only by the interchange of the dummy indices i and ℓ

$$g^{i\ell} g_{\ell k,i} = g^{\ell i} g_{ik,\ell} = g^{i\ell} g_{ki,\ell}. \tag{11.309}$$

So the contracted connection collapses to

$$\Gamma^i_{ki} = \tfrac{1}{2} g^{i\ell} g_{i\ell,k}. \tag{11.310}$$

There is a nice formula for this last expression. To derive it, let $\underline{g} \equiv g_{i\ell}$ be the 4×4 matrix whose elements are those of the covariant metric tensor $g_{i\ell}$. Its determinant, like that of any matrix, is the cofactor sum (1.195) along any row or column, that is, over ℓ for fixed i or over i for fixed ℓ

$$\det(\underline{g}) = \sum_{i \text{ or } \ell} g_{i\ell}\, C_{i\ell} \tag{11.311}$$

in which the cofactor $C_{i\ell}$ is $(-1)^{i+\ell}$ times the determinant of the reduced matrix consisting of the matrix $\underline{g}$ with row i and column ℓ omitted. Thus the partial derivative of $\det \underline{g}$ with respect to the $i\ell$th element $g_{i\ell}$ is

$$\frac{\partial \det(\underline{g})}{\partial g_{i\ell}} = C_{i\ell} \tag{11.312}$$

in which we allow $g_{i\ell}$ and $g_{\ell i}$ to be independent variables for the purposes of this differentiation. The inverse $g^{i\ell}$ of the metric tensor $\underline{g}$, like the inverse

(1.197) of any matrix, is the transpose of the cofactor matrix divided by its determinant $\det(\underline{g})$

$$g^{i\ell} = \frac{C_{\ell i}}{\det(\underline{g})} = \frac{1}{\det(\underline{g})} \frac{\partial \det(\underline{g})}{\partial g_{\ell i}}. \tag{11.313}$$

Using this formula and the chain rule, we may write the derivative of the determinant $\det(\underline{g})$ as

$$\det(\underline{g})_{,k} = \frac{\partial \det(\underline{g})}{\partial g_{i\ell}} \, g_{i\ell,k} = \det(\underline{g}) \, g^{\ell i} \, g_{i\ell,k} \tag{11.314}$$

and so since $g_{i\ell} = g_{\ell i}$, the contracted connection (11.310) is

$$\Gamma^i_{ki} = \tfrac{1}{2} g^{i\ell} g_{i\ell,k} = \frac{\det(\underline{g})_{,k}}{2\det(\underline{g})} = \frac{|\det(\underline{g})|_{,k}}{2|\det(\underline{g})|} = \frac{g_{,k}}{2g} = \frac{(\sqrt{g})_{,k}}{\sqrt{g}} \tag{11.315}$$

in which $g \equiv \left|\det(\underline{g})\right|$ is the absolute value of the determinant of the metric tensor.

Thus from (11.307 & 11.315), we arrive at our formula for the covariant divergence of a contravariant vector:

$$\nabla \cdot V = V^i_{;i} = V^i_{,i} + \Gamma^i_{ki} V^k = V^k_{,k} + \frac{(\sqrt{g})_{,k}}{\sqrt{g}} V^k = \frac{(\sqrt{g}\, V^k)_{,k}}{\sqrt{g}}. \tag{11.316}$$

Example 11.26 (Maxwell's inhomogeneous equations)**.** An important application of this divergence formula (11.316) is the generally covariant form (11.232) of Maxwell's inhomogeneous equations

$$\frac{1}{\sqrt{g}} \left(\sqrt{g} F^{k\ell}\right)_{,\ell} = \mu_0 j^k. \tag{11.317}$$

□

Example 11.27 (Energy-momentum tensor)**.** Another application is to the divergence of the symmetric energy-momentum tensor $T^{ij} = T^{ji}$

$$\begin{aligned} T^{ij}_{;i} &= T^{ij}_{,i} + \Gamma^i_{ki} T^{kj} + \Gamma^j_{mi} T^{im} \\ &= \frac{(\sqrt{g} T^{kj})_k}{\sqrt{g}} + \Gamma^j_{mi} T^{im}. \end{aligned} \tag{11.318}$$

□

More formally, the Hodge dual (11.226) of the 1-form $V = V_i\, dx^i$ is

$$\begin{aligned} *V = V_i * dx^i &= V_i \frac{1}{3!}\, g^{ik}\, \eta_{k\ell mn}\, dx^\ell \wedge dx^m \wedge dx^n \\ &= \frac{1}{3!}\, \sqrt{g}\, V^k\, \epsilon_{k\ell mn}\, dx^\ell \wedge dx^m \wedge dx^n \end{aligned} \tag{11.319}$$

in which g is the absolute value of the determinant of the metric tensor g_{ij}. The exterior derivative now gives

$$d * V = \frac{1}{3!} \left(\sqrt{g}\, V^k\right)_{,p} \epsilon_{k\ell mn}\, dx^p \wedge dx^\ell \wedge dx^m \wedge dx^n. \tag{11.320}$$

So using (11.226) to apply a second Hodge star, we get (exercise 11.20)

$$\begin{aligned} * d * V &= \frac{1}{3!} \left(\sqrt{g}\, V^k\right)_{,p} \epsilon_{k\ell mn} * \left(dx^p \wedge dx^\ell \wedge dx^m \wedge dx^n\right) \\ &= \frac{1}{3!} \left(\sqrt{g}\, V^k\right)_{,p} \epsilon_{k\ell mn}\, g^{pt}\, g^{\ell u}\, g^{mv}\, g^{nw} \eta_{tuvw} \\ &= \frac{1}{3!} \left(\sqrt{g}\, V^k\right)_{,p} \epsilon_{k\ell mn}\, g^{pt}\, g^{\ell u}\, g^{mv}\, g^{nw} \epsilon_{tuvw} \sqrt{g} \\ &= \frac{1}{3!} \left(\sqrt{g}\, V^k\right)_{,p} \epsilon_{k\ell mn} \frac{\sqrt{g}}{\det g_{ij}}\, \epsilon^{p\ell mn} \\ &= \frac{s}{\sqrt{g}} \left(\sqrt{g}\, V^k\right)_{,p} \delta^p_k = \frac{s}{\sqrt{g}} \left(\sqrt{g}\, V^k\right)_{,k} \end{aligned} \tag{11.321}$$

So in our spacetime with $\det g_{ij} = -g$

$$- * d * V = \frac{1}{\sqrt{g}} \left(\sqrt{g}\, V^k\right)_{,k}. \tag{11.322}$$

In 3-space the Hodge star (11.215) of a 1-form $V = V_i\, dx^i$ is

$$* V = V_i * dx^i = V_i \frac{1}{2}\, g^{i\ell}\, \eta_{\ell jk}\, dx^j \wedge dx^k = \frac{1}{2} \sqrt{g}\, V^\ell\, \epsilon_{\ell jk}\, dx^j \wedge dx^k. \tag{11.323}$$

Applying the exterior derivative, we get the invariant form

$$d * V = \frac{1}{2} \left(\sqrt{g}\, V^\ell\right)_{,p} \epsilon_{\ell jk}\, dx^p \wedge dx^j \wedge dx^k. \tag{11.324}$$

We add a star by using the definition (11.215) of the Hodge dual in a 3-space in which the determinant $\det g_{ij}$ is positive and the identity (exercise 11.19)

$$\epsilon_{\ell jk}\, \epsilon^{pjk} = 2\, \delta^p_\ell \tag{11.325}$$

as well as the definition (1.184) of the determinant

$$
\begin{aligned}
\,d\,\,V &= \frac{1}{2}\left(\sqrt{g}\,V^{\ell}\right)_{,p}\,\epsilon_{\ell jk}\,*\left(dx^p\wedge dx^j\wedge dx^k\right)\\
&= \frac{1}{2}\left(\sqrt{g}\,V^{\ell}\right)_{,p}\,\epsilon_{\ell jk}\,g^{pt}g^{ju}g^{kv}\eta_{tuv}\\
&= \frac{1}{2}\left(\sqrt{g}\,V^{\ell}\right)_{,p}\,\epsilon_{\ell jk}\,g^{pt}g^{ju}g^{kv}\epsilon_{tuv}\sqrt{g}\\
&= \frac{1}{2}\left(\sqrt{g}\,V^{\ell}\right)_{,p}\,\epsilon_{\ell jk}\,\epsilon^{pjk}\frac{\sqrt{g}}{\det g_{ij}}\\
&= \frac{1}{\sqrt{g}}\left(\sqrt{g}\,V^{\ell}\right)_{,p}\,\delta^p_{\ell} = \frac{1}{\sqrt{g}}\left(\sqrt{g}\,V^p\right)_{,p}.
\end{aligned}
\tag{11.326}
$$

Example 11.28 (Divergence in Orthogonal Coordinates)**.** In two orthogonal coordinates, equations (11.177 & 11.184) imply that $\sqrt{g} = h_1 h_2$ and $V^k = \overline{V}_k/h_k$, and so the divergence (11.316) of a vector $\overline{V}$ is

$$
\nabla\cdot V = \frac{1}{h_1 h_2}\sum_{k=1}^{2}\left(\frac{h_1 h_2}{h_k}\,\overline{V}_k\right)_{,k}
\tag{11.327}
$$

which in polar coordinates (section 11.24) with $h_r = 1$ and $h_\theta = r$, is

$$
\nabla\cdot V = \frac{1}{r}\left[\left(r\,\overline{V}_r\right)_{,r} + \left(\overline{V}_\theta\right)_{,\theta}\right] = \frac{1}{r}\left[\left(r\,\overline{V}_r\right)_{,r} + \overline{V}_{\theta,\theta}\right].
\tag{11.328}
$$

In three orthogonal coordinates, equations (11.177 & 11.184) give $\sqrt{g} = h_1 h_2 h_3$ and $V^k = \overline{V}_k/h_k$, and so the divergence (11.316) of a vector V is (6.29)

$$
\nabla\cdot V = \frac{1}{h_1 h_2 h_3}\sum_{k=1}^{3}\left(\frac{h_1 h_2 h_3}{h_k}\,\overline{V}_k\right)_{,k}.
\tag{11.329}
$$

In cylindrical coordinates (section 11.25), $h_\rho = 1$, $h_\phi = \rho$, and $h_z = 1$; so

$$
\begin{aligned}
\nabla\cdot V &= \frac{1}{\rho}\left[\left(\rho\,\overline{V}_\rho\right)_{,\rho} + \left(\overline{V}_\phi\right)_{,\phi} + \left(\rho\,\overline{V}_z\right)_{,z}\right]\\
&= \frac{1}{\rho}\left[\left(\rho\,\overline{V}_\rho\right)_{,\rho} + \overline{V}_{\phi,\phi} + \rho\,\overline{V}_{z,z}\right].
\end{aligned}
\tag{11.330}
$$

In spherical coordinates (section 11.26), $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$, $g = |\det \underline{g}| = r^4\sin^2\theta$ and the inverse g^{ij} of the metric tensor is

$$
(g^{ij}) = \begin{pmatrix} 1 & 0 & 0\\ 0 & r^{-2} & 0\\ 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix}.
\tag{11.331}
$$

So our formula (11.327) gives us

$$\begin{aligned}\nabla\cdot V &= \frac{1}{r^2\sin\theta}\left[\left(r^2\sin\theta\,\overline{V}_r\right)_{,r}+\left(r\sin\theta\,\overline{V}_\theta\right)_{,\theta}+\left(r\,\overline{V}_\phi\right)_{,\phi}\right]\\ &= \frac{1}{r^2\sin\theta}\left[\sin\theta\,\left(r^2\overline{V}_r\right)_{,r}+r\,\left(\sin\theta\,\overline{V}_\theta\right)_{,\theta}+r\overline{V}_{\phi,\phi}\right]\end{aligned}\tag{11.332}$$

as the divergence $\nabla\cdot V$. □

11.41 The Covariant Laplacian

In flat 3-space, we write the laplacian as $\nabla\cdot\nabla=\nabla^2$ or as $\triangle$. In euclidian coordinates, both mean $\partial_x^2+\partial_y^2+\partial_z^2$. In flat minkowski space, one often turns the triangle into a square and writes the 4-laplacian as $\Box=\triangle-\partial_0^2$.

The gradient $f_{,k}$ of a scalar field f is a covariant vector, and $f^{,i}=g^{ik}f_{,k}$ is its contravariant form. The **invariant laplacian** $\Box f$ of a scalar field f is the covariant divergence $f^{,i}_{;i}$. We may use our formula (11.316) for the divergence of a contravariant vector to write it in these equivalent ways

$$\Box f = f^{,i}_{;i} = (g^{ik}f_{,k})_{;i} = \frac{(\sqrt{g}\,f^{,i})_{,i}}{\sqrt{g}} = \frac{(\sqrt{g}\,g^{ik}f_{,k})_{,i}}{\sqrt{g}}.\tag{11.333}$$

To find the laplacian $\Box f$ in terms of forms, we apply the exterior derivative to the Hodge dual (11.226) of the 1-form $df=f_{,i}dx^i$

$$\begin{aligned}d*df &= d\left(f_{,i}*dx^i\right)=d\left(\frac{1}{3!}\,f_{,i}\,g^{ik}\,\eta_{k\ell mn}\,dx^\ell\wedge dx^m\wedge dx^n\right)\\ &= \frac{1}{3!}\left(f^{,k}\sqrt{g}\right)_{,p}\epsilon_{k\ell mn}\,dx^p\wedge dx^\ell\wedge dx^m\wedge dx^n\end{aligned}\tag{11.334}$$

and then add a star using (11.226)

$$\begin{aligned}*\,d*df &= \frac{1}{3!}\left(f^{,k}\sqrt{g}\right)_{,p}\epsilon_{k\ell mn}\,*\left(dx^p\wedge dx^\ell\wedge dx^m\wedge dx^n\right)\\ &= \frac{1}{3!}\left(f^{,k}\sqrt{g}\right)_{,p}\epsilon_{k\ell mn}\,g^{pt}g^{\ell u}g^{mv}g^{nw}\sqrt{g}\,\epsilon_{tuvw}.\end{aligned}\tag{11.335}$$

The definition (1.184) of the determinant now gives (exercise 11.20)

$$\begin{aligned}*\,d*df &= \frac{1}{3!}\left(f^{,k}\sqrt{g}\right)_{,p}\epsilon_{k\ell mn}\,\epsilon^{p\ell mn}\frac{\sqrt{g}}{\det g}\\ &= \left(f^{,k}\sqrt{g}\right)_{,p}\delta^p_k\,\frac{s}{\sqrt{g}}=\frac{s}{\sqrt{g}}\,\left(f^{,k}\sqrt{g}\right)_{,k}.\end{aligned}\tag{11.336}$$

In our spacetime $\det g_{ij} = sg = -g$, and so the laplacian is

$$\Box f = - * d * df = \frac{1}{\sqrt{g}} \left(f^{,k} \sqrt{g} \right)_{,k}. \tag{11.337}$$

Example 11.29 (Invariant Laplacians)**.** In two orthogonal coordinates, equations (11.177 & 11.178) imply that $\sqrt{g} = \sqrt{|\det(g_{ij})|} = h_1 h_2$ and that $f^{,i} = g^{ik} f_{,k} = h_i^{-2} f_{,i}$, and so the laplacian (11.333) of a scalar f is

$$\triangle f = \frac{1}{h_1 h_2} \left(\sum_{i=1}^{2} \frac{h_1 h_2}{h_i^2} f_{,i} \right)_{,i}. \tag{11.338}$$

In polar coordinates, where $h_1 = 1$, $h_2 = r$, and $g = r^2$, the laplacian is

$$\triangle f = \frac{1}{r} \left[(r f_{,r})_{,r} + \left(r^{-1} f_{,\theta} \right)_{,\theta} \right] = f_{,rr} + r^{-1} f_{,r} + r^{-2} f_{,\theta\theta}. \tag{11.339}$$

In three orthogonal coordinates, equations (11.177 & 11.178) imply that $\sqrt{g} = \sqrt{|\det(g_{ij})|} = h_1 h_2 h_3$ and that $f^{,i} = g^{ik} f_{,k} = h_i^{-2} f_{,i}$, and so the laplacian (11.333) of a scalar f is (6.33)

$$\triangle f = \frac{1}{h_1 h_2 h_3} \left(\sum_{i=1}^{3} \frac{h_1 h_2 h_3}{h_i^2} f_{,i} \right)_{,i}. \tag{11.340}$$

In cylindrical coordinates (section 11.25), $h_\rho = 1$, $h_\phi = \rho$, $h_z = 1$, $g = \rho^2$, and the laplacian is

$$\triangle f = \frac{1}{\rho} \left[(\rho\, f_{,\rho})_{,\rho} + \frac{1}{\rho} f_{,\phi\phi} + \rho\, f_{,zz} \right] = f_{,\rho\rho} + \frac{1}{\rho} f_{,\rho} + \frac{1}{\rho^2} f_{,\phi\phi} + f_{,zz}. \tag{11.341}$$

In spherical coordinates (section 11.26), $h_r = 1$, $h_\theta = r$, $h_\phi = r \sin\theta$, and $g = |\det \underline{g}| = r^4 \sin^2\theta$. So (11.340) gives us the laplacian of f as (6.35)

$$\begin{aligned} \triangle f &= \frac{\left(r^2 \sin\theta f_{,r}\right)_{,r} + \left(\sin\theta f_{,\theta}\right)_{,\theta} + \left(f_{,\phi}/\sin\theta\right)_{,\phi}}{r^2 \sin\theta} \\ &= \frac{\left(r^2 f_{,r}\right)_{,r}}{r^2} + \frac{\left(\sin\theta f_{,\theta}\right)_{,\theta}}{r^2 \sin\theta} + \frac{f_{,\phi\phi}}{r^2 \sin^2\theta}. \end{aligned} \tag{11.342}$$

If the function f is a function only of the radial variable r, then the laplacian is simply

$$\triangle f(r) = \frac{1}{r^2} \left[r^2 f'(r) \right]' = \frac{1}{r} \left[r f(r) \right]'' = f''(r) + \frac{2}{r} f'(r) \tag{11.343}$$

in which the primes denote r-derivatives. $\Box$

11.42 The Principle of Stationary Action

It follows from a path-integral formulation of quantum mechanics that the classical motion of a particle is given by the **principle of stationary action** $\delta S = 0$. In the simplest case of a free non-relativistic particle, the lagrangian is $L = m\dot{\boldsymbol{x}}^2/2$ and the action is

$$S = \int_{t_1}^{t_2} \frac{m}{2}\dot{\boldsymbol{x}}^2\, dt. \tag{11.344}$$

The classical trajectory is the one that when varied slightly by $\delta\boldsymbol{x}$ (with $\delta\boldsymbol{x}(t_1) = \delta\boldsymbol{x}(t_2) = \mathbf{0}$) does not change the action to first order in $\delta\boldsymbol{x}$. We first note that the change $\delta\dot{\boldsymbol{x}}$ in the velocity is the time derivative of the change in the path

$$\delta\dot{\boldsymbol{x}} = \dot{\boldsymbol{x}}' - \dot{\boldsymbol{x}} = \frac{d}{dt}(\boldsymbol{x}' - \boldsymbol{x}) = \frac{d}{dt}\delta\boldsymbol{x}. \tag{11.345}$$

So since $\delta\boldsymbol{x}(t_1) = \delta\boldsymbol{x}(t_2) = \mathbf{0}$, the stationary path satisfies

$$\begin{aligned} 0 = \delta S &= \int_{t_1}^{t_2} m\dot{\boldsymbol{x}}\cdot\delta\dot{\boldsymbol{x}}\, dt = \int_{t_1}^{t_2} m\dot{\boldsymbol{x}}\cdot\frac{d\delta\boldsymbol{x}}{dt}\, dt \\ &= \int_{t_1}^{t_2}\left[m\frac{d}{dt}\left(\dot{\boldsymbol{x}}\cdot\delta\boldsymbol{x}\right) - m\ddot{\boldsymbol{x}}\cdot\delta\boldsymbol{x}\right] dt \\ &= m\left[\dot{\boldsymbol{x}}\cdot\delta\boldsymbol{x}\right]_{t_1}^{t_2} - m\int_{t_1}^{t_2}\ddot{\boldsymbol{x}}\cdot\delta\boldsymbol{x}\, dt = -m\int_{t_1}^{t_2}\ddot{\boldsymbol{x}}\cdot\delta\boldsymbol{x}\, dt. \end{aligned} \tag{11.346}$$

If the first-order change in the action is to vanish for arbitrary small variations $\delta\boldsymbol{x}$ in the path, then the acceleration must vanish

$$\ddot{\boldsymbol{x}} = \mathbf{0} \tag{11.347}$$

which is the classical equation of motion for a free particle.

