11
Tensors and Local Symmetries

11.1 Points and Coordinates

A point on a curved surface or in a curved space also is a point in a higher-dimensional flat space called an embedding space. For instance, a point on a sphere also is a point in three-dimensional euclidean space and in four-dimensional spacetime. One always can add extra dimensions, but it’s simpler to use as few as possible, three in the case of a sphere.

On a sufficiently small scale, any reasonably smooth space locally looks like n-dimensional euclidean space. Such a space is called a manifold. Incidentally, according to Whitney’s embedding theorem, every n-dimensional connected, smooth manifold can be embedded in 2n-dimensional euclidean space $\mathbb{R}^{2n}$. So the embedding space for such spaces in general relativity has no more than eight dimensions.

We use coordinates to label points. For example, we can choose a polar axis and a meridian and label a point on the sphere by its polar and azimuthal angles $(\theta, \phi)$ with respect to that axis and meridian. If we use a different axis and meridian, then the coordinates $(\theta', \phi')$ for the same point will change. Points are physical, coordinates are metaphysical. When we change our system of coordinates, the points don’t change, but their coordinates do.

Most points $p$ have unique coordinates $x^i(p)$ and $x'^i(p)$ in their coordinate systems. For instance, polar coordinates $(\theta, \phi)$ are unique for all points on a sphere — except the north and south poles which are labeled by $\theta = 0$ and $\theta = \pi$ and all $0 \leq \phi < 2\pi$. By using more than one coordinate system, one usually can arrange to label every point uniquely in some coordinate system. In the flat three-dimensional space in which the sphere is a surface, each point of the sphere has unique coordinates, $\vec{p} = (x, y, z)$. 
We will use coordinate systems that represent the points of a manifold uniquely and smoothly at least in local patches, so that the maps

\[ x^i = x^i(p) = x^i(p(x)) = x^i(x) \] (11.1)

and

\[ x^i = x^i(p) = x^i(p(x')) = x^i(x') \] (11.2)

are well defined, differentiable, and one to one in the patches. We’ll often group the \( n \) coordinates \( x^i \) together and write them collectively as \( x \) without a superscript. Since the coordinates \( x(p) \) label the point \( p \), we sometimes will call them “the point \( x \).” But \( p \) and \( x \) are different. The point \( p \) is unique with infinitely many coordinates \( x, x', x'', \ldots \) in infinitely many coordinate systems.

### 11.2 Scalars

A **scalar** is a quantity \( B \) that is the same in all coordinate systems

\[ B' = B. \] (11.3)

If it also depends upon the coordinates \( x \) of the spacetime point \( p \), and

\[ B'(x') = B(x) \] (11.4)

then it is a **scalar field**.

### 11.3 Contravariant Vectors

The change \( dx^i \) due to changes in the unprimed coordinates is

\[ dx^i = \sum_j \frac{\partial x^h}{\partial x^j} dx^j. \] (11.5)

This rule defines **contravariant vectors**: a quantity \( A^i \) is a contravariant vector if it transforms like \( dx^i \)

\[ A'^i = \sum_j \frac{\partial x'^h}{\partial x^j} A^j. \] (11.6)

The coordinate differentials \( dx^i \) form a contravariant vector. A contravariant vector \( A^i(x) \) that depends on the coordinates \( x \) and transforms as

\[ A'^i(x') = \sum_j \frac{\partial x'^i}{\partial x^j} A^j(x) \] (11.7)

is a **contravariant vector field**.
11.4 Covariant Vectors

The chain rule for partial derivatives
\[ \frac{\partial}{\partial x'^i} = \sum_j \frac{\partial x^j}{\partial x'^i} \frac{\partial}{\partial x^j} \]  
(11.8)
deﬁnes covariant vectors: a vector \( C_i \) that transforms as
\[ C'_i = \sum_j \frac{\partial x'^i}{\partial x'^j} C_j \]  
(11.9)
is a covariant vector. A covariant vector \( C_i(x) \) that depends on the coordinates \( x \) and transforms as
\[ C'_i(x') = \sum_j \frac{\partial x'^i}{\partial x'^j} C_j(x) \]  
(11.10)
is a covariant vector ﬁeld.

Example 11.1 (Gradient of a Scalar). The derivatives of a scalar ﬁeld form a covariant vector ﬁeld. For by using the chain rule to differentiate the equation \( B'(x') = B(x) \) that deﬁnes a scalar ﬁeld, one ﬁnds
\[ \frac{\partial B'(x')}{\partial x'^i} = \frac{\partial B(x)}{\partial x'^i} = \sum_j \frac{\partial x^j}{\partial x'^i} \frac{\partial B(x)}{\partial x^j} \]  
(11.11)
which shows that the gradient \( \partial B(x)/\partial x^j \) is a covariant vector ﬁeld. □

11.5 Euclidean Space in Euclidean Coordinates

If we use euclidean coordinates to describe points in euclidean space, then covariant and contravariant vectors are the same.