If the particle is moving under the influence of a potential $V(\boldsymbol{x})$, then the action is

$$S = \int_{t_1}^{t_2}\left(\frac{m}{2}\dot{\boldsymbol{x}}^2 - V(\boldsymbol{x})\right) dt. \tag{11.348}$$

Since $\delta V(\boldsymbol{x}) = \nabla V(\boldsymbol{x})\cdot\delta\boldsymbol{x}$, the principle of stationary action requires that

$$0 = \delta S = \int_{t_1}^{t_2}\left(-m\ddot{\boldsymbol{x}} - \nabla V\right)\cdot\delta\boldsymbol{x}\, dt \tag{11.349}$$

or

$$m\ddot{\boldsymbol{x}} = -\nabla V \tag{11.350}$$

which is the classical equation of motion for a particle of mass m in a potential V.

The action for a free particle of mass m in special relativity is

$$S = -m\int_{\tau_1}^{\tau_2} d\tau = -\int_{t_1}^{t_2} m\sqrt{1-\dot{\boldsymbol{x}}^2}\,dt \tag{11.351}$$

where $c = 1$ and $\dot{\boldsymbol{x}} = d\boldsymbol{x}/dt$. The requirement of stationary action is

$$0 = \delta S = -\delta\int_{t_1}^{t_2} m\sqrt{1-\dot{\boldsymbol{x}}^2}\,dt = m\int_{t_1}^{t_2} \frac{\dot{\boldsymbol{x}}\cdot\delta\dot{\boldsymbol{x}}}{\sqrt{1-\dot{\boldsymbol{x}}^2}}\,dt \tag{11.352}$$

But $1/\sqrt{1-\dot{\boldsymbol{x}}^2} = dt/d\tau$ and so

$$\begin{aligned} 0 = \delta S &= m\int_{t_1}^{t_2} \frac{d\boldsymbol{x}}{dt}\cdot\frac{d\delta\boldsymbol{x}}{dt}\,\frac{dt}{d\tau}\,dt = m\int_{\tau_1}^{\tau_2} \frac{d\boldsymbol{x}}{dt}\cdot\frac{d\delta\boldsymbol{x}}{dt}\,\frac{dt}{d\tau}\,\frac{dt}{d\tau}\,d\tau \\ &= m\int_{\tau_1}^{\tau_2} \frac{d\boldsymbol{x}}{d\tau}\cdot\frac{d\delta\boldsymbol{x}}{d\tau}\,d\tau. \end{aligned} \tag{11.353}$$

So integrating by parts, keeping in mind that $\delta\boldsymbol{x}(\tau_2) = \delta\boldsymbol{x}(\tau_1) = \mathbf{0}$, we have

$$0 = \delta S = m\int_{\tau_1}^{\tau_2} \left[\frac{d}{d\tau}\left(\frac{d\boldsymbol{x}}{d\tau}\cdot\delta\boldsymbol{x}\right) - \frac{d^2\boldsymbol{x}}{d\tau^2}\cdot\delta\boldsymbol{x}\right] d\tau = -m\int_{\tau_1}^{\tau_2} \frac{d^2\boldsymbol{x}}{d\tau^2}\cdot\delta\boldsymbol{x}\,d\tau. \tag{11.354}$$

To have this hold for arbitrary $\delta\boldsymbol{x}$, we need

$$\frac{d^2\boldsymbol{x}}{d\tau^2} = \mathbf{0} \tag{11.355}$$

which is the equation of motion for a free particle in special relativity.

What about a charged particle in an electromagnetic field A_i? Its action is

$$S = -m\int_{\tau_1}^{\tau_2} d\tau + q\int_{x_1}^{x_2} A_i(x)\,dx^i = \int_{\tau_1}^{\tau_2} \left(-m + qA_i(x)\frac{dx^i}{d\tau}\right) d\tau. \tag{11.356}$$

We now treat the first term in a four-dimensional manner

$$\delta d\tau = \delta\sqrt{-\eta_{ik}dx^i dx^k} = \frac{-\eta_{ik}dx^i\delta dx^k}{\sqrt{-\eta_{ik}dx^i dx^k}} = -u_k\delta dx^k = -u_k d\delta x^k \tag{11.357}$$

in which $u_k = dx_k/d\tau$ is the 4-velocity (11.65) and η is the Minkowski metric (11.26) of flat spacetime. The variation of the other term is

$$\delta\left(A_i\,dx^i\right) = (\delta A_i)\,dx^i + A_i\,\delta dx^i = A_{i,k}\delta x^k\,dx^i + A_i\,d\delta x^i \tag{11.358}$$

Putting them together, we get for δS

$$\delta S = \int_{\tau_1}^{\tau_2} \left(m u_k \frac{d\delta x^k}{d\tau} + q A_{i,k} \delta x^k \frac{dx^i}{d\tau} + q A_i \frac{d\delta x^i}{d\tau} \right) d\tau. \tag{11.359}$$

After integrating by parts the last term, dropping the boundary terms, and changing a dummy index, we get

$$\begin{aligned} \delta S &= \int_{\tau_1}^{\tau_2} \left(- m \frac{du_k}{d\tau} \delta x^k + q A_{i,k} \delta x^k \frac{dx^i}{d\tau} - q \frac{dA_k}{d\tau} \delta x^k \right) d\tau \\ &= \int_{\tau_1}^{\tau_2} \left[- m \frac{du_k}{d\tau} + q \left(A_{i,k} - A_{k,i} \right) \frac{dx^i}{d\tau} \right] \delta x^k \, d\tau. \end{aligned} \tag{11.360}$$

If this first-order variation of the action is to vanish for arbitrary δx^k, then the particle must follow the path

$$0 = -m \frac{du_k}{d\tau} + q \left(A_{i,k} - A_{k,i} \right) \frac{dx^i}{d\tau} \qquad \text{or} \qquad \frac{dp^k}{d\tau} = q F^{ki} u_i \tag{11.361}$$

which is the Lorentz force law (11.95).

11.43 A Particle in a Gravitational Field

The invariant action for a particle of mass m moving along a path $x^i(t)$ is

$$S = -m \int_{\tau_1}^{\tau_2} d\tau = -m \int \Big(- g_{i\ell} dx^i dx^\ell \Big)^{\frac{1}{2}} . \tag{11.362}$$

Proceeding as in Eq.(11.357), we compute the variation $\delta d\tau$ as

$$\begin{aligned} \delta d\tau = \delta \sqrt{-g_{i\ell} dx^i dx^\ell} &= \frac{-\delta(g_{i\ell}) dx^i dx^\ell - 2 g_{i\ell} dx^i \delta dx^\ell}{2\sqrt{-g_{i\ell} dx^i dx^\ell}} \\ &= -\tfrac{1}{2} g_{i\ell,k} \delta x^k u^i u^\ell d\tau - g_{i\ell} u^i \delta dx^\ell \\ &= -\tfrac{1}{2} g_{i\ell,k} \delta x^k u^i u^\ell d\tau - g_{i\ell} u^i d\delta x^\ell \end{aligned} \tag{11.363}$$

in which $u^\ell = dx^\ell/d\tau$ is the 4-velocity (11.65). The condition of stationary action then is

$$0 = \delta S = -m \int_{\tau_1}^{\tau_2} \delta d\tau = m \int_{\tau_1}^{\tau_2} \left(\tfrac{1}{2} g_{i\ell,k} \delta x^k u^i u^\ell + g_{i\ell} u^i \frac{d\delta x^\ell}{d\tau} \right) d\tau \tag{11.364}$$

which we integrate by parts keeping in mind that $\delta x^\ell(\tau_2) = \delta x^\ell(\tau_1) = 0$

$$\begin{aligned} 0 &= m\int_{\tau_1}^{\tau_2}\left(\tfrac{1}{2}g_{i\ell,k}\delta x^k u^i u^\ell - \frac{d(g_{i\ell}u^i)}{d\tau}\delta x^\ell\right)d\tau \\ &= m\int_{\tau_1}^{\tau_2}\left(\tfrac{1}{2}g_{i\ell,k}\delta x^k u^i u^\ell - g_{i\ell,k}u^i u^k\delta x^\ell - g_{i\ell}\frac{du^i}{d\tau}\delta x^\ell\right)d\tau. \end{aligned} \tag{11.365}$$

Now interchanging the dummy indices ℓ and k on the second and third terms, we have

$$0 = m\int_{\tau_1}^{\tau_2}\left(\tfrac{1}{2}g_{i\ell,k}u^i u^\ell - g_{ik,\ell}u^i u^\ell - g_{ik}\frac{du^i}{d\tau}\right)\delta x^k d\tau \tag{11.366}$$

or since δx^k is arbitrary

$$0 = \tfrac{1}{2}g_{i\ell,k}u^i u^\ell - g_{ik,\ell}u^i u^\ell - g_{ik}\frac{du^i}{d\tau}. \tag{11.367}$$

If we multiply this equation of motion by g^{rk} and note that $g_{ik,\ell}u^i u^\ell = g_{\ell k,i}u^i u^\ell$, then we find

$$0 = \frac{du^r}{d\tau} + \tfrac{1}{2}g^{rk}\left(g_{ik,\ell} + g_{\ell k,i} - g_{i\ell,k}\right)u^i u^\ell. \tag{11.368}$$

So using the symmetry $g_{i\ell} = g_{\ell i}$ and the formula (11.302) for $\Gamma^r_{i\ell}$, we get

$$0 = \frac{du^r}{d\tau} + \Gamma^r_{i\ell}\,u^i u^\ell \qquad \text{or} \qquad 0 = \frac{d^2x^r}{d\tau^2} + \Gamma^r_{i\ell}\,\frac{dx^i}{d\tau}\,\frac{dx^\ell}{d\tau} \tag{11.369}$$

which is the geodesic equation. In empty space, particles fall along **geodesics independently of their masses**.

The right-hand side of the geodesic equation (11.369) is a contravariant vector because (Weinberg, 1972) under general coordinate transformations, the inhomogeneous terms arising from $\ddot{x}^r$ cancel those from $\Gamma^r_{i\ell}\dot{x}^i\dot{x}^\ell$. Here and often in what follows we'll use dots to mean proper-time derivatives.

The action for a particle of mass m and charge q in a gravitational field $\Gamma^r_{i\ell}$ and an electromagnetic field A_i is

$$S = -m\int\Big(-g_{i\ell}dx^i dx^\ell\Big)^{\frac{1}{2}} + q\int_{\tau_1}^{\tau_2}A_i(x)\,dx^i \tag{11.370}$$

because the interaction $q\int A_i dx^i$ is invariant under general coordinate transformations. By (11.360 & 11.366), the first-order change in S is

$$\delta S = m\int_{\tau_1}^{\tau_2}\left[\tfrac{1}{2}g_{i\ell,k}u^i u^\ell - g_{ik,\ell}u^i u^\ell - g_{ik}\frac{du^i}{d\tau} + q\left(A_{i,k} - A_{k,i}\right)u^i\right]\delta x^k d\tau \tag{11.371}$$

and so by combining the Lorentz force law (11.361) and the geodesic equation (11.369) and by writing $F^{ri}\dot{x}_i$ as $F^r{}_i\,\dot{x}^i$, we have

$$0 = \frac{d^2x^r}{d\tau^2} + \Gamma^r_{i\ell}\,\frac{dx^i}{d\tau}\,\frac{dx^\ell}{d\tau} - \frac{q}{m}\,F^r{}_i\,\frac{dx^i}{d\tau} \tag{11.372}$$

as the equation of motion of a particle of mass m and change q. It is striking how nearly perfect the electromagnetism of Faraday and Maxwell is.

The action of the electromagnetic field interacting with an electric current j^k in a gravitational field is

$$S = \int \left[-\tfrac{1}{4}\,F_{k\ell}\,F^{k\ell} + \mu_0\,A_k\,j^k\right]\sqrt{g}\,d^4x \tag{11.373}$$

in which $\sqrt{g}\,d^4x$ is the invariant volume element. After an integration by parts, the first-order change in the action is

$$\delta S = \int \left[-\frac{\partial}{\partial x^\ell}\left(F^{k\ell}\,\sqrt{g}\right) + \mu_0\,j^k\,\sqrt{g}\right]\delta A_k\,d^4x, \tag{11.374}$$

and so the inhomogeneous Maxwell equations in a gravitational field are

$$\frac{\partial}{\partial x^\ell}\left(\sqrt{g}\,F^{k\ell}\right) = \mu_0\,\sqrt{g}\,j^k. \tag{11.375}$$

11.44 Equivalence principle and geodesic equation

The **principle of equivalence** (section 11.19) says that in any gravitational field, one may choose free-fall coordinates in which all physical laws take the same form as in special relativity without acceleration or gravitation—at least over a suitably small volume of spacetime. Within this volume and in these coordinates, things behave as they would at rest deep in empty space far from any matter or energy. The volume must be small enough so that the gravitational field is constant throughout it.

Example 11.30 (Elevators)**.** When a modern elevator starts going down from a high floor, it accelerates downward at something less than the local acceleration of gravity. One feels less pressure on one's feet; one feels lighter. After accelerating downward for a few seconds, the elevator assumes a constant downward speed, and then one feels the normal pressure of one's weight on one's feet. The elevator seems to be slowing down for a stop, but actually it has just stopped accelerating downward.

What if the cable snapped, and a frightened passenger dropped his laptop?

He could catch it very easily as it would not seem to fall because the elevator, the passenger, and the laptop would all fall at the same rate. The physics in the falling elevator would be the same as if the elevator were at rest in empty space far from any gravitational field. The laptop's clock would tick as fast as it would at rest in the absence of gravity. □

The transformation from arbitrary coordinates x^k to free-fall coordinates y^i changes the metric $g_{j\ell}$ to the diagonal metric η_{ik} of flat spacetime $\eta = \mathrm{diag}(-1, 1, 1, 1)$, which has two indices and is not a Levi-Civita tensor. Algebraically, this transformation is a congruence (1.314)

$$\eta_{ik} = \frac{\partial x^j}{\partial y^i}\, g_{j\ell}\, \frac{\partial x^\ell}{\partial y^k}. \tag{11.376}$$

The geodesic equation (11.369) follows from the **principle of equivalence** (Weinberg, 1972; Hobson et al., 2006). Suppose a particle is moving under the influence of gravitation alone. Then one may choose free-fall coordinates $y(x)$ so that the particle obeys the force-free equation of motion

$$\frac{d^2 y^i}{d\tau^2} = 0 \tag{11.377}$$

with $d\tau$ the proper time $d\tau^2 = -\eta_{ik}\, dy^i dy^k$. The chain rule applied to $y^i(x)$ in (11.377) gives

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(\frac{\partial y^i}{\partial x^k}\frac{dx^k}{d\tau}\right) \\ &= \frac{\partial y^i}{\partial x^k}\frac{d^2 x^k}{d\tau^2} + \frac{\partial^2 y^i}{\partial x^k \partial x^\ell}\frac{dx^k}{d\tau}\frac{dx^\ell}{d\tau}. \end{aligned} \tag{11.378}$$

We multiply by $\partial x^m/\partial y^i$ and use the identity

$$\frac{\partial x^m}{\partial y^i}\frac{\partial y^i}{\partial x^k} = \delta^m_k \tag{11.379}$$

to write the equation of motion (11.377) in the x-coordinates

$$\frac{d^2 x^m}{d\tau^2} + \Gamma^m_{k\ell}\frac{dx^k}{d\tau}\frac{dx^\ell}{d\tau} = 0. \tag{11.380}$$

This is the geodesic equation (11.369) in which the affine connection is

$$\Gamma^m_{k\ell} = \frac{\partial x^m}{\partial y^i}\frac{\partial^2 y^i}{\partial x^k \partial x^\ell}. \tag{11.381}$$

11.45 Weak, Static Gravitational Fields

Newton's equations describe slow motion in a weak, static gravitational. Because the motion is slow, we neglect u^i compared to u^0 and simplify the geodesic equation (11.369) to

$$0 = \frac{du^r}{d\tau} + \Gamma^r_{00}\,(u^0)^2. \tag{11.382}$$

Because the gravitational field is static, we neglect the time derivatives $g_{k0,0}$ and $g_{0k,0}$ in the connection formula (11.302) and find for Γ^r_{00}

$$\Gamma^r_{00} = \tfrac{1}{2}\,g^{rk}\,(g_{0k,0} + g_{0k,0} - g_{00,k}) = -\tfrac{1}{2}\,g^{rk}\,g_{00,k} \tag{11.383}$$

with $\Gamma^0_{00} = 0$. Because the field is weak, the metric can differ from η_{ij} by only a tiny tensor $g_{ij} = \eta_{ij} + h_{ij}$ so that to first order in $|h_{ij}| \ll 1$ we have $\Gamma^r_{00} = -\frac{1}{2}\,h_{00,r}$ for $r = 1, 2, 3$. With these simplifications, the geodesic equation (11.369) reduces to

$$\frac{d^2x^r}{d\tau^2} = \tfrac{1}{2}\,(u^0)^2\,h_{00,r} \qquad \text{or} \qquad \frac{d^2x^r}{d\tau^2} = \frac{1}{2}\left(\frac{dx^0}{d\tau}\right)^2 h_{00,r}. \tag{11.384}$$

So for slow motion, the ordinary acceleration is described by Newton's law

$$\frac{d^2\boldsymbol{x}}{dt^2} = \frac{c^2}{2}\,\nabla h_{00}. \tag{11.385}$$

If ϕ is his potential, then for slow motion in weak static fields

$$g_{00} = -1 + h_{00} = -1 - 2\phi/c^2 \qquad \text{and so} \qquad h_{00} = -2\phi/c^2. \tag{11.386}$$

Thus, if the particle is at a distance r from a mass M, then $\phi = -GM/r$ and $h_{00} = -2\phi/c^2 = 2GM/rc^2$ and so

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\nabla\phi = \nabla\frac{GM}{r} = -GM\frac{\boldsymbol{r}}{r^3}. \tag{11.387}$$

How weak are the static gravitational fields we know about? The dimensionless ratio ϕ/c^2 is 10^{-39} on the surface of a proton, 10^{-9} on the Earth, 10^{-6} on the surface of the sun, and 10^{-4} on the surface of a white dwarf.

11.46 Gravitational Time Dilation

Suppose we have a system of coordinates x^i with a metric g_{ik} and a clock at rest in this system. Then the proper time $d\tau$ between ticks of the clock is

$$d\tau = (1/c)\sqrt{-g_{ij}\,dx^i\,dx^j} = \sqrt{-g_{00}}\,dt \tag{11.388}$$

where dt is the time between ticks in the x^i coordinates, which is the laboratory frame in the gravitational field g_{00}. By the principle of equivalence (section **??**), the proper time $d\tau$ between ticks is the same as the time between ticks when the same clock is at rest deep in empty space.

If the clock is in a weak static gravitational field due to a mass M at a distance r, then

$$-g_{00} = 1 + 2\phi/c^2 = 1 - 2GM/c^2 r \tag{11.389}$$

is a little less than unity, and the interval of proper time between ticks

$$d\tau = \sqrt{-g_{00}}\, dt = \sqrt{1 - 2GM/c^2 r}\, dt \tag{11.390}$$

is slightly less than the interval dt between ticks in the coordinate system of an observer at x in the rest frame of the clock and the mass, and in its gravitational field. Since $dt > d\tau$, the laboratory time dt between ticks is greater than the proper or intrinsic time $d\tau$ between ticks of the clock unaffected by any gravitational field. Clocks near big masses run slow.

Now suppose we have two identical clocks at different heights above sea level. The time T_ℓ for the lower clock to make N ticks will be longer than the time T_u for the upper clock to make N ticks. The ratio of the clock times will be

$$\frac{T_\ell}{T_u} = \frac{\sqrt{1 - 2GM/c^2(r+h)}}{\sqrt{1 - 2GM/c^2 r}} \approx 1 + \frac{gh}{c^2}. \tag{11.391}$$

Now imagine that a photon going down passes the upper clock which measures its frequency as ν_u and then passes the lower clock which measures its frequency as ν_ℓ. The slower clock will measure a higher frequency. The ratio of the two frequencies will be the same as the ratio of the clock times

$$\frac{\nu_\ell}{\nu_u} = 1 + \frac{gh}{c^2}. \tag{11.392}$$

As measured by the lower, slower clock, the photon is blue shifted.

Example 11.31 (Pound, Rebka, and Mössbauer)**.** Pound and Rebka in 1960 used the Mössbauer effect to measure the blue shift of light falling down a 22.6 m shaft. They found

$$\frac{\nu_\ell - \nu_u}{\nu} = \frac{gh}{c^2} = 2.46 \times 10^{-15} \tag{11.393}$$

(Robert Pound 1919–2010, Glen Rebka 1931–, Rudolf Mössbauer 1929–2011). □

Example 11.32 (Redshift of the Sun)**.** A photon emitted with frequency ν_0 at a distance r from a mass M would be observed at spatial infinity to have frequency ν

$$\nu = \nu_0 \sqrt{-g_{00}} = \nu_0 \sqrt{1 - 2MG/c^2 r} \tag{11.394}$$

for a redshift of $\Delta\nu = \nu_0 - \nu$. Since the Sun's dimensionless potential $\phi_\odot/c^2$ is $-MG/c^2 r = -2.12 \times 10^{-6}$ at its surface, sunlight is shifted to the red by 2 parts per million. □

11.47 Curvature

The **curvature tensor** or **Riemann tensor** is

$$R^i_{mnk} = \Gamma^i_{mn,k} - \Gamma^i_{mk,n} + \Gamma^i_{kj}\,\Gamma^j_{nm} - \Gamma^i_{nj}\,\Gamma^j_{km} \tag{11.395}$$

which we may write as the commutator

$$\begin{aligned} R^i_{mnk} &= (R_{nk})^i{}_m = [\partial_k + \Gamma_k, \partial_n + \Gamma_n]^i{}_m \\ &= (\Gamma_{n,k} - \Gamma_{k,n} + \Gamma_k\,\Gamma_n - \Gamma_n\,\Gamma_k)^i{}_m \end{aligned} \tag{11.396}$$

in which the Γ's are treated as matrices

$$\Gamma_k = \begin{pmatrix} \Gamma^0_{k\,0} & \Gamma^0_{k\,1} & \Gamma^0_{k\,2} & \Gamma^0_{k\,3} \\ \Gamma^1_{k\,0} & \Gamma^1_{k\,1} & \Gamma^1_{k\,2} & \Gamma^1_{k\,3} \\ \Gamma^2_{k\,0} & \Gamma^2_{k\,1} & \Gamma^2_{k\,2} & \Gamma^2_{k\,3} \\ \Gamma^3_{k\,0} & \Gamma^3_{k\,1} & \Gamma^3_{k\,2} & \Gamma^3_{k\,3} \end{pmatrix} \tag{11.397}$$

with $(\Gamma_k\,\Gamma_n)^i{}_m = \Gamma^i_{kj}\,\Gamma^j_{nm}$ and so forth. Just as there are two conventions for the Faraday tensor F_{ik} which differ by a minus sign, so too there are two conventions for the curvature tensor R^i_{mnk}. Weinberg (Weinberg, 1972) uses the definition (11.395); Carroll (Carroll, 2003), Padmanabhan (Padmanabhan, 2010), Schutz (Schutz, 2009), and Zee (Zee, 2013) use an extra minus sign.

The Ricci tensor is a contraction of the curvature tensor

$$R_{mk} = R^n_{mnk} \tag{11.398}$$

and the scalar curvature is a further contraction

$$R = g^{mk}\,R_{mk}. \tag{11.399}$$

Example 11.33 (Curvature of a Sphere)**.** While in four-dimensional space-time indices run from 0 to 3, on the sphere they are just θ and ϕ. There are only eight possible affine connections, and because of the symmetry (11.245) in their lower indices $\Gamma^i_{\theta\phi} = \Gamma^i_{\phi\theta}$, only six are independent.

The point $\boldsymbol{p}$ on a sphere of radius r has cartesian coordinates

$$\boldsymbol{p} = r\ (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) \tag{11.400}$$

so the two 3-vectors are

$$\begin{aligned} \boldsymbol{e}_\theta &= \frac{\partial \boldsymbol{p}}{\partial\theta} = r\ (\cos\theta\ \cos\phi,\ \cos\theta\ \sin\phi,\ -\sin\theta) = r\,\hat{\boldsymbol{\theta}} \\ \boldsymbol{e}_\phi &= \frac{\partial \boldsymbol{p}}{\partial\phi} = r\sin\theta\ (-\sin\phi,\ \cos\phi,\ 0) = r\sin\theta\,\hat{\boldsymbol{\phi}}. \end{aligned} \tag{11.401}$$

The embedding metric is the 3×3 identity matrix, so the metric g_{ij} made of the dot products $g_{ij} = \boldsymbol{e}_i\cdot\boldsymbol{e}_j$ is

$$(g_{ij}) = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix}. \tag{11.402}$$

Differentiating the vectors $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$, we find

$$e_{\theta,\theta} = -\,r\,(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) = -r\,\hat{\boldsymbol{r}} \tag{11.403}$$

$$e_{\theta,\phi} = r\cos\theta\,(-\sin\phi,\ \cos\phi,\ 0) = r\cos\theta\,\hat{\boldsymbol{\phi}} \tag{11.404}$$

$$e_{\phi,\theta} = e_{\theta,\phi} \tag{11.405}$$

$$e_{\phi,\phi} = -\,r\sin\theta\,(\cos\phi,\ \sin\phi,\ 0)\,. \tag{11.406}$$

The metric with upper indices (g^{ij}) is the inverse of the metric (g_{ij})

$$(g^{ij}) = \begin{pmatrix} r^{-2} & 0 \\ 0 & r^{-2}\sin^{-2}\theta \end{pmatrix} \tag{11.407}$$

so the dual vectors $\boldsymbol{e}^i$ are

$$\begin{aligned} \boldsymbol{e}^{\,\theta} &= r^{-1}\ (\cos\theta\ \cos\phi,\ \cos\theta\ \sin\phi,\ -\sin\theta) = r^{-1}\hat{\boldsymbol{\theta}} \\ \boldsymbol{e}^{\,\phi} &= = \frac{1}{r\sin\theta}\,(-\sin\phi,\ \cos\phi,\ 0) = \frac{1}{r\sin\theta}\,\hat{\boldsymbol{\phi}}. \end{aligned} \tag{11.408}$$

The affine connections are given by (11.239) as

$$\Gamma^i_{jk} = \Gamma^i_{kj} = \boldsymbol{e}^{\,i}\cdot\boldsymbol{e}_{j,k}. \tag{11.409}$$

Since both $\boldsymbol{e}^\theta$ and $\boldsymbol{e}^{\,\phi}$ are perpendicular to $\hat{r}$, the affine connections $\Gamma^\theta_{\theta\theta}$ and $\Gamma^\phi_{\theta\theta}$ both vanish. Also, $e_{\phi,\phi}$ is orthogonal to $\hat{\phi}$, so $\Gamma^\phi_{\phi\phi} = 0$ as well. Similarly, $e_{\theta,\phi}$ is perpendicular to $\hat{\theta}$, so $\Gamma^\theta_{\theta\phi} = \Gamma^\theta_{\phi\theta}$ also vanishes.

The two nonzero affine connections are

$$\Gamma^{\phi}_{\theta\phi} = \boldsymbol{e}^{\,\phi} \cdot e_{\theta,\phi} = r^{-1} \sin^{-1}\theta\, \hat{\boldsymbol{\phi}} \cdot r\cos\theta\, \hat{\boldsymbol{\phi}} = \cot\theta \tag{11.410}$$

and

$$\begin{aligned}\Gamma^{\theta}_{\phi\phi} &= \boldsymbol{e}^{\,\theta} \cdot e_{\phi,\phi} \\ &= -\sin\theta\,(\cos\theta\,\cos\phi,\, \cos\theta\,\sin\phi,\, -\sin\theta) \cdot (\cos\phi,\, \sin\phi,\, 0) \\ &= -\sin\theta\cos\theta. \end{aligned}\tag{11.411}$$

In terms of the two non-zero affine connections $\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$ and $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$, the two Christoffel matrices (11.397) are

$$\Gamma_{\theta} = \begin{pmatrix} 0 & 0 \\ 0 & \Gamma^{\phi}_{\theta\phi} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \cot\theta \end{pmatrix} \tag{11.412}$$

and

$$\Gamma_{\phi} = \begin{pmatrix} 0 & \Gamma^{\theta}_{\phi\phi} \\ \Gamma^{\phi}_{\phi\theta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sin\theta\cos\theta \\ \cot\theta & 0 \end{pmatrix}. \tag{11.413}$$

Their commutator is

$$[\Gamma_{\theta}, \Gamma_{\phi}] = \begin{pmatrix} 0 & \cos^2\theta \\ \cot^2\theta & 0 \end{pmatrix} = -[\Gamma_{\phi}, \Gamma_{\theta}] \tag{11.414}$$

and both $[\Gamma_{\theta}, \Gamma_{\theta}]$ and $[\Gamma_{\phi}, \Gamma_{\phi}]$ vanish.

So the commutator formula (11.396) gives for Riemann's curvature tensor

$$\begin{aligned} R^{\theta}_{\theta\theta\theta} &= [\partial_{\theta} + \Gamma_{\theta}, \partial_{\theta} + \Gamma_{\theta}]^{\theta}{}_{\theta} = 0 \\ R^{\phi}_{\theta\phi\theta} &= [\partial_{\theta} + \Gamma_{\theta}, \partial_{\phi} + \Gamma_{\phi}]^{\phi}{}_{\theta} = (\Gamma_{\phi,\theta})^{\phi}{}_{\theta} + [\Gamma_{\theta}, \Gamma_{\phi}]^{\phi}{}_{\theta} \\ &= (\cot\theta)_{,\theta} + \cot^2\theta = -1 \\ R^{\theta}_{\phi\theta\phi} &= [\partial_{\phi} + \Gamma_{\phi}, \partial_{\theta} + \Gamma_{\theta}]^{\theta}{}_{\phi} = -(\Gamma_{\phi,\theta})^{\theta}{}_{\phi} + [\Gamma_{\ \phi}, \Gamma_{\theta}]^{\theta}{}_{\phi} \\ &= \cos^2\theta - \sin^2\theta - \cos^2\theta = -\sin^2\theta \\ R^{\phi}_{\phi\phi\phi} &= [\partial_{\phi} + \Gamma_{\phi}, \partial_{\phi} + \Gamma_{\phi}]^{\phi}{}_{\phi} = 0. \end{aligned}\tag{11.415}$$

The Ricci tensor (11.398) is the contraction $R_{mk} = R^{n}_{mnk}$, and so

$$\begin{aligned} R_{\theta\theta} &= R^{\theta}_{\theta\theta\theta} + R^{\phi}_{\theta\phi\theta} = -1 \\ R_{\phi\phi} &= R^{\theta}_{\phi\theta\phi} + R^{\phi}_{\phi\phi\phi} = -\sin^2\theta. \end{aligned}\tag{11.416}$$

The curvature scalar (11.399) is the contraction $R = g^{km} R_{mk}$, and so since $g^{\theta\theta} = r^{-2}$ and $g^{\phi\phi} = r^{-2}\sin^{-2}\theta$, it is

$$R = g^{\theta\theta}\, R_{\theta\theta} + g^{\phi\phi}\, R_{\phi\phi} = -r^{-2} - \sin^2\theta\; r^{-2}\sin^{-2}\theta = -\frac{2}{r^2} \tag{11.417}$$

for a 2-sphere of radius r.

Gauss invented a formula for the curvature K of a surface; for all two-dimensional surfaces, his $K = -R/2$. □

Example 11.34 (Curvature of a cylindrical hyperboloid)**.** The points of a cylindrical hyperboloid in 3-space satisfy $z^2 = x^2 + y^2 - r^2$ and may be parameterized as $p = r(\cosh\theta\cos\phi, \cosh\theta\sin\phi, \sinh\theta)$. The (orthogonal) coordinate basis vectors are

$$\begin{aligned} \boldsymbol{e}_\theta &= p_{,\theta} = r(\sinh\theta\cos\phi, \sinh\theta\sin\phi, \cosh\theta) \\ \boldsymbol{e}_\phi &= p_{,\phi} = r(-\cosh\theta\sin\phi, \cosh\theta\cos\phi, 0). \end{aligned} \tag{11.418}$$

The embedding metric is the 3×3 identity matrix, so the metric $g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$ is

$$(g_{ij}) = r^2 \begin{pmatrix} \cosh^2\theta + \sinh^2\theta & 0 \\ 0 & \cosh^2\theta \end{pmatrix}. \tag{11.419}$$

Tristan Hubsch's Mathematica package GREAT.m can compute the scalar curvature for us. One enters

```
x = {theta, phi};
(met = {{r^2*(Cosh[theta]^2 + Sinh[theta]^2), 0},
             {0, r^2*Cosh[theta]^2}}) // MatrixForm;
SCurvature[met, x]
```

and gets $R = 2/(r\cosh(2\theta))^2$ after adjusting for Hubsch's sign convention. Python has sympy and gravipy. □

11.48 Einstein's equations

If we make an action that is a scalar, invariant under general coordinate transformations, and then apply to it the principle of stationary action, we will get tensor field equations that are invariant under general coordinate transformations. If the metric of spacetime is among the fields of the action, then the resulting theory will be a possible theory of gravity. If we make the action as simple as possible, it will be Einstein's theory.

To make the action of the gravitational field, we need a scalar. Apart from the volume 4-form $*1 = \sqrt{|\det g_{ik}|}\, d^4x = \sqrt{g}\, d^4x = \sqrt{g}\, c\, dt\, d^3x$, the simplest scalar we can form from the metric tensor and its first and second derivatives is the scalar curvature R which gives us the **Einstein-Hilbert action**

$$S_{EH} = -\frac{c^3}{16\pi G}\int R\sqrt{g}\, d^4x = -\frac{c^3}{16\pi G}\int g^{ik} R_{ik}\sqrt{g}\, d^4x \tag{11.420}$$

in which $G = 6.7087 \times 10^{-39}\,\hbar c\,(\mathrm{GeV}/c^2)^{-2} = 6.6742 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ is Newton's constant.