Euclidean space has a natural inner product (section 1.6), the usual dot-product, which is real and symmetric. In a euclidean space of \( n \) dimensions, we may choose any \( n \) ﬁxed, orthonormal basis vectors \( e_i \)
\[ (e_i, e_j) \equiv e_i \cdot e_j = \sum_{k=1}^{n} e_i^k e_j^k = \delta_{ij} \]  
(11.12)
and use them to represent any point \( p \) as the linear combination
\[ p = \sum_{i=1}^{n} e_i x^i. \]  
(11.13)
The coefficients $x^i$ are the euclidean coordinates in the $e_i$ basis. Since the basis vectors $e_i$ are orthonormal, each $x^i$ is an inner product or dot product

$$x^i = e_i \cdot p = \sum_{j=1}^{n} e_i \cdot e_j x^j = \sum_{j=1}^{n} \delta_{ij} x^j. \quad (11.14)$$

The dual vectors $e^i$ are defined as those vectors whose inner products with the $e_j$ are $(e^i, e_j) = \delta^j_i$. In this section, they are the same as the vectors $e_i$, and so we shall not bother to distinguish $e^i$ from $e_i = e^i$.

If we use different orthonormal vectors $e'_i$ as a basis

$$p = \sum_{i=1}^{n} e'_i x^i \quad (11.15)$$

then we get new euclidean coordinates $x'_i = e'_i \cdot p$ for the same point $p$. These two sets of coordinates are related by the equations

$$x'^i = e'_i \cdot p = \sum_{j=1}^{n} e'_i \cdot e_j x^j \quad (11.16)$$

$$x^j = e_j \cdot p = \sum_{k=1}^{n} e_j \cdot e'_k x'^k. \quad (11.17)$$

Because the basis vectors $e$ and $e'$ are all independent of $x$, the coefficients $\partial x'^i / \partial x^j$ of the transformation laws for contravariant (11.6) and covariant (11.9) vectors are

$$\text{contravariant} \quad \frac{\partial x'^i}{\partial x^j} = e'_i \cdot e_j \quad \text{and} \quad \frac{\partial x^j}{\partial x'^i} = e_j \cdot e'_i \quad (11.18)$$

But the dot-product (1.82) is symmetric, and so these are the same:

$$\frac{\partial x'^i}{\partial x^j} = e'_i \cdot e_j = e_j \cdot e'_i = \frac{\partial x^j}{\partial x'^i}. \quad (11.19)$$

Contravariant and covariant vectors transform the same way in euclidean space with euclidean coordinates.

The relations between $x'^i$ and $x^j$ imply that

$$x'^i = \sum_{j,k=1}^{n} (e'_i \cdot e_j) (e_j \cdot e'_k) x'^k. \quad (11.20)$$

Since this holds for all coordinates $x'^i$, we have

$$\sum_{j=1}^{n} (e'_i \cdot e_j) (e_j \cdot e'_k) = \delta_{ik}. \quad (11.20)$$
The coefficients $e_i' \cdot e_j$ form an orthogonal matrix, and the linear operator

$$\sum_{i=1}^{n} e_i e_i'^\top = \sum_{i=1}^{n} |e_i\rangle\langle e_i'|$$  (11.21)

is an orthogonal (real, unitary) transformation. The change $x \to x'$ is a rotation plus a possible reflection (exercise 11.2).

**Example 11.2** (A Euclidean Space of Two Dimensions). In two-dimensional euclidean space, one can describe the same point by euclidean $(x, y)$ and polar $(r, \theta)$ coordinates. The derivatives

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{\partial y}{\partial r}$$  (11.22)

respect the symmetry (11.18), but (exercise 11.1) these derivatives

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} \neq \frac{\partial x}{\partial \theta} = -y \quad \text{and} \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} \neq \frac{\partial y}{\partial \theta} = x$$  (11.23)

do not because polar coordinates are not euclidean.

### 11.6 Summation Conventions

When a given index is repeated in a product, that index usually is being summed over. So to avoid distracting summation symbols, one writes

$$A_i B_i \equiv \sum_{i=1}^{n} A_i B_i.$$  (11.24)

The sum is understood to be over the relevant range of indices, usually from 0 or 1 to 3 or $n$. Where the distinction between covariant and contravariant indices matters, an index that appears twice in the same monomial, once as a subscript and once as a superscript, is a dummy index that is summed over as in

$$A_i B^i \equiv \sum_{i=1}^{n} A_i B^i.$$  (11.25)

These summation conventions make tensor notation almost as compact as matrix notation. They make equations easier to read and write.
Example 11.3 (The Kronecker Delta). The summation convention and the chain rule imply that
\[ \frac{\partial x^i}{\partial x^k} \frac{\partial x^k}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \] (11.26)

The repeated index \( k \) has disappeared in this contraction. \( \square \)

11.7 Minkowski Space

Minkowski space has one time dimension, labeled by \( k = 0 \), and \( n \) space dimensions. In special relativity \( n = 3 \), and the Minkowski metric \( \eta \)
\[ \eta_{k\ell} = \eta^{k\ell} = \begin{cases} -1 & \text{if } k = \ell = 0 \\ 1 & \text{if } 1 \leq k = \ell \leq 3 \\ 0 & \text{if } k \neq \ell \end{cases} \] (11.27)
defines an inner product between points \( p \) and \( q \) with coordinates \( x^k_p \) and \( x^\ell_q \) as
\[ (p, q) = p \cdot q = p^k \eta_{k\ell} q^\ell = (q, p). \] (11.28)

If one time component vanishes, the Minkowski inner product reduces to the euclidean dot-product (1.82).