If $\delta g^{ik}(x)$ is a tiny local change in the inverse metric, then the rule $\delta \det A = \det A \,\mathrm{Tr}(A^{-1}\delta A)$ (1.41, valid for any nonsingular, nondefective matrix A) together with the identity $0 = \delta(g^{ik} g_{k\ell}) = \delta g^{ik}\, g_{k\ell} + g^{ik}\, \delta g_{k\ell}$ imply that

$$\delta\sqrt{g} = -\frac{\delta \det g}{2\sqrt{g}} = -\frac{(\det g) g^{ik}\, \delta g_{ik}}{2\sqrt{g}} = \frac{1}{2}\sqrt{g}\; g_{ik}\, \delta g^{ik}. \tag{11.421}$$

So the first-order change in the action density is

$$\begin{aligned} \delta\left(g^{ik}\, R_{ik}\, \sqrt{g}\right) &= R_{ik}\, \sqrt{g}\; \delta g^{ik} + g^{ik}\, R_{ik}\; \delta\sqrt{g} + g^{ik}\, \sqrt{g}\; \delta R_{ik} \\ &= \left(R_{ik} - \frac{1}{2}\,\sqrt{g}\; g_{ik}\right) \delta g^{ik} + g^{ik}\, \sqrt{g}\; \delta R_{ik}. \end{aligned} \tag{11.422}$$

The product $g^{ik}\delta R_{ik}$ is a scalar, so we can evaluate it in any coordinate system. In a local inertial frame, where $\Gamma^a_{bc} = 0$ and g_{de} is constant, this invariant variation of the Ricci tensor (11.398) is

$$\begin{aligned} g^{ik}\delta R_{ik} &= g^{ik}\, \delta\left(\Gamma^n_{in,k} - \Gamma^n_{ik,n}\right) = g^{ik}\left(\partial_k\, \delta\Gamma^n_{in} - \partial_n\, \delta\Gamma^n_{ik}\right) \\ &= g^{ik}\, \partial_k\, \delta\Gamma^n_{in} - g^{in}\, \partial_k\, \delta\Gamma^k_{in} = \partial_k\left(g^{ik}\, \delta\Gamma^n_{in} - g^{in}\, \delta\Gamma^k_{in}\right). \end{aligned} \tag{11.423}$$

By (11.247), the variations $\delta\Gamma^n_{in}$ and $\delta\Gamma^k_{in}$ are tensors although the connections themselves aren't. Thus, we can evaluate this invariant variation of the Ricci tensor in any coordinate system by replacing the derivatives with covariant ones getting

$$g^{ik}\delta R_{ik} = \left(g^{ik}\, \delta\Gamma^n_{in} - g^{in}\, \delta\Gamma^k_{in}\right)_{;k} \tag{11.424}$$

which we recognize as the covariant divergence (11.316) of a contravariant vector. The last term in the first-order change (11.422) in the action density is therefore a surface term whose variation vanishes for tiny local changes δg^{ik} of the metric

$$\sqrt{g}\, g^{ik}\delta R_{ik} = \left[\sqrt{g}\left(g^{ik}\, \delta\Gamma^n_{in} - g^{in}\, \delta\Gamma^k_{in}\right)\right]_{,k}. \tag{11.425}$$

Hence the variation of S_{EH} is simply

$$\delta S_{EH} = -\frac{c^3}{16\pi G}\int\left(R_{ik} - \frac{1}{2}\, g_{ik} R\right)\sqrt{g}\; \delta g^{ik}\, d^4x. \tag{11.426}$$

The principle of least action $\delta S_{EH} = 0$ now gives us **Einstein's equations**

for empty space:

$$R_{ik} - \frac{1}{2}\, g_{ik}\, R = 0. \tag{11.427}$$

The tensor $G_{ik} = R_{ik} - \frac{1}{2} g_{ik}\, R$ is Einstein's tensor.

Taking the trace of Einstein's equations (11.427), we find that the scalar curvature R and the Ricci tensor R_{ik} are zero in empty space:

$$R = 0 \quad \text{and} \quad R_{ik} = 0. \tag{11.428}$$

The **energy-momentum tensor** T_{ik} is the source of the gravitational field. It is defined so that the change in the action of the matter fields due to a tiny local change $\delta g^{ik}(x)$ in the metric is

$$\delta S_m = -\frac{1}{2c} \int T_{ik} \sqrt{g}\ \delta g^{ik}\, d^4x = \frac{1}{2c} \int T^{ik} \sqrt{g}\ \delta g_{ik}\, d^4x \tag{11.429}$$

in which the identity $\delta g^{ik} = -\, g^{ij} g^{\ell k} \delta g_{j\ell}$ explains the sign change. Now the principle of least action $\delta S = \delta S_{EH} + \delta S_m = 0$ yields Einstein's equations in the presence of matter and energy

$$R_{ik} - \frac{1}{2}\, g_{ik}\, R = -\, \frac{8\pi G}{c^4} T_{ik}. \tag{11.430}$$

Taking the trace of both sides, we get

$$R = \frac{8\pi G}{c^4}\, T \quad \text{and} \quad R_{ik} = -\, \frac{8\pi G}{c^4} \left(T_{ik} - \frac{T}{2} g_{ik} \right). \tag{11.431}$$

11.49 Energy-momentum tensor

The action S_m of the matter fields is a scalar that is invariant under general coordinate transformations. In particular, a tiny local general coordinate transformation $x'^a = x^a + \epsilon^a(x)$ leaves S_m invariant

$$0 = \delta S_m = \int \delta \left(L(\phi_i(x)) \sqrt{g(x)} \right)\, d^4x. \tag{11.432}$$

The vanishing change $\delta S_m = \delta S_{m\phi} + \delta S_{mg}$ has a part $\delta S_{m\phi}$ due to the changes in the fields $\delta\phi_i(x)$ and a part δS_{mg} due to the change in the metric δg^{ik}. The principle of stationary action tells us that the change $\delta S_{m\phi}$ is zero as long as the fields obey the classical equations of motion. The definition (11.429) of the energy-momentum tensor now tells us that

$$0 = \delta S_m = \delta S_{mg} = \frac{1}{2c} \int T^{ik} \sqrt{g}\ \delta g_{ik}\, d^4x. \tag{11.433}$$

We take the change in S_m to be

$$\begin{aligned}\delta S_m &= \int L'(\phi_i'(x'))\,\sqrt{g'(x')}\,d^4x' - \int L(\phi_i(x))\,\sqrt{g(x)}\,d^4x \\ &= \int L'(\phi_i'(x))\,\sqrt{g'(x)}\,d^4x - \int L(\phi_i(x))\,\sqrt{g(x)}\,d^4x.\end{aligned} \tag{11.434}$$

So using the identity $\delta g^{ik}\,g_{k\ell} = -g^{ik}\,\delta g_{k\ell}$, the definition (11.263) of the covariant derivative of a covariant vector, and the formula (11.302) for the connection in terms of the metric, we find that to lowest order in $\epsilon^a(x)$, the change in the metric is

$$\begin{aligned}\delta g_{ik} &= g_{ik}'(x) - g_{ik}(x) = g_{ik}'(x') - g_{ik}(x) - (g_{ik}'(x') - g_{ik}'(x)) \\ &= (\delta_i^a - \epsilon_{,i}^a)(\delta_k^b - \epsilon_{,k}^b)g_{ab} - \epsilon^c g_{ik,c} \\ &= -\,g_{ib}\,\epsilon_{,k}^b - g_{ak}\,\epsilon_{,i}^a - \epsilon^c\,g_{ik,c} \\ &= -\,g_{ib}(g^{bc}\,\epsilon_c)_{,k} - g_{ak}(g^{ac}\,\epsilon_c)_{,i} - \epsilon^c\,g_{ik,c} \\ &= -\,\epsilon_{i,k} - \epsilon_{k,i} - \epsilon_c\,g_{ib}\,g_{,k}^{bc} - \epsilon_c\,g_{ak}\,g_{,i}^{ac} - \epsilon^c\,g_{ik,c} \\ &= -\,\epsilon_{i,k} - \epsilon_{k,i} + \epsilon_c\,g^{bc}\,g_{ib,k} + \epsilon_c\,g^{ac}\,g_{ak,i} - \epsilon^c\,g_{ik,c} \\ &= -\,\epsilon_{i,k} - \epsilon_{k,i} + \epsilon_c\,g^{ac}\,(g_{ia,k} + g_{ak,i} - g_{ik,a}) \\ &= -\,\epsilon_{i,k} - \epsilon_{k,i} + \epsilon_c\,\Gamma_{ik}^c + \epsilon_c\,\Gamma_{ki}^c = -\,\epsilon_{i;k} - \epsilon_{k;i}.\end{aligned} \tag{11.435}$$

Combining this result (11.435) with the vanishing (11.433) of the change δS_{mg}, we have

$$0 = \int T^{ik}\,\sqrt{g}\,(\epsilon_{i;k} + \epsilon_{k;i})\,d^4x. \tag{11.436}$$

Since the energy-momentum tensor is symmetric, we may combine the two terms, integrate by parts, divide by $\sqrt{g}$, and so find that the covariant divergence of the energy-momentum tensor is zero

$$0 = T_{;k}^{ik} = T_{,k}^{ik} + \Gamma_{ak}^k\,T^{ia} + \Gamma_{ak}^i\,T^{ak} = \frac{1}{\sqrt{g}}(\sqrt{g}T^{ik})_{,k} + \Gamma_{ak}^i\,T^{ak} \tag{11.437}$$

when the fields obey their equations of motion. In a given inertial frame, only the total energy, momentum, and angular momentum of both the matter and the gravitational field are conserved.

11.50 Perfect fluids

In many astrophysical and most cosmological models, the energy-momentum tensor is assumed to be that of a **perfect fluid**, which is isotropic in its rest frame, does not conduct heat, and has zero viscosity. For a perfect fluid of

pressure p and density ρ with 4-velocity u^i (defined by (11.65)), the energy-momentum or **energy-momentum** tensor T_{ij} is

$$T_{ij} = p\, g_{ij} + (\frac{p}{c^2} + \rho)\, u_i\, u_j \tag{11.438}$$

in which g_{ij} is the spacetime metric.

An important special case is the energy-momentum tensor due to a nonzero value of the energy density of the vacuum. In this case $p = -c^2\rho$ and the energy-momentum tensor is

$$T_{ij} = p\, g_{ij} = -c^2 \rho\, g_{ij} \tag{11.439}$$

in which $T_{00} = c^2\rho$ is the (presumably constant) value of the energy density of the ground state of the theory. This energy density ρ is a plausible candidate for the **dark-energy** density. It is equivalent to a **cosmological constant** $\Lambda = 8\pi G\rho$.

On small scales, such as that of our solar system, one may neglect matter and dark energy. So in empty space and on small scales, the energy-momentum tensor vanishes $T_{ij} = 0$ along with its trace and the scalar curvature $T = 0 = R$, and Einstein's equations (11.431) are

$$R_{ij} = 0. \tag{11.440}$$

11.51 Standard Form

Tensor equations are independent of the choice of coordinates, so it's wise to choose coordinates that simplify one's work. For a **static** and **isotropic** gravitational field, this choice is the **standard form** (Weinberg, 1972, ch. 8)

$$ds^2 = -B(r)\, c^2\, dt^2 + A(r)\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{11.441}$$

in which $B(r)$ and $A(r)$ are functions that one may find by solving the field equations (11.430). Since $ds^2 = -c^2\, d\tau^2 = g_{ij}\, dx^i dx^j$, the nonzero components of the metric tensor are $g_{rr} = A(r)$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2 \sin^2\theta$, and $g_{00} = -B(r)$, and those of its inverse are $g^{rr} = A^{-1}(r)$, $g^{\theta\theta} = r^{-2}$, $g^{\phi\phi} = r^{-2} \sin^{-2}\theta$, and $g^{00} = -B^{-1}(r)$. By differentiating the metric tensor and using (11.302), one gets the components of the connection $\Gamma^i_{k\ell}$, such as $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$, and the components (11.398) of the Ricci tensor R_{ij},

such as (Weinberg, 1972, ch. 8)

$$R_{rr} = \frac{B''(r)}{2B(r)} - \frac{1}{4}\left(\frac{B'(r)}{B(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{A'(r)}{A(r)}\right) \tag{11.442}$$

in which the primes mean d/dr.

11.52 Schwarzschild's Solution

If one ignores the small dark-energy parameter Λ, one may solve Einstein's field equations (11.440) in empty space

$$R_{ij} = 0 \tag{11.443}$$

outside a mass M for the standard form of the Ricci tensor. One finds (Weinberg, 1972) that $A(r)\,B(r) = 1$ and that $r\,B(r) = r$ plus a constant, and one determines the constant by invoking the Newtonian limit $g_{00} = -B \to -1 + 2MG/c^2 r$ as $r \to \infty$. In 1916, Schwarzschild found the solution

$$ds^2 = -\left(1 - \frac{2MG}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2MG}{c^2 r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{11.444}$$

which one can use to analyze orbits around a star. The singularity in

$$g_{rr} = \left(1 - \frac{2MG}{c^2 r}\right)^{-1} \tag{11.445}$$

at the Schwarzschild radius $r_{\rm s} = 2MG/c^2$ is an artifact of the coordinates; the scalar curvature R and other invariant curvatures are not singular at the Schwarzschild radius. Moreover, for the Sun, the Schwarzschild radius $r_{\rm s} = 2M_\odot G/c^2$ is only 2.95 km, far less than the radius of the Sun, which is 6.955×10^5 km. So the surface at $r_{\rm s} = 2M_\odot G/c^2$ is far from the empty space in which Schwarzschild's metric applies (Karl Schwarzschild, 1873–1916).

11.53 Black Holes

Suppose an uncharged, spherically symmetric star of mass M has collapsed within a sphere of radius r_b less than its Schwarzschild radius $r_{\rm s} = 2MG/c^2$. Then for $r > r_b$, the Schwarzschild metric (11.444) is correct. By Eq.(11.388), the apparent time dt of a process of proper time $d\tau$ at $r \geq 2MG/c^2$ is

$$dt = d\tau/\sqrt{-g_{00}} = d\tau/\sqrt{1 - \frac{2MG}{c^2 r}}. \tag{11.446}$$

The apparent time dt becomes infinite as $r \to 2MG/c^2$. To outside observers, the star seems frozen in time.

Due to the gravitational redshift (11.394), light of frequency ν_p emitted at $r \geq 2MG/c^2$ will have frequency ν

$$\nu = \nu_p \sqrt{-g_{00}} = \nu_p \sqrt{1 - \frac{2MG}{c^2 r}} \tag{11.447}$$

when observed at great distances. Light coming from the surface at $r_{\rm s} = 2MG/c^2$ is redshifted to zero frequency $\nu = 0$. The star is black. It is a black hole with a surface or horizon at its Schwarzschild radius $r_{\rm s} = 2MG/c^2$, although there is no singularity there. If the radius of the Sun were less than its Schwarzschild radius of 2.95 km, then the Sun would be a black hole. The radius of the Sun is 6.955×10^5 km.

Black holes are not really black. Stephen Hawking (1942–) has shown that the intense gravitational field of a black hole of mass M radiates at temperature

$$T = \frac{\hbar\, c^3}{8\pi\, k\, G\, M} \tag{11.448}$$

in which $k = 8.617343 \times 10^{-5}\,\mathrm{eV\ K^{-1}}$ is Boltzmann's constant, and $\hbar$ is Planck's constant $h = 6.6260693 \times 10^{-34}\,\mathrm{J\,s}$ divided by 2π, $\hbar = h/(2\pi)$.

The black hole is entirely converted into radiation after a time

$$t = \frac{5120\,\pi\, G^2}{\hbar\, c^4}\, M^3 \tag{11.449}$$

proportional to the cube of its mass.

11.54 Cosmology

Astrophysical observations tell us that on the largest observable scales, space is **flat** or very nearly flat; that the visible universe contains at least 10^{90} particles; and that the cosmic microwave background radiation is isotropic to one part in 10^5 apart from a Hubble-Doppler shift due the motion of the Earth at 371 km/s towards the constellation Leo. These and other observations suggest that potential energy expanded our universe by $\exp(60) = 10^{26}$ during a brief period of **inflation** that could have been as short as 10^{-35} s. The potential energy that powered inflation almost immediately became the radiation of the **Big Bang**. During and after inflation, negative gravitational potential energy kept the total energy constant.

Within the first three minutes, some of that radiation became hydrogen,

helium, neutrinos, and **dark matter** Weinberg (1988, 2010). But the **era of radiation**, during which most of the energy of the visible universe was radiation, lasted for 50,952 years.

Because the momentum of a particle but not its mass falls with the expansion of the universe, the era of radiation gradually gave way to an **era of matter**. The universe changed from radiation dominated to matter dominated as its temperature kT dropped below 0.81 eV. After 380,000 years,

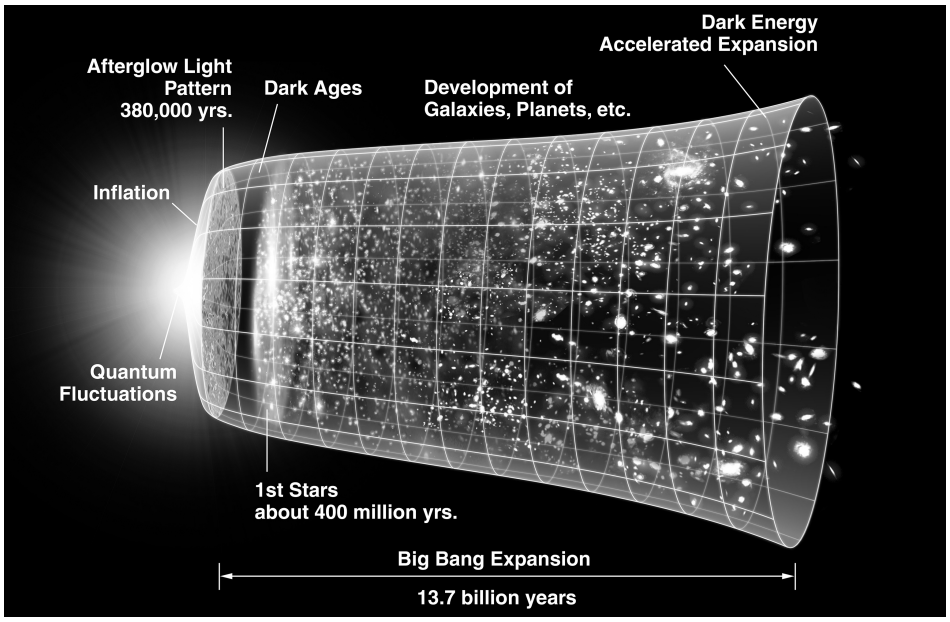

Figure 11.1 NASA/WMAP's timeline of the known universe

the universe had cooled to 3000 K or $kT = 0.26$ eV, and less than 1% of the atoms were ionized. Photons no longer scattered off a plasma of electrons and ions. The universe became **transparent**. The photons that last scattered just before this **initial transparency** became the **cosmic microwave background radiation** or **CMBR** that now surrounds us, redshifted to $T_0 = 2.7255 \pm 0.0006\,\mathrm{K}$.

Between 10 and 17 million years after the Big Bang, the temperature of the known universe fell from 373 to 273 K. If by then the supernovas of very early, very heavy stars had produced carbon, nitrogen, and oxygen, biochemistry may have started during this period of 7 million years.

The era of matter lasted for 10.19 billion years.

The **era of dark energy** began about 3.6 billon years ago as matter ceased to be the dominant form of energy. Since then, most of the energy

of the visible universe has been of an unknown kind called dark energy. Dark energy may be the energy of the vacuum; classically, it seems to be equivalent to a **cosmological constant**. Dark energy has been accelerating the expansion of the universe for the past 3.6 billion years and may continue to do so forever.

It is now 13.799 ± 0.021 billion years after the Big Bang or the time of infinite redshift and zero scale factor. The present value Ade *et al.* (2015) of the Hubble frequency H is the **Hubble constant** $H_0 = 67.74 \pm 0.46$ $\mathrm{km\,(s\,Mpc)^{-1}} = 2.1953 \times 10^{-18}\ \mathrm{s}^{-1}$, one parsec being $3.085\,677\,581\,49 \times 10^{16}$ m or about 3.262 light-years. The present **critical mass density** $\rho_{c0} = 3H_0^2/8\pi G = 8.6197 \times 10^{-27}\ \mathrm{kg\,m^{-3}}$ is the present density of a universe that is isotropic, homogeneous, and flat. The ratio $\Omega_0 = \rho_0/\rho_{c0}$ of the present density ρ_0 of the universe to the present critical mass density ρ_{c0} is $\Omega_0 = 1.0000 \pm 0.0088$. The ratio $\Omega_{\Lambda 0} = \rho_{\Lambda 0}/\rho_{c0}$ of the present **dark-energy density** $\rho_{\Lambda 0}$ to the critical density is $\Omega_{\Lambda 0} = 0.6911 \pm 0.0062$. The ratio $\Omega_{m0} = \rho_{m0}/\rho_{c0}$ of the present total matter density ρ_{m0} to the present critical density ρ_{c0} is $\Omega_{m0} = 0.3089 \pm 0.0062$. The ratio $\Omega_{b0} = \rho_{b0}/\rho_{c0}$ of the present density of baryons ρ_{b0} to the present critical density ρ_{c0} is $\Omega_{b0} = 0.0486 \pm 0.0007$. Baryons account for 4.9% of the density of the universe and only 15.7 % of the matter density, the rest being **dark matter**, which interacts very little with light.

Einstein's equations (11.431) are second-order, non-linear partial differential equations for 10 unknown functions $g_{ij}(x)$ in terms of the energy-momentum tensor $T_{ij}(x)$ throughout the universe, which of course we don't know. The problem is not quite hopeless, however. The ability to choose arbitrary coordinates, the appeal to symmetry, and the choice of a reasonable form for T_{ij} all help.

Hubble showed us that the universe is expanding. The cosmic microwave background radiation looks the same in all spatial directions (apart from a Hubble-Doppler shift due to the motion of the Earth at 371 km/s toward the constellation Leo). Observations of clusters of galaxies reveal a universe that is homogeneous on suitably large scales of distance. So it is plausible that the universe is **homogeneous** and **isotropic** in space, but not in time. One may show (Carroll, 2003) that for a universe of such symmtery, the line element in **comoving coordinates** (in which the Hubble-Doppler shift is isotropic) is

$$ds^2 = -\,c^2 dt^2 + a^2 \left[\frac{dr^2}{1 - k\,r^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right] \tag{11.450}$$

in which the **dimensionless** function of time $a(t)$ is the **scale factor** and k is a constant whose dimension is inverse squared length.

Whitney's embedding theorem tells us that any smooth four-dimensional manifold can be embedded in a flat space of eight dimensions with a suitable **signature**. We need only four or five dimensions to embed the spacetime described by the line element (11.450). If the universe is closed, then the signature is $(-1, 1, 1, 1, 1)$, and our three-dimensional space is the **3-sphere** which is the surface of a four-dimensional sphere in four space dimensions. The points of the universe then are

$$\boldsymbol{p} = (ct,\ a \sin\chi \sin\theta \cos\phi,\ a \sin\chi \sin\theta \sin\phi,\ a \sin\chi \cos\theta,\ a \cos\chi) \quad (11.451)$$

in which $0 \le \chi \le \pi$, $0 \le \theta \le \pi$, and $0 \le \phi \le 2\pi$. If the universe is flat, then the embedding space is flat, four-dimensional Minkowski space with points

$$\boldsymbol{p} = (ct,\ ar \sin\theta \cos\phi,\ ar \sin\theta \sin\phi,\ ar \cos\theta) = (ct, ax, ay, az) \quad (11.452)$$

in which $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$. If the universe is open, then the embedding space is a flat five-dimensional space with signature $(-1, 1, 1, 1, -1)$, and our three-dimensional space is a hyperboloid in a flat Minkowski space of one time and three space dimensions. The points of the universe then are

$$\boldsymbol{p} = (ct,\ a \sinh\chi \sin\theta \cos\phi,\ a \sinh\chi \sin\theta \sin\phi,\ a \sinh\chi \cos\theta,\ a \cosh\chi) \quad (11.453)$$

in which $0 \le \chi \le \infty$, $0 \le \theta \le \pi$, and $0 \le \phi \le 2\pi$.

In all three cases, the corresponding **Robertson-Walker metric** is

$$g_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2/(1-kr^2) & 0 & 0 \\ 0 & 0 & a^2\, r^2 & 0 \\ 0 & 0 & 0 & a^2\, r^2\, \sin^2\theta \end{pmatrix} \quad (11.454)$$

in which the coordinates (ct, r, θ, ϕ) are numbered $(0, 1, 2, 3)$, and k is a constant with units of $1/\text{length}^2$. One always may choose coordinates (exercise 11.31) such that k numerically is 0 or ± 1. This constant determines whether the spatial universe is **open** $k < 0$, **flat** $k = 0$, or **closed** $k > 0$. The **scale factor** a, which is a function of time $a(t)$, tells us how space expands and contracts. These coordinates are called **comoving** because a point at rest (fixed r, θ, ϕ) sees the same Hubble-Doppler shift in all directions.

The metric (11.454) is diagonal; its inverse g^{ij} also is diagonal; and so we may use our formula (11.302) to compute the affine connections $\Gamma^k_{i\ell}$, such as

$$\Gamma^0_{\ell\ell} = \tfrac{1}{2} g^{0k} \left(g_{\ell k,\ell} + g_{\ell k,\ell} - g_{\ell\ell,k} \right) = \tfrac{1}{2} g^{00} \left(g_{\ell 0,\ell} + g_{\ell 0,\ell} - g_{\ell\ell,0} \right) = \tfrac{1}{2} g_{\ell\ell,0} \tag{11.455}$$

so that

$$\Gamma^0_{11} = \frac{a\dot{a}/c}{1 - kr^2} \qquad \Gamma^0_{22} = a\dot{a}\, r^2/c \quad \text{and} \quad \Gamma^0_{33} = a\dot{a}\, r^2 \sin^2\theta/c. \tag{11.456}$$

in which a dot means a time derivative. The other Γ^0_{ij}'s vanish. Similarly, for *fixed* $\ell = 1$, 2, or 3

$$\begin{aligned} \Gamma^\ell_{0\ell} &= \tfrac{1}{2} g^{\ell k} \left(g_{0k,\ell} + g_{\ell k,0} - g_{0\ell,k} \right) \\ &= \tfrac{1}{2} g^{\ell\ell} \left(g_{0\ell,\ell} + g_{\ell\ell,0} - g_{0\ell,\ell} \right) \\ &= \tfrac{1}{2} g^{\ell\ell}\, g_{\ell\ell,0} = \frac{\dot{a}}{ca} = \Gamma^\ell_{\ell 0} \quad \text{no sum over } \ell. \end{aligned} \tag{11.457}$$

The other nonzero Γ's are

$$\Gamma^1_{22} = -r\left(1 - kr^2\right) \qquad \Gamma^1_{33} = -r\left(1 - kr^2\right)\sin^2\theta \tag{11.458}$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r} = \Gamma^2_{21} = \Gamma^3_{31} \tag{11.459}$$

$$\Gamma^2_{33} = -\sin\theta\,\cos\theta \qquad \Gamma^3_{23} = \cot\,\theta = \Gamma^3_{32}. \tag{11.460}$$

Our formulas (11.398 & 11.396) for the Ricci and curvature tensors give

$$R_{00} = R^n_{0n0} = [\partial_0 + \Gamma_0, \partial_n + \Gamma_n]^n{}_0. \tag{11.461}$$

Clearly the commutator of Γ_0 with itself vanishes, and one may use the formulas (11.456–11.460) for the other connections to check that

$$[\Gamma_0, \Gamma_n]^n{}_0 = \Gamma^n_{0\,k}\,\Gamma^k_{n\,0} - \Gamma^n_{n\,k}\,\Gamma^k_{0\,0} = 3\left(\frac{\dot{a}}{ca}\right)^2 \tag{11.462}$$

and that

$$\partial_0\,\Gamma^n_{n\,0} = 3\,\partial_0\left(\frac{\dot{a}}{ca}\right) = 3\,\frac{\ddot{a}}{c^2 a} - 3\left(\frac{\dot{a}}{ca}\right)^2 \tag{11.463}$$

while $\partial_n \Gamma^n_{0\,0} = 0$. So the 00-component of the Ricci tensor is

$$R_{00} = 3\,\frac{\ddot{a}}{c^2 a}. \tag{11.464}$$

Similarly, one may show that the other nonzero components of Ricci's tensor are

$$R_{11} = -\frac{A}{1 - kr^2} \qquad R_{22} = -r^2 A \quad \text{and} \quad R_{33} = -r^2 A \sin^2\theta \tag{11.465}$$

in which $A = a\ddot{a}/c^2 + 2\dot{a}^2/c^2 + 2k$. The scalar curvature (11.399) is

$$R = g^{ab} R_{ba} = -\frac{6}{a^2}\left(\frac{a\ddot{a}}{c^2} + \frac{\dot{a}^2}{c^2} + k\right). \tag{11.466}$$

In comoving coordinates such as those of the Robertson-Walker metric (11.454), the 4-velocity (11.65) is $u^i = (c, 0, 0, 0)$, and so the energy-momentum tensor (11.438) is

$$T_{ij} = \begin{pmatrix} c^2\rho & 0 & 0 & 0 \\ 0 & p\, g_{11} & 0 & 0 \\ 0 & 0 & p\, g_{22} & 0 \\ 0 & 0 & 0 & p\, g_{33} \end{pmatrix}. \tag{11.467}$$

Its trace is

$$T = g^{ij}\, T_{ij} = -c^2\rho + 3p. \tag{11.468}$$

Thus successively using our formulas (11.454) for $g_{00} = -1$, (11.464) for $R_{00} = 3\ddot{a}/(c^2 a)$, (11.467) for T_{ij}, and (11.468) for T, we can write the 00 Einstein equation (11.431) as the second-order equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{11.469}$$

which is nonlinear because ρ and $3p$ depend upon a. The sum $\rho + 3p$ determines the acceleration $\ddot{a}$ of the scale factor $a(t)$. The apparently positive sum $c^2\rho + 3p$ can be negative, and when it is, it accelerates the expansion of the universe.

Because of the isotropy of the metric, the three nonzero spatial Einstein equations (11.431) give us only one relation

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{c^2 k}{a^2} = 4\pi G\left(\rho - \frac{p}{c^2}\right). \tag{11.470}$$

Using the 00-equation (11.469) to eliminate the second derivative $\ddot{a}$, we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho - \frac{c^2 k}{a^2} \tag{11.471}$$

which is a first-order nonlinear equation. It and the second-order equation (11.469) are known as the **Friedmann equations**.

The LHS of the first-order Friedmann equation (11.493) is the square of the **Hubble rate**

$$H = \frac{\dot{a}}{a} \tag{11.472}$$

which is an inverse time or a frequency. Its present value H_0 is the **Hubble constant**. In terms of H, Friedmann's first-order equation (11.493) is

$$H^2 = \frac{8\pi G}{3}\,\rho - \frac{c^2 k}{a^2}. \tag{11.473}$$

The energy density of a flat universe with $k = 0$ is the **critical energy density**

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{11.474}$$

The ratio of the present energy density ρ_0 to the present critical energy density ρ_{c0} is Ω_0

$$\Omega_0 = \frac{\rho_0}{\rho_{c0}} = \frac{8\pi G}{3H_0^2}\,\rho_0. \tag{11.475}$$

From (11.473), we see that Ω is

$$\Omega = 1 + \frac{c^2 k}{(aH)^2} = 1 + \frac{c^2 k}{\dot{a}^2}. \tag{11.476}$$

Thus $\Omega = 1$ both in a flat universe ($k = 0$) and as $aH \to \infty$. One use of inflation is to expand a by 10^{26} so as to force Ω almost exactly to unity.

Something like inflation is needed because in a universe in which the energy density is due to matter and/or radiation, the present value of Ω

$$\Omega_0 = 1.0000 \pm 0.0062 \tag{11.477}$$

is unlikely. To see why, we note that conservation of energy ensures that a^3 times the matter density ρ_m is constant. Radiation redshifts by a, so energy conservation implies that a^4 times the radiation density ρ_r is constant. So with $n = 3$ for matter and 4 for radiation, $\rho\, a^n \equiv 3\,F^2/8\pi G$ is a constant. In terms of F and n, Friedmann's first-order equation (11.493) is

$$\dot{a}^2 = \frac{8\pi G}{3}\,\rho\, a^2 - c^2 k = \frac{F^2}{a^{n-2}} - c^2 k \tag{11.478}$$

In the small-a limit of the early Universe, we have

$$\dot{a} = F/a^{(n-2)/2} \qquad \text{or} \qquad a^{(n-2)/2} da = F\, dt \tag{11.479}$$

which we integrate to $a \sim t^{2/n}$ so that $\dot{a} \sim t^{2/n-1}$. Now (11.476) says that

$$|\Omega - 1| = \frac{c^2|k|}{\dot{a}^2} \propto t^{2-4/n} = \begin{cases} t & \text{radiation} \\ t^{2/3} & \text{matter} \end{cases}. \tag{11.480}$$

Thus, Ω deviated from unity at least as fast as $t^{2/3}$ during the early Universe. At this rate, the inequality $|\Omega_0 - 1| < 0.009$ could last 13.8 billion years only

if Ω at $t = 1$ second had been unity to within two parts in 10^{14}. The only *known* explanation for such early flatness is inflation.

The relation (11.476) between Ω and aH shows that $k = 0$ is equivalent to $\Omega = 1$, that $\Omega > 1 \iff k > 0$, and that $\Omega < 1 \iff k < 0$. And writing the same relation (11.476) as $(aH)^2 = c^2k/(\Omega - 1)$, we see that as $\Omega \to 1$ the product $aH \to \infty$, which is the essence of flatness since curvature vanishes as the scale factor $a \to \infty$. Imagine blowing up a balloon.

Staying for the moment with a universe without inflation and with an energy density composed of radiation and/or matter, we note that the first-order equation (11.478) in the form $\dot{a}^2 = F^2/a^{n-2} - c^2k$ tells us that for a closed ($k > 0$) universe, in the limit $a \to \infty$ we'd have $\dot{a}^2 \to -1$ which is impossible. Thus a closed universe cannot expand indefinitely.

The first-order equation Friedmann (11.493) says that $\rho\, a^2 \geq 3c^2k/8\pi G$. So in a closed universe ($k > 0$), the energy density ρ is positive and increases without limit as $a \to 0$ as in a collapse. In open ($k < 0$) and flat ($k = 0$) universes, the same Friedmann equation (11.493) in the form $\dot{a}^2 = 8\pi G\rho a^2/3 - c^2k$ tells us that if ρ is positive, then $\dot{a}^2 > 0$, which means that $\dot{a}$ never vanishes. Hubble told us that $\dot{a} > 0$ now. So if our universe is open or flat, then it always expands.

Due to the expansion of the universe, the wave-length of radiation grows with the scale factor $a(t)$. A photon emitted at time t and scale factor $a(t)$ with wave-length $\lambda(t)$ will be seen now at time t_0 and scale factor $a(t_0)$ to have a longer wave-length $\lambda(t_0)$

$$\frac{\lambda(t_0)}{\lambda(t)} = \frac{a(t_0)}{a(t)} = z + 1 \tag{11.481}$$

in which the **redshift** z is the ratio

$$z = \frac{\lambda(t_0) - \lambda(t)}{\lambda(t)} = \frac{\Delta\lambda}{\lambda}. \tag{11.482}$$

Now $H = \dot{a}/a = da/(adt)$ implies $dt = da/(aH)$, and $z = a_0/a - 1$ implies $dz = -a_0 da/a^2$, so we find

$$dt = -\frac{dz}{(1+z)H(z)} \tag{11.483}$$

which relates time intervals to redshift intervals. An on-line calculator is available for macroscopic intervals (Wright, 2006).

11.55 Density and pressure

The 0-th component of the energy-momentum conservation law (11.437) is

$$0 = T^{0a}_{;a} = \partial_a T^{0a} + \Gamma^a_{ca} T^{0c} + \Gamma^0_{ca} T^{ca}. \tag{11.484}$$

The perfect-fluid energy-momentum tensor (11.467) is diagonal, and for a Robinson-Walker metric (11.454) our connection formulas (11.455) and (11.457) respectively tell us that $\Gamma^0_{00} = -\frac{1}{2} g_{00,0} = 0$, and that $\Gamma^a_{0a} = 3\dot{a}/(ca)$. Thus

$$\dot{\rho} = -\frac{3\dot{a}}{a}(\rho + \frac{p}{c^2}), \quad \text{and so} \quad \frac{d\rho}{da} = -\frac{3}{a}\left(\rho + \frac{p}{c^2}\right). \tag{11.485}$$

The energy density ρ is composed of fractions ρ_i each contributing its own partial pressure p_i according to its own **equation of state**

$$p_i = c^2 w_i \rho_i \tag{11.486}$$

in which w_i is a constant. The rate of change (11.486) of the density ρ_i is then

$$\frac{d\rho_i}{da} = -\frac{3}{a}\,(1 + w_i)\,\rho_i. \tag{11.487}$$

In terms of the present density ρ_{i0} and scale factor a_0, the solution is

$$\rho_i = \rho_{i0}\left(\frac{a_0}{a}\right)^{3(1+w_i)}. \tag{11.488}$$

There are three important kinds of density. The dark-energy density ρ_Λ is assumed to be like a cosmological constant Λ or like the energy density of the vacuum, so it is independent of the scale factor a

$$\rho_\Lambda = \rho_{\Lambda 0} \tag{11.489}$$

and has $w_\Lambda = -1$. The matter density ρ_m is assumed to have no pressure, $w_m = 0$, and so the matter density falls inversely with the volume

$$\rho_m = \rho_{m0}\left(\frac{a_0}{a}\right)^3. \tag{11.490}$$

The density of radiation ρ_r has $w_r = 1/3$ because wavelengths scale with the scale factor, and so there's an extra factor of a

$$\rho_r = \rho_{r0}\left(\frac{a_0}{a}\right)^4. \tag{11.491}$$

The total density ρ varies with a as

$$\rho = \rho_{\Lambda 0} + \rho_{m0}\left(\frac{a_0}{a}\right)^3 + \rho_{r0}\left(\frac{a_0}{a}\right)^4. \tag{11.492}$$

In simple cosmological models, only the dominant component of the density is considered.