We can use different sets \( \{e_i\} \) and \( \{e'_i\} \) of \( n + 1 \) Lorentz-orthonormal basis vectors
\[ (e_i, e_j) = e_i \cdot e_j = e^k_i \eta_{k\ell} e^\ell_j = \eta_{ij} = e'_i \cdot e'_j = (e'_i, e'_j) \] (11.29)
to represent any point \( p \) in the space either as a linear combination of the vectors \( e_i \) with coefficients \( x^i \) or as a linear combination of the vectors \( e'_i \) with coefficients \( x'^i \)
\[ p = e_i x^i = e'_i x'^i. \] (11.30)

The dual vectors, which carry upper indices, are defined as
\[ e^i = \eta^{ij} e_j \quad \text{and} \quad e'^i = \eta^{ij} e'_j. \] (11.31)

They are orthonormal to the vectors \( e_i \) and \( e'_i \) because
\[ (e^i, e_j) = e^i \cdot e_j = \eta^{ik} e_k \cdot e_j = \eta^{ik} \eta_{k\ell} = \delta^i_j \] (11.32)
and similarly \( (e'^i, e'_j) = e'^i \cdot e'_j = \delta^i_j \). Since the square of the matrix \( \eta \) is the identity matrix \( \eta_{kl} \eta^{lj} = \delta^i_l \), it follows that
\[ e_i = \eta_{ij} e^j \quad \text{and} \quad e'_i = \eta_{ij} e'^j. \] (11.33)
The metric \( \eta \) raises (11.31) and lowers (11.33) the index of a basis vector. The component \( x^i \) is related to the components \( x^j \) by the linear map

\[
x^i = e^i \cdot p = e^i \cdot e_j x^j.
\] (11.34)

Such a map from a 4-vector \( x \) to a 4-vector \( x' \) is a Lorentz transformation

\[
x'^i = L^i_j x^j \quad \text{with matrix} \quad L^i_j = e^i \cdot e_j.
\] (11.35)

The inner product \((p, q)\) of two points \( p = e_i x^i = e'_i x'^i \) and \( q = e_k y^k = e'_k y'^k \) is physical and so is invariant under Lorentz transformations

\[
(p, q) = x^i y^k e_i \cdot e_k = \eta_{ik} x^i y^k = x^i y^k e'_i \cdot e'_k = \eta_{ik} x'^i y'^k.
\] (11.36)

With \( x'^i = L^i_r x^r \) and \( y'^k = L^k_s y^s \), this invariance is

\[
\eta_{rs} x'^r y'^s = \eta_{ik} L^i_r x^r L^k_s y^s = \eta_{ik} x'^i y'^k
\] (11.37)

or since \( x^r \) and \( y^s \) are arbitrary

\[
\eta_{rs} = \eta_{ik} L^i_r L^k_s = L^i_r \eta_{ik} L^k_s.
\] (11.38)

In matrix notation, a left index labels a row, and a right index labels a column. Transposition interchanges rows and columns \( L^i_r = L^T_r i \), so

\[
\eta_{rs} = L^T_r i \eta_{ik} L^k_s \quad \text{or} \quad \eta = L^T \eta L
\] (11.39)

in matrix notation. In such matrix products, the height of an index—whether it is up or down—determines whether it is contravariant or covariant but does not affect its place in its matrix.

**Example 11.4 (A Boost).** The matrix

\[
L = \begin{pmatrix}
\gamma & \sqrt{\gamma^2 - 1} & 0 & 0 \\
\sqrt{\gamma^2 - 1} & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] (11.40)

where \( \gamma = 1/\sqrt{1 - v^2/c^2} \) represents a Lorentz transformation that is a boost in the \( x \)-direction. Boosts and rotations are Lorentz transformations. Working with \( 4 \times 4 \) matrices can get tedious, so students are advised to think in terms of scalars, like \( p \cdot x = p^i \eta_{ij} x^j = p \cdot x - Et \) whenever possible.

If the basis vectors \( e \) and \( e' \) are independent of \( p \) and of \( x \), then the coefficients of the transformation law (11.6) for contravariant vectors are

\[
\frac{\partial x'^i}{\partial x^j} = e'^i \cdot e_j.
\] (11.41)
Similarly, the component $x^j$ is $x^j = e^j \cdot p = e^j \cdot e'_i x^i$, so the coefficients of the transformation law (11.9) for covariant vectors are

$$\frac{\partial x^j}{\partial x'^i} = e^j \cdot e'_i. \quad (11.42)$$

Using $\eta$ to raise and lower the indices in the formula (11.41) for the coefficients of the transformation law (11.6) for contravariant vectors