11.56 How the scale factor evolves with time

The first-order Friedmann equation (11.493) expresses the square of the instantaneous Hubble rate $H = \dot{a}/a$ in terms of the density ρ and the scale factor $a(t)$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho - \frac{c^2 k}{a^2} \tag{11.493}$$

in which $k = \pm 1$ or 0. The critical density ρ_c is the one that satisfies this equation for a flat ($k = 0$) universe. Its present value is

$$\rho_{0c} = \frac{3H_0^2}{8\pi G}. \tag{11.494}$$

Dividing Friedmann's equation by the square of the present Hubble rate H_0^2, we get

$$\frac{H^2}{H_0^2} = \frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{H_0^2}\left(\frac{8\pi G}{3}\,\rho - \frac{c^2 k}{a^2}\right) = \frac{\rho}{\rho_{0c}} - \frac{c^2 k}{a^2 H_0^2} \tag{11.495}$$

in which ρ is the total density (11.492)

$$\begin{aligned} \frac{H^2}{H_0^2} &= \frac{\rho_\Lambda}{\rho_{0c}} + \frac{\rho_r}{\rho_{0c}} + \frac{\rho_m}{\rho_{0c}} - \frac{c^2 k}{a^2 H_0^2} \\ &= \frac{\rho_{\Lambda 0}}{\rho_{0c}} + \frac{\rho_{r0}}{\rho_{0c}}\,\frac{a_0^4}{a^4} + \frac{\rho_{m0}}{\rho_{0c}}\,\frac{a_0^3}{a^3} - \frac{c^2 k}{a_0^2 H_0^2}\,\frac{a_0^2}{a^2}. \end{aligned} \tag{11.496}$$

These density ratios are called $\Omega_{\Lambda 0}$, Ω_{r0}, Ω_{m0}, and $\Omega_{k0} \equiv -c^2 k/a_0^2 H_0^2$, and in terms of them this formula (11.496) for H^2/H_0^2 is

$$\frac{H^2}{H_0^2} = \Omega_{\Lambda 0} + \Omega_{k0}\,\frac{a_0^2}{a^2} + \Omega_{m0}\,\frac{a_0^3}{a^3} + \Omega_{r0}\,\frac{a_0^4}{a^4}. \tag{11.497}$$

Since $H = \dot{a}/a$, we have $dt = H_0^{-1}(da/a)(H_0/H)$, and so with $x = a/a_0$, the time interval dt is

$$dt = \frac{1}{H_0}\,\frac{dx}{x}\,\frac{1}{\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\,x^{-2} + \Omega_{m0}\,x^{-3} + \Omega_{r0}\,x^{-4}}}. \tag{11.498}$$

Integrating and setting the origin of time $t(0) = 0$ at $x = a/a_0 = 0$, we find that the time $t(a/a_0)$ during which the ratio $a(t)/a_0$ grew from 0 to $a(t)/a_0$

is

$$t(a/a_0) = \frac{1}{H_0}\int_0^{a/a_0} \frac{dx}{\sqrt{\Omega_{\Lambda 0}\, x^2 + \Omega_{k0} + \Omega_{m0}\, x^{-1} + \Omega_{r0}\, x^{-2}}}. \tag{11.499}$$

The definition (11.481) of the redshift gives $a/a_0 = 1/(z+1)$. So this formula (11.499) for $t(a/a_0)$ also says that a photon emitted at time $t(a/a_0)$ will be seen now with redshift $z(t) = a_0/a - 1$.

The Planck Collaboration's values (Ade *et al.* (2015)) for the density ratios $\Omega_{\Lambda 0}$, Ω_{k0}, and Ω_{m0} are

$$\begin{aligned} \Omega_{\Lambda 0} &= 0.6911 \pm 0.0062 \\ \Omega_{k0} &= 0.0008 \pm 0.004 \\ \Omega_{m0} &= 0.3089 \pm 0.0062. \end{aligned} \tag{11.500}$$

We use may use the present temperature $T_0 = 2.7255$ K of the cosmic microwave background radiation and our formula (4.98) for the energy density of photons to estimate the mass density of photons as

$$\rho_\gamma = \frac{8\pi^5\,(k_B T_0)^4}{15 h^3 c^5} = 4.6451 \times 10^{-31}\ \text{kg m}^{-3}. \tag{11.501}$$

Adding in three kinds of neutrinos and antineutrinos at $T_{0\nu} = (4/11)^{1/3}\, T_0$, we get for the present density of massless and nearly massless particles (Weinberg, 2010, section 2.1)

$$\rho_r = \left[1 + 3\left(\frac{7}{8}\right)\left(\frac{4}{11}\right)^{4/3}\right]\ \rho_\gamma = 7.8099 \times 10^{-31}\ \text{kg m}^{-3}. \tag{11.502}$$

The fraction Ω_{r0} the present energy density that is due to radiation is then

$$\Omega_{r0} = \frac{\rho_{r0}}{\rho_{c0}} = 9.0606 \times 10^{-5}. \tag{11.503}$$

By putting the Ω values (11.500 & 11.503) into the integral (11.499) and integrating, we may get the time as a function of the scale factor. Figure 11.2 plots the reduced scale factor $a(t)/a_0$ (solid) and the redshift $z(t)$ (...) as functions of the time t in Gyr since the time of infinite redshift. The age of the universe is 13.8 Gyr (vertical line). A photon emitted with wavelength λ at time t now has wavelength $\lambda_0 = (a_0/a(t))\,\lambda$. The change is its wavelength is $\Delta\lambda = \lambda\, z(t)$.

Example 11.35 ($w = -1/3$, No Acceleration)**.** If $w = -1/3$, then $p = w\,\rho = -\rho/3$ and $\rho + 3p = 0$. The second-order Friedmann equation (11.469) then tells us that $\ddot{a} = 0$. The scale factor does not accelerate.

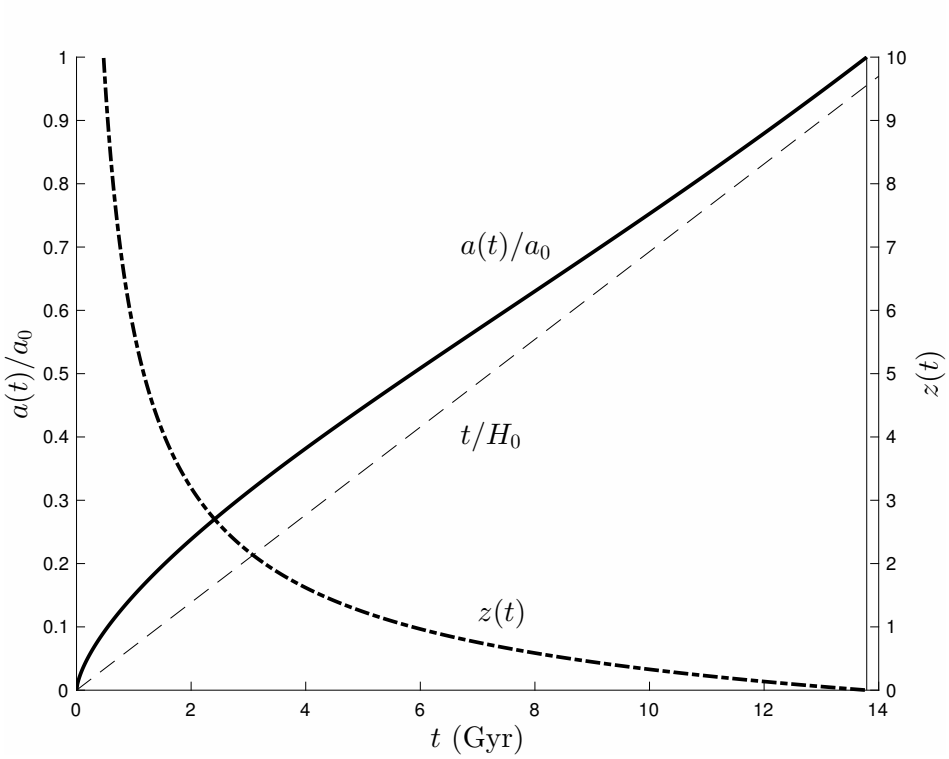

Figure 11.2 The reduced scale factor $a(t)/a_0$ (solid), the redshift $z(t)$ (dot-dash), and the fraction t/H_0 (dashed) are plotted as functions of the time (11.499) in Gyr since the time of infinite redshift. The age of the universe is 13.8 Gyr (vertical line). A photon emitted with wavelength λ at time t now has wavelength $\lambda_0 = (a_0/a(t)\,\lambda$ and redshift $z(t) = \Delta\lambda/\lambda$.

To find its constant speed, we use its equation of state (11.488)

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)} = \rho_0 \left(\frac{a_0}{a}\right)^2 . \tag{11.504}$$

Now all the terms in Friedmann's first-order equation (11.493) have a common factor of $1/a^2$ which cancels leaving us with the square of the constant speed

$$\dot{a}^2 = \frac{8\pi G}{3}\,\rho_0\,a_0^2 - c^2 k \tag{11.505}$$

in which $\rho_0\,a_0^2$ must exceed $3c^2k/8\pi G$. Since $\dot{a} = aH$ is constant, the scale factor grows linearly with time like the dashed line $a(t) = a_0\,H_0\,t$ in figure 11.2 if we set $a(0) = 0$. □

11.57 Inflation ($w = -1$)

Inflation occurs when the ground state of the theory has a positive and constant potential-energy density $\rho > 0$ that dwarfs the energy densities of the matter and radiation. The **internal energy** of the universe then is proportional to its volume $U = c^2 \rho V$, and the pressure p as given by the thermodynamic relation

$$p = -\frac{\partial U}{\partial V} = -c^2 \rho \tag{11.506}$$

is **negative**. The equation of state (11.486) tells us that in this case $w = -1$. The second-order Friedmann equation (11.469) becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) = \frac{8\pi G\rho}{3} \equiv g^2 \tag{11.507}$$

By it and the first-order Friedmann equation (11.493) and by choosing $t = 0$ as the time at which the scale factor a is zero or minimal, one may show (exercise 11.38) that in a flat ($k = 0$), closed ($k > 0$), and open ($k < 0$), universes, the scale factor varies as

$$a(t) = a(0)\, e^{g\,t} \qquad \text{flat,}\ \ k = 0 \tag{11.508}$$

$$a(t) = \frac{\cosh g\,t}{g} \qquad \text{closed,}\ \ k > 0 \tag{11.509}$$

$$a(t) = \frac{\sinh g\,t}{g} \qquad \text{open,}\ \ k < 0. \tag{11.510}$$

A **de Sitter** universe is flat, has $a(t) = a(0)\exp(gt)$, and can be infinitely old.

Studies of the cosmic microwave background radiation suggest that inflation did occur in the **very early** universe—possibly on a time scale as short as 10^{-35} s. The origin of the vacuum energy density ρ that drove inflation is unknown. In **chaotic inflation**, a scalar field ϕ fluctuated to a mean value $\langle\phi\rangle$ very different from the one $\langle 0|\phi|0\rangle$ that minimizes the energy density of the vacuum. When $\langle\phi\rangle$ settled to $\langle 0|\phi|0\rangle$, the potential energy of the vacuum was released as radiation in a **Big Bang**.

Anti-de Sitter models have $w = -1$ and a **negative potential energy** $\rho = -p < 0$.

11.58 The era of radiation ($w = 1/3$)

Until a redshift of $z = 3408.3$ or $50,953$ years after the time of infinite redshift our universe was dominated by radiation. During *The First Three Minutes* (Weinberg, 1988) of the era of radiation, the quarks and gluons formed hadrons, which decayed into protons and neutrons. As the neutrons decayed ($\tau = 885.7$ s), they and the protons formed the light elements—principally hydrogen, deuterium, and helium in a process called **big-bang nucleosynthesis**.

We can guess the value of w for radiation by noticing that the energy-momentum tensor of the electromagnetic field (in SI units)

$$T^{ab} = \frac{1}{\mu_0}\left(F^a{}_c F^{bc} - \frac{1}{4} g^{ab} F_{cd} F^{cd}\right) \tag{11.511}$$

is traceless

$$T = T^a{}_a = \frac{1}{\mu_0}\left(F^a{}_c F_a{}^c - \frac{1}{4}\delta^a_a F_{cd} F^{cd}\right) = 0. \tag{11.512}$$

But by (11.468) its trace must be $T = 3p - c^2\rho$. So for radiation $p = c^2\rho/3$ and $w = 1/3$. The relation (11.491) between the density and the scale factor for radiation then is

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^4. \tag{11.513}$$

The density drops both with the volume a^3 and with the scale factor a due to a redshift; so it drops as $1/a^4$.

Early in the era of radiation, we can ignore the matter and vacuum densities and focus on the radiation density so that the quantity

$$f^2 \equiv \frac{8\pi G \rho a^4}{3} \tag{11.514}$$

is a constant. The Friedmann equations (11.469 & 11.470) then are

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) = -\frac{8\pi G\rho}{3} \quad \text{or} \quad \ddot{a} = -\frac{f^2}{a^3} \tag{11.515}$$

and

$$\dot{a}^2 + c^2 k = \frac{f^2}{a^2}. \tag{11.516}$$

With calendars chosen so that $a(0) = 0$, this last equation (11.516) tells

us that for flat ($k = 0$), closed ($k = 1$), and open ($k = -1$) universes

$$a(t) = \sqrt{2f\,t} \qquad \text{flat,}\ \ k = 0 \tag{11.517}$$

$$a(t) = \sqrt{2f\,t - kc^2t^2} \quad \text{closed,}\ \ k > 0 \tag{11.518}$$

$$a(t) = \sqrt{2f\,t + kc^2t^2} \quad \text{open,}\ \ k < 0, \tag{11.519}$$

as we saw in (6.488). The scale factor (11.518) of a closed universe of radiation has a maximum $a = f/(c\sqrt{k})$ at $t = f/(kc^2)$ and falls back to zero at $t = 2f/(kc^2)$.

In all three cases, the scale factor rises as the square-root of the time $a(t) \approx \sqrt{2ft}$ at early times. The density of radiation is proportional both to the fourth power of the temperature $\rho \propto T^4$ and to the inverse fourth power of the scale factor $\rho \sim 1/a^4(t) \sim 1/t^2$. Thus the temperature falls as $T \sim 1/\sqrt{t}$.

Weinberg gives more accurate estimates. When the temperature was in the range $10^{12} > T > 10^{10}\,\mathrm{K}$ or $m_\mu c^2 > kT > m_e c^2$, where m_μ is the mass of the muon and m_e that of the electron, the radiation was mostly electrons, positrons, photons, and neutrinos, and the relation between the time t and the temperature T was (Weinberg, 2010, ch. 3)

$$t = 0.994\ \ \mathrm{sec} \times \left[\frac{10^{10}\,\mathrm{K}}{T}\right]^2 + \ \mathrm{constant}. \tag{11.520}$$

By 10^9 K, the positrons had annihilated with electrons, and the neutrinos fallen out of equilibrium. Between 10^9 K and $10^6\,\mathrm{K}$, when the energy density of nonrelativistic particles became relevant, the time-temperature relation was (Weinberg, 2010, ch. 3)

$$t = 1.78\ \ \mathrm{sec} \times \left[\frac{10^{10}\,\mathrm{K}}{T}\right]^2 + \ \mathrm{constant}'. \tag{11.521}$$

At times of tens of thousands of years, the matter density ρ_m became important because it drops with the cube of the scale factor as $\rho_m(t) = \Omega_{m0}\,a_0^3/a^3(t)$ while the radiation density $\rho_r(t)$ drops with the fourth power of the scale factor as $\rho_r(t) = \Omega_{r0}\,a_0^4/a^4(t)$. The era of radiation ended and the era of matter began when these densities were equal, $\rho_m(t) = \rho_r(t)$. We can estimate this time by using the density ratios Ω_{m0} and Ω_{r0} (11.500 & 11.503). We find

$$\frac{a}{a_0} = \frac{\rho_{r,0}}{\rho_{m,0}} = \frac{\Omega_{r0}}{\Omega_{m0}} = \frac{9.0606 \times 10^{-5}}{0.3089} = 2.93318 \times 10^{-4} \tag{11.522}$$

which is a redshift of $z = a_0/a - 1 = 3408.3$. Our integral (11.499) gives

the time as $t(3408.3) = 50,953$ years after the time of infinite redshift. The temperature then was $T = 9,400$ K or $kT = 0.81$ eV.

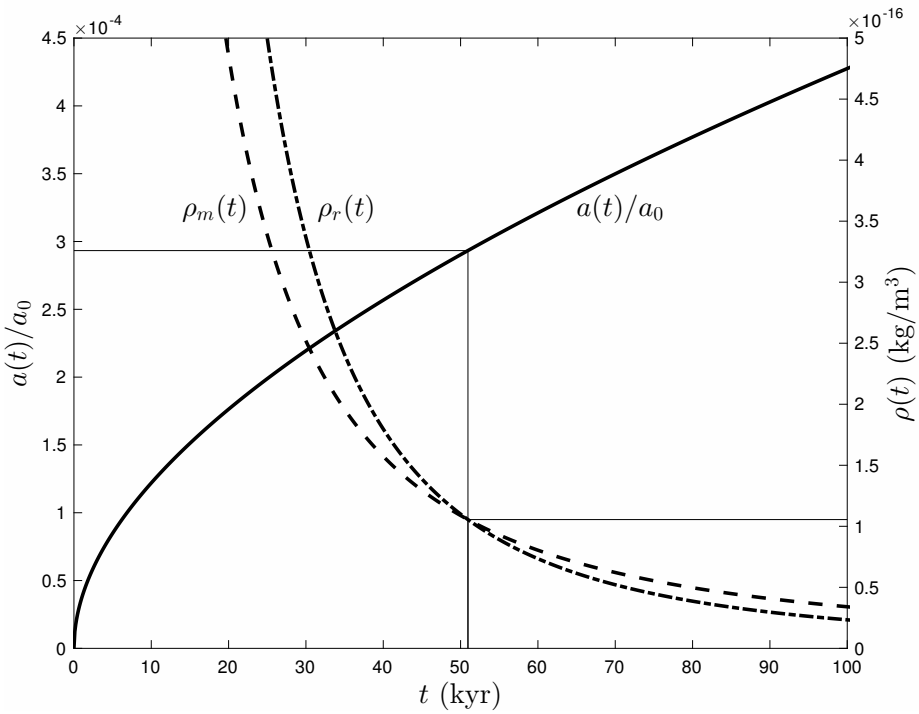

Figure 11.3 The scale factor ratio a/a_0 (solid), the radiation density ρ_r (dotdash), and the matter density ρ_m (dashed) are plotted as functions of the time (11.499) in kyr after the time of infinite redshift. The era of radiation ends at $t = 50,953$ years when the two densities are equal to 1.055×10^{-16} kg/m^3, and $a/a_0 = 2.933 \times 10^{-4}$.

11.59 The era of matter ($w = 0$)

A universe composed only of **dust** or **non-relativistic collisionless matter** has no pressure. Thus $p = w\rho = 0$ with $\rho \neq 0$, and so $w = 0$. Conservation of energy (11.485), or equivalently (11.490), implies that the matter density falls with the volume as

$$\rho_m = \rho_{m0} \left(\frac{a_0}{a}\right)^3 . \tag{11.523}$$

The era of matter began about 50,953 after the time of infinite redshift when the matter density ρ_m first exceeded the radiation density ρ_r. Most of the matter is of an unknown kind that interacts very weakly with photons and is called **dark matter**. Baryons amount to only about 15.7% of ρ_m, so if they were the principal kind of matter, the era of radiation would have lasted much longer.

Near the middle of the era of matter, at $t \sim 5$ Gyr, the radiation density and the dark-energy density are unimportant, and we need take only the matter density into account in order to approximate the evolution of the scale factor. The density then varies as $\rho \sim \rho_m \propto 1/a^3$, and so the quantity

$$f = \frac{4\pi G \rho_m a^3}{3} \tag{11.524}$$

is constant. The resulting Friedmann equations (11.469 & 11.470) are

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_m + \frac{3p}{c^2}\right) = -\frac{4\pi G \rho_m}{3} \quad \text{or} \quad \ddot{a} = -\frac{f}{a^2} \tag{11.525}$$

and

$$\dot{a}^2 + c^2 k = 2f/a. \tag{11.526}$$

For a flat universe, $k = 0$, we get the **Einstein-de Sitter model**

$$a(t) = \left[3\sqrt{\frac{f}{2}}\, t\right]^{2/3}. \tag{11.527}$$

The scale factors of open and closed universes also rise as $a(t) \sim t^{2/3}$ as long as $a < 2f/(c^2|k|)$.

Transparency: Some 380,000 years after inflation at a redshift of $z = 1090$, the universe had cooled to about $T = 3000$ K or $kT = 0.26\,\text{eV}$—a temperature at which less than 1% of the hydrogen is ionized. Ordinary matter became a gas of neutral atoms rather than a plasma of ions and electrons, and the universe suddenly became **transparent** to light. This moment of last scattering and first transparency often is (inexplicably) called **recombination**.

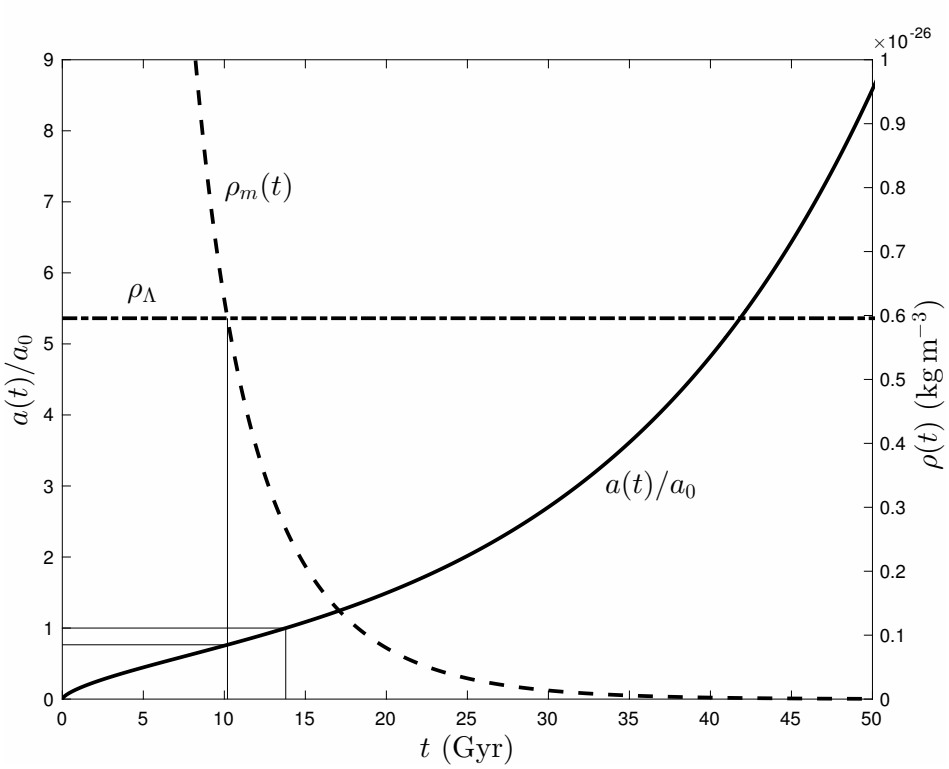

Figure 11.4 The reduced scale factor a/a_0 (solid), the vacuum density ρ_Λ (dotdash), and the matter density ρ_m (dashed) are plotted as functions of the time (11.499) in Gyr after the time of infinite redshift. The era of matter ends at $t = 10.193$ Gyr (first vertical line) when the two densities are equal. The present time t_0 is 13.8 Gyr (second vertical line) at which $a(t)/a_0 = 1$.

11.60 The era of dark energy ($w = -1$)

About 3.606 billion years ago or 10.193 Gyr after inflation at a redshift of $z = 0.3079$, the matter density, falling as $1/a^3$, dropped below the **dark-energy** density $\rho_\Lambda = 6.0084 \times 10^{-27}$ kg/m^3 or (2.256 meV)^4. The age of the universe is 13.799 billion years. For the past 3.606 billion years, this constant energy density, called **dark energy**, has accelerated the expansion of the universe approximately as in the de Sitter model (11.510)

$$a(t) = a(t_m)\, \exp\left((t - t_m)\sqrt{8\pi G\rho_v/3}\right) \tag{11.528}$$

in which $t_m = (10.427 \pm 0.07) \times 10^9$ years.

11.61 Yang-Mills theory

The gauge transformation of an **abelian** gauge theory like electrodynamics multiplies a *single* charged field by a spacetime-dependent *phase factor* $\phi'(x) = \exp(iq\theta(x))\,\phi(x)$. Yang and Mills generalized this gauge transformation to one that multiplies a *vector* ϕ of matter fields by a spacetime dependent *unitary matrix* $U(x)$

$$\phi'_a(x) = \sum_{b=1}^{n} U_{ab}(x)\,\phi_b(x) \quad \text{or} \quad \phi'(x) = U(x)\,\phi(x) \tag{11.529}$$

and showed how to make the action of the theory invariant under such **non-abelian** gauge transformations. (The fields ϕ are scalars for simplicity.)

Since the matrix U is unitary, inner products like $\phi^\dagger(x)\,\phi(x)$ are automatically invariant

$$\Big(\phi^\dagger(x)\,\phi(x)\Big)' = \phi^\dagger(x)U^\dagger(x)U(x)\phi(x) = \phi^\dagger(x)\phi(x). \tag{11.530}$$

But inner products of derivatives $\partial^i\phi^\dagger\,\partial_i\phi$ are not invariant because the derivative acts on the matrix $U(x)$ as well as on the field $\phi(x)$.

Yang and Mills made derivatives $D_i\phi$ that transform like the fields ϕ

$$(D_i\phi)' = U\,D_i\phi. \tag{11.531}$$

To do so, they introduced **gauge-field matrices** A_i that play the role of the connections Γ_i in general relativity and set

$$D_i = \partial_i + A_i \tag{11.532}$$

in which A_i like ∂_i is antihermitian. They required that under the gauge transformation (11.529), the gauge-field matrix A_i transform to A'_i in such a way as to make the derivatives transform as in (11.531)

$$(D_i\phi)' = \big(\partial_i + A'_i\big)\,\phi' = \big(\partial_i + A'_i\big)\,U\phi = U\,D_i\phi = U\,(\partial_i + A_i)\,\phi. \tag{11.533}$$

So they set

$$\big(\partial_i + A'_i\big)\,U\phi = U\,(\partial_i + A_i)\,\phi \quad \text{or} \quad (\partial_i U)\,\phi + A'_i\,U\phi = UA_i\,\phi. \tag{11.534}$$

and made the gauge-field matrix A_i transform as

$$A'_i = UA_iU^{-1} - (\partial_i U)\,U^{-1}. \tag{11.535}$$

Thus under the gauge transformation (11.529), the derivative $D_i\phi$ transforms as in (11.531), like the vector ϕ in (11.529), and the inner product of covariant derivatives

$$\Big[\big(D^i\phi\big)^\dagger\,D_i\phi\Big]' = \big(D^i\phi\big)^\dagger\,U^\dagger UD_i\phi = \big(D^i\phi\big)^\dagger\,D_i\phi \tag{11.536}$$

remains invariant.

To make an invariant action density for the gauge-field matrices A_i, they used the transformation law (11.533) which implies that $D'_i U\phi = UD_i\,\phi$ or $D'_i = UD_i\,U^{-1}$. So they defined their generalized Faraday tensor as

$$F_{ik} = [D_i, D_k] = \partial_i A_k - \partial_k A_i + [A_i, A_k] \tag{11.537}$$

so that it transforms covariantly

$$F'_{ik} = UF_{ik}U^{-1}. \tag{11.538}$$

They then generalized the action density $F_{ik}F^{ik}$ of electrodynamics to the trace $\mathrm{Tr}\left(F_{ik}F^{ik}\right)$ of the square of the Faraday matrices which is invariant under gauge transformations since

$$\mathrm{Tr}\left(UF_{ik}U^{-1}UF^{ik}U^{-1}\right) = \mathrm{Tr}\left(UF_{ik}F^{ik}U^{-1}\right) = \mathrm{Tr}\left(F_{ik}F^{ik}\right). \tag{11.539}$$

As an action density for fermionic matter fields, they replaced the ordinary derivative in Dirac's formula $\overline{\psi}(\gamma^i\partial_i + m)\psi$ by the covariant derivative (11.532) to get $\overline{\psi}(\gamma^i D_i + m)\psi$ (Chen-Ning Yang 1922–, Robert L. Mills 1927–1999).

In an abelian gauge theory, the square of the 1-form $A = A_i\,dx^i$ vanishes $A^2 = A_i\,A_k\,dx^i \wedge dx^k = 0$, but in a nonabelian gauge theory the gauge fields are matrices, and $A^2 \neq 0$. The sum $dA + A^2$ is the Faraday 2-form

$$\begin{aligned} F = dA + A^2 &= (\partial_i\,A_k + A_i\,A_k)\;dx^i \wedge dx^k \\ &= \tfrac{1}{2}\left(\partial_i\,A_k - \partial_k\,A_i + [A_i,\,A_k]\right)\,dx^i \wedge dx^k = \tfrac{1}{2}F_{ik}\,dx^i \wedge dx^k. \end{aligned} \tag{11.540}$$

The scalar matter fields ϕ may have self-interactions described by a potential $V(\phi)$ such as $V(\phi) = \lambda(\phi^\dagger\phi - m^2/\lambda)^2$ which is positive unless $\phi^\dagger\phi = m^2/\lambda$. The kinetic action of these fields is $(D^i\phi)^\dagger D_i\phi$. At low temperatures, these scalar fields assume mean values $\langle 0|\phi|0\rangle = \phi_0$ in the vacuum with $\phi_0^\dagger\phi_0 = m^2/\lambda$ so as to minimize their potential energy density $V(\phi)$ and their kinetic action $(D^i\phi)^\dagger D_i\phi = (\partial^i\phi + A^i\phi)^\dagger(\partial_i\phi + A_i\phi)$ is approximately $\phi_0^\dagger\,A^i\,A_i\,\phi_0$. The gauge-field matrix $A^i_{ab} = i\,t^\alpha_{ab}A^i_\alpha$ is a linear combination of the generators t^α of the gauge group. So the action of the scalar fields contains the term $\phi_0^\dagger\,A^i\,A_i\,\phi_0 = -M^2_{\alpha\beta}\,A^i_\alpha\,A_{i\beta}$ in which the mass-squared matrix for the gauge fields is $M^2_{\alpha\beta} = \phi_0^{*a}\,t^\alpha_{ab}\,t^\beta_{bc}\,\phi_0^c$. This **Higgs mechanism** gives masses to those linear combinations $b_{\beta i}\,A_\beta$ of the gauge fields for which $M^2_{\alpha\beta}\,b_{\beta i} = m_i^2 b_{\alpha i} \neq 0$.

The Higgs mechanism also gives masses to the fermions. The mass term m in the Yang-Mills-Dirac action is replaced by something like $c\,\phi$ in which c is a constant, different for each fermion. In the vacuum and at low temperatures,

each fermion acquires as its mass $c\,\phi_0$. On 4 July 2012, physicists at CERN's Large Hadron Collider announced the discovery of a Higgs-like particle with a mass near 125 GeV/c^2 (Peter Higgs 1929–).

11.62 Spin-one-half fields in general relativity

The flat-space action density (10.305) for a spin-one-half field ψ is

$$L = -\bar{\psi}\left[\gamma^a\left(\partial_a + A_a\right) + m\right]\psi \tag{11.541}$$

in which a is a flat-space index, A_a is a matrix of gauge fields, ψ is a 4-component Dirac or Majorana field, $\bar{\psi} = \psi^\dagger\beta = i\psi^\dagger\gamma^0$, and m is a constant or a mean value of a scalar field. One may use the tetrad fields $e^a_\mu(x)$ of section 11.20 to turn the flat-space indices a into curved-space indices. Since derivatives and gauge fields intrinsically are generally covariant vectors, one replaces $\gamma^a\left(\partial_a + A_a\right)$ by $\gamma^a\, e_a{}^\mu\left(\partial_\mu + A_\mu\right)$. The next step is to correct for the effect of the derivative ∂_μ on the field ψ by making the derivative generally covariant as well as gauge covariant. The required Einstein connection is (Weinberg, 1972, sec. 12.5)

$$E_\mu = \frac{i}{2} J^{ab}\, e_a{}^\nu\, e_{b\nu;\mu} \tag{11.542}$$

in which the generators (10.300) of the Lorentz group J^{ab} are the commutators of Dirac's 4×4 gamma matrices (10.303)

$$J^{ab} = \frac{-i}{4}\left[\gamma^a, \gamma^b\right], \tag{11.543}$$

the covariant derivative of the tetrad $e_{b\nu}$ is

$$e_{b\nu;\mu} = e_{b\nu,\mu} - e_{b\sigma}\,\Gamma^\sigma_{\nu\mu}, \tag{11.544}$$

and the Levi-Civita affine connection (11.239) is $\Gamma^\sigma_{\nu\mu} = e^\sigma{}_a e^a{}_{\mu,\nu}$. Thus the generally covariant, gauge-covariant action density of a spin-one-half field is

$$L' = -\overline{\psi}\left[\gamma^a\, e_a{}^\mu\left(\partial_\mu + A_\mu + E_\mu\right)\right]\psi. \tag{11.545}$$

11.63 Gauge Theory and Vectors

This section is optional on a first reading.

We can formulate Yang-Mills theory in terms of vectors as we did relativity. To accomodate noncompact groups, we will generalize the unitary matrices $U(x)$ of the Yang-Mills gauge group to nonsingular matrices $V(x)$ that act on n matter fields $\psi^a(x)$ as

$$\psi'^a(x) = \sum_{a=1}^{n} V^a{}_b(x)\, \psi^b(x). \tag{11.546}$$

The field

$$\Psi(x) = \sum_{a=1}^{n} e_a(x)\, \psi^a(x) \tag{11.547}$$

will be gauge invariant $\Psi'(x) = \Psi(x)$ if the vectors $e_a(x)$ transform as

$$e'_a(x) = \sum_{b=1}^{n} e_b(x)\, V^{-1b}{}_a(x). \tag{11.548}$$

In what follows, we will sum over repeated indices from 1 to n and often will suppress explicit mention of the spacetime coordinates. In this compressed notation, the field Ψ is gauge invariant because

$$\Psi' = e'_a\, \psi'^a = e_b\, V^{-1b}{}_a\, V^a{}_c\, \psi^c = e_b\, \delta^b{}_c\, \psi^c = e_b\, \psi^b = \Psi \tag{11.549}$$

which is $e'^{\mathsf{T}}\psi' = e^{\mathsf{T}} V^{-1} V \psi = e^{\mathsf{T}}\psi$ in matrix notation.

The inner product of two basis vectors is an internal "metric tensor"

$$e^*_a \cdot e_b = \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} e^{\alpha *}_a\, \eta_{\alpha\beta} e^\alpha_b = \sum_{\alpha=1}^{N} e^{\alpha *}_a\, e^\alpha_b = g_{ab} \tag{11.550}$$

in which for simplicity I used the the N-dimensional identity matrix for the metric η. As in relativity, we'll assume the matrix g_{ab} to be nonsingular. We then can use its inverse to construct dual vectors $e^a = g^{ab} e_b$ that satisfy $e^{a\dagger} \cdot e_b = \delta^a_b$.

The free Dirac action density of the invariant field Ψ

$$\overline{\Psi}(\gamma^i\partial_i + m)\Psi = \overline{\psi}_a e^{a\dagger}(\gamma^i\partial_i + m) e_b \psi^b = \overline{\psi}_a \left[\gamma^i(\delta^a{}_b \partial_i + e^{a\dagger}\cdot e_{b,i}) + m\delta^a{}_b\right]\psi^b \tag{11.551}$$

is the full action of the component fields ψ^b

$$\overline{\Psi}(\gamma^i\partial_i + m)\Psi = \overline{\psi}_a(\gamma^i D^a_{i\,b} + m\,\delta^a{}_b)\psi^b = \overline{\psi}_a \left[\gamma^i(\delta^a{}_b\partial_i + A^a_{i\,b}) + m\,\delta^a{}_b\right]\psi^b \tag{11.552}$$

if we identify the gauge-field matrix as $A^a_{i\,b} = e^{a\dagger}\cdot e_{b,i}$ in harmony with the definition (11.239) of the affine connection $\Gamma^k_{i\ell} = e^k \cdot e_{\ell,i}$.

Under the gauge transformation $e'_a = e_b\, V^{-1b}{}_a$, the metric matrix transforms as

$$g'_{ab} = V^{-1c*}{}_a\, g_{cd}\, V^{-1d}{}_b \qquad \text{or as} \qquad g' = V^{-1\dagger}\, g\, V^{-1} \tag{11.553}$$

in matrix notation. Its inverse goes as $g'^{-1} = V\, g^{-1}\, V^\dagger$.

The gauge-field matrix $A^a_{i\,b} = e^{a\dagger} \cdot e_{b,i} = g^{ac} e^\dagger_c \cdot e_{b,i}$ transforms as

$$A'^a_{i\,b} = g'^{ac} e'^\dagger_a \cdot e'_{b,i} = V^a{}_c A^c_{id} V^{-1d}{}_b + V^a{}_c V^{-1c}{}_{b,i} \tag{11.554}$$

or as $A'_i = V A_i V^{-1} + V \partial_i V^{-1} = V A_i V^{-1} - (\partial_i V)\, V^{-1}$.

By using the identity $e^{a\dagger} \cdot e_{c,i} = -\, e^{a\dagger}_{,i} \cdot e_c$, we may write (exercise 11.45) the Faraday tensor as

$$F^a_{ijb} = [D_i, D_j]^a{}_b = e^{a\dagger}_{,i} \cdot e_{b,j} - e^{a\dagger}_{,i} \cdot e_c\; e^{c\dagger} \cdot e_{b,j} - e^{a\dagger}_{,j} \cdot e_{b,i} + e^{a\dagger}_{,j} \cdot e_c\; e^{c\dagger} \cdot e_{b,i}. \tag{11.555}$$

If $n = N$, then

$$\sum_{c=1}^{n} e^\alpha_c\; e^{\beta c*} = \delta^{\alpha\beta} \quad \text{and} \quad F^a_{ijb} = 0. \tag{11.556}$$

The Faraday tensor vanishes when $n = N$ because the dimension of the embedding space is too small to allow the tangent space to have different orientations at different points x of spacetime. The Faraday tensor, which represents internal curvature, therefore must vanish. One needs at least three dimensions in which to bend a sheet of paper. The embedding space must have $N > 2$ dimensions for $SU(2)$, $N > 3$ for $SU(3)$, and $N > 5$ for $SU(5)$.

The covariant derivative of the internal metric matrix

$$g_{;i} = g_{,i} - g A_i - A^\dagger_i g \tag{11.557}$$

does not vanish and transforms as $(g_{;i})' = V^{-1\dagger} g_{,i} V^{-1}$. A suitable action density for it is the trace $\mathrm{Tr}(g_{;i} g^{-1} g^{;i} g^{-1})$. If the metric matrix assumes a (constant, hermitian) mean value g_0 in the vacuum at low temperatures, then its action is

$$m^2 \mathrm{Tr}\left[(g_0 A_i + A^\dagger_i g_0) g_0^{-1} (g_0 A^i + A^{i\dagger} g_0) g_0^{-1}\right] \tag{11.558}$$

which is a mass term for the matrix of gauge bosons

$$W_i = g_0^{1/2}\, A_i\, g_0^{-1/2} + g_0^{-1/2}\, A^\dagger_i\, g_0^{1/2}. \tag{11.559}$$

This mass mechanism also gives masses to the fermions. To see how, we write the Dirac action density (11.552) as

$$\overline{\psi}_a \left[\gamma^i (\delta^a{}_b \partial_i + A^a_{i\,b}) + m\, \delta^a{}_b\right] \psi^b = \overline{\psi}^a \left[\gamma^i (g_{ab} \partial_i + g_{ac} A^c_{i\,b}) + m\, g_{ab}\right] \psi^b. \tag{11.560}$$

Each fermion now gets a mass $m\,c_i$ proportional to an eigenvalue c_i of the hermitian matrix g_0.

This mass mechanism does not leave behind scalar bosons. Whether Nature ever uses it is unclear.

11.64 Geometry

This section is optional on a first reading.

In gauge theory, what plays the role of spacetime? Could it be the group manifold? Let us consider the gauge group $SU(2)$ whose group manifold is the 3-sphere in flat euclidian 4-space. A point on the 3-sphere is

$$p = \left(\pm\sqrt{1-\boldsymbol{r}^2},\, r^1,\, r^2,\, r^3\right) \tag{11.561}$$

as explained in example 10.31. The coordinates $r^a = r_a$ are not vectors. The three basis vectors are

$$e_a = \frac{\partial p}{\partial r^a} = \left(\mp\frac{r_a}{\sqrt{1-\boldsymbol{r}^2}},\, \delta^1_a,\, \delta^2_a,\, \delta^3_a\right) \tag{11.562}$$

and so the metric $g_{ab} = e_a \cdot e_b$ is

$$g_{ab} = \frac{r_a\, r_b}{1-\boldsymbol{r}^2} + \delta_{ab} \tag{11.563}$$

or

$$\parallel g \parallel = \frac{1}{1-\boldsymbol{r}^2}\begin{pmatrix} 1-r_2^2-r_3^2 & r_1\, r_2 & r_1\, r_3 \\ r_2\, r_1 & 1-r_1^2-r_3^2 & r_2\, r_3 \\ r_3\, r_1 & r_3\, r_2 & 1-r_1^2-r_2^2 \end{pmatrix}. \tag{11.564}$$

The inverse matrix is

$$g^{bc} = \delta_{bc} - r_b\, r_c. \tag{11.565}$$

The dual vectors

$$e^b = g^{bc} e_c = \left(\mp r_b\sqrt{1-\boldsymbol{r}^2},\ \delta^b_1 - r_b r_1,\ \delta^b_2 - r_b r_2,\ \delta^b_3 - r_b r_3\right) \tag{11.566}$$

satisfy $e^b \cdot e_a = \delta^b_a$.

There are two kinds of affine connections $e^b \cdot e_{a,c}$ and $e^b \cdot e_{a,i}$. If we differentiate e_a with respect to an $SU(2)$ coordinate r_c, then

$$E^b_{c\,a} = e^b \cdot e_{a,c} = r_b\left(\delta_{ac} + \frac{r_a\, r_c}{1-\boldsymbol{r}^2}\right) \tag{11.567}$$

in which we used E (for Einstein) instead of Γ for the affine connection. If we differentiate e_a with respect to a spacetime coordinate x^i, then

$$E^b_{i\,a} = e^b \cdot e_{a,i} = e^b \cdot e_{a,c}\; r^c_{,i} = r_b\; r^c_{,i} \left(\delta_{ac} + \frac{r_a\, r_c}{1 - \boldsymbol{r}^2} \right). \tag{11.568}$$

But if the group coordinates r_a are functions of the spacetime coordinates x^i, then there are 4 new basis 4-vectors $e_i = e_a r_{a,i}$. The metric then is a 7×7 matrix $\| \, g \, \|$ with entries $g_{a,b} = e_a \cdot e_b$, $g_{a,k} = e_a \cdot e_k$, $g_{i,b} = e_i \cdot e_b$, and $g_{i,k} = e_i \cdot e_k$ or

$$\| \, g \, \| = \begin{pmatrix} g_{a,b} & g_{a,b}\, r_{b,k} \\ g_{a,b}\, r_{a,i} & g_{a,b}\, r_{a,i}\, r_{b,k}. \end{pmatrix} \tag{11.569}$$

Further Reading

The classics *Gravitation and Cosmology* (Weinberg, 1972), *Gravitation* (Misner et al., 1973), *Cosmology* (Weinberg, 2010), *General Theory of Relativity* (Dirac, 1996), *Gravitation: Foundations and Frontiers* (Padmanabhan, 2010) and the very accessible *Spacetime and Geometry* (Carroll, 2003), *A First Course in General Relativity* (Schutz, 2009), and *Einstein Gravity in a Nutshell* (Zee, 2013) are of special interest.

Exercises

11.1 Compute the derivatives (11.22 & 11.23).

11.2 Show that the transformation $x \to x'$ defined by (11.17) is a rotation and a reflection.

11.3 Show that the matrix (11.39) satisfies the Lorentz condition (11.38).

11.4 If $\eta = L\,\eta\,L^{\mathsf{T}}$, show that $\Lambda = L^{-1}$ satisfies the definition (11.38) of a Lorentz transformation $\eta = \Lambda^{\mathsf{T}} \eta\, \Lambda$.

11.5 The LHC is designed to collide 7 TeV protons against 7 TeV protons for a total collision energy of 14 TeV. Suppose one used a linear accelerator to fire a beam of protons at a target of protons at rest at one end of the accelerator. What energy would you need to see the same physics as at the LHC?

11.6 Use Gauss's law and the Maxwell-Ampère law (11.86) to show that the microscopic (total) current 4-vector $j = (c\rho, \boldsymbol{j})$ obeys the continuity equation $\dot{\rho} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$.

11.7 Show that if M_{ik} is a covariant second-rank tensor with no particular symmetry, then only its antisymmetric part contributes to the 2-form

$M_{ik}\, dx^i \wedge dx^k$ and only its symmetric part contributes to the quantity $M_{ik}\, dx^i dx^k$.

11.8 In rectangular coordinates, use the Levi-Civita identity (1.455) to derive the curl-curl equations (11.89).

11.9 Derive the Bianchi identity (11.91) from the definition (11.78) of the Faraday field-strength tensor, and show that it implies the two homogeneous Maxwell equations (11.81).

11.10 Show that if A is a p-form, then $d(AB) = dA \wedge B + (-1)^p A \wedge dB$.

11.11 Show that if $\omega = a_{ij} dx^i \wedge dx^j/2$ with $a_{ij} = -a_{ji}$, then

$$d\omega = \frac{1}{3!}\left(\partial_k a_{ij} + \partial_i a_{jk} + \partial_j a_{ki}\right) dx^i \wedge dx^j \wedge dx^k. \tag{11.570}$$

11.12 (a) Derive the metric (**??**) for the hyperboloid H^2 from the inner products of $\boldsymbol{e}_r$ and $\boldsymbol{e}_\phi$ by using the metric $\eta = \mathrm{diag}(1, 1, -1)$. (b) Derive the line element (**??**) and metric (**??**) of the hyperboloid H^2 from its line element (11.153) in the embedding space $\mathbb{R}^3$.

11.13 Using tensor notation throughout, derive (11.170) from (11.168 & 11.169).

11.14 Use the flat-space formula (11.191) to compute the change $\boldsymbol{dp}$ due to $d\rho$, $d\phi$, and dz, and so derive the expressions (11.192) for the orthonormal basis vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{z}}$.

11.15 Similarly, derive (11.200) from (11.198).

11.16 Use the definition (11.215) to show that in flat 3-space, the dual of the Hodge dual is the identity: $**dx^i = dx^i$ and $**(dx^i \wedge dx^k) = dx^i \wedge dx^k$.

11.17 Use the definition of the Hodge star (11.226) to derive (a) two of the four identities (11.227) and (b) the other two.

11.18 Show that Levi-Civita's 4-symbol obeys the identity (11.231).

11.19 Show that $\epsilon_{\ell mn}\, \epsilon^{pmn} = 2\, \delta^p_\ell$.

11.20 Show that $\epsilon_{k\ell mn}\, \epsilon^{p\ell mn} = 3!\, \delta^p_k$.

11.21 (a) Using the formulas (11.200) for the basis vectors of spherical coordinates in terms of those of rectangular coordinates, compute the derivatives of the unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ with respect to the variables r, θ, and ϕ and express them in terms of the basis vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$. (b) Using the formulas of (a) and our expression (6.28) for the gradient in spherical coordinates, derive the formula (11.342) for the laplacian $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$.

11.22 Consider the torus with coordinates θ, ϕ labeling the arbitrary point

$$\boldsymbol{p} = (\cos\phi(R + r\sin\theta), \sin\phi(R + r\sin\theta), r\cos\theta) \tag{11.571}$$

in which $R > r$. Both θ and ϕ run from 0 to 2π. (a) Find the basis vectors e_θ and e_ϕ. (b) Find the metric tensor and its inverse.

11.23 For the same torus, (a) find the dual vectors e^θ and e^ϕ and (b) find the nonzero connections Γ^i_{jk} where i, j, & k take the values θ & ϕ.

11.24 For the same torus, (a) find the two Christoffel matrices Γ_θ and Γ_ϕ, (b) find their commutator $[\Gamma_\theta, \Gamma_\phi]$, and (c) find the elements $R^\theta_{\theta\theta\theta}$, $R^\phi_{\theta\phi\theta}$, $R^\theta_{\phi\theta\phi}$, and $R^\phi_{\phi\phi\phi}$ of the curvature tensor.

11.25 Find the curvature scalar R of the torus with points (11.571). **Hint:** In these four problems, you may imitate the corresponding calculation for the sphere in Sec. 11.47.

11.26 By differentiating the identity $g^{ik}\, g_{k\ell} = \delta^i_\ell$, show that $\delta g^{ik} = -g^{is} g^{kt} \delta g_{st}$ or equivalently that $dg^{ik} = -\, g^{is} g^{kt} dg_{st}$.

11.27 Just to get an idea of the sizes involved in black holes, imagine an isolated sphere of matter of uniform density ρ that as an initial condition is all at rest within a radius r_b. Its radius will be less than its Schwarzschild radius if

$$r_b < \frac{2MG}{c^2} = 2\left(\frac{4}{3}\pi r_b^3 \rho\right)\frac{G}{c^2}. \tag{11.572}$$

If the density ρ is that of water under standard conditions (1 gram per cc), for what range of radii r_b might the sphere be or become a black hole? Same question if ρ is the density of dark energy.

11.28 For the points (11.451), derive the metric (11.454) with $k = 1$. Don't forget to relate $d\chi$ to dr.

11.29 For the points (11.452), derive the metric (11.454) with $k = 0$.

11.30 For the points (11.453), derive the metric (11.454) with $k = -1$. Don't forget to relate $d\chi$ to dr.

11.31 Suppose the constant k in the Roberson-Walker metric (11.450 or 11.454) is some number other than 0 or ± 1. Find a coordinate transformation such that in the new coordinates, the Roberson-Walker metric has $k = k/|k| = \pm 1$. Hint: You also can change the scale factor a.

11.32 Derive the affine connections in Eq.(11.458).

11.33 Derive the affine connections in Eq.(11.459).

11.34 Derive the affine connections in Eq.(11.460).

11.35 Derive the spatial Einstein equation (11.470) from (**??**, 11.454, 11.465, 11.467, & 11.468).

11.36 Assume there had been no inflation, no era of radiation, and no dark energy. In this case, the magnitude of the difference $|\Omega - 1|$ would have increased as $t^{2/3}$ over the past 13.8 billion years. Show explicitly how

close to unity Ω would have had to have been at $t = 1\,\mathrm{s}$ so as to satisfy the observational constraint $|\Omega_0 - 1| < 0.036$ on the present value of Ω.

11.37 Derive the relation (11.488) between the energy density ρ and the Robertson-Walker scale factor $a(t)$ from the conservation law (11.485) and the equation of state $p_i = w_i \rho_i$.

11.38 Use the Friedmann equations (11.469 & 11.493) for constant $\rho = -p$ and $k = 1$ to derive (11.509) subject to the boundary condition that $a(t)$ has its minimum at $t = 0$.

11.39 Use the Friedmann equations (11.469 & 11.493) with $w = -1$, ρ constant, and $k = -1$ to derive (11.510) subject to the boundary condition that $a(0) = 0$.

11.40 Use the Friedmann equations (11.469 & 11.493) with $w = -1$, ρ constant, and $k = 0$ to derive (11.508). Show why a linear combination of the two solutions (11.508) does not work.

11.41 Use the conservation equation (11.514) and the Friedmann equations (11.469 & 11.493) with $w = 1/3$, $k = 0$, and $a(0) = 0$ to derive (11.517).

11.42 Show that if the matrix $U(x)$ is nonsingular, then

$$(\partial_i U)\, U^{-1} = -U\, \partial_i U^{-1}. \tag{11.573}$$

11.43 The gauge-field matrix is a linear combination $A_k = -ig\, t^b\, A_k^b$ of the generators t^b of a representation of the gauge group. The generators obey the commutation relations

$$[t^a, t^b] = i f_{abc} t^c \tag{11.574}$$

in which the f_{abc} are the structure constants of the gauge group. Show that under a gauge transformation (11.535)

$$A_i' = U A_i U^{-1} - (\partial_i U)\, U^{-1} \tag{11.575}$$

by the unitary matrix $U = \exp(-ig\lambda^a t^a)$ in which λ^a is infinitesimal, the gauge-field matrix A_i transforms as

$$-ig A_i'^a t^a = -ig A_i^a t^a - ig^2 f_{abc} \lambda^a A_i^b t^c + ig \partial_i \lambda^a t^a. \tag{11.576}$$

Show further that the gauge field transforms as

$$A_i'^a = A_i^a - \partial_i \lambda^a - g f_{abc} A_i^b \lambda^c. \tag{11.577}$$

11.44 Show that if the vectors $e_a(x)$ are orthonormal, then $e^{a\dagger} \cdot e_{c,i} = -e_{,i}^{a\dagger} \cdot e_c$.

11.45 Use the identity of exercise 11.44 to derive the formula (11.555) for the nonabelian Faraday tensor.

11.46 Using the tricks of section 11.40, show that $\delta\sqrt{g} = -\frac{1}{2}\sqrt{g}\, g_{ik}\, \delta g^{ik}$. This relation and the definition (11.399) $R = g^{ik} R^n_{ink}$ imply that the first-order change in the Einstein-Hilbert action is (11.426) apart from an irrelevant surface term (Carroll, 2003, chap 4.3) due to $g^{ik}\sqrt{g}\ \delta R^n_{ink}$.

11.47 Write Dirac's action density in the explicitly hermitian form $L_D = -\frac{1}{2}\overline{\psi}\gamma^i\partial_i\psi - \frac{1}{2}\left[\overline{\psi}\gamma^i\partial_i\psi\right]^\dagger$ in which the field ψ has the invariant form $\psi = e_a\psi_a$ and $\overline{\psi} = i\psi^\dagger\gamma^0$. Use the identity $\left[\overline{\psi}_a\gamma^i\psi_b\right]^\dagger = -\overline{\psi}_b\gamma^i\psi_a$ to show that the gauge-field matrix A_i defined as the coefficient of $\overline{\psi}_a\gamma^i\psi_b$ as in $\overline{\psi}_a\gamma^i(\partial_i + iA_{iab})\psi_b$ is hermitian $A^*_{iab} = A_{iba}$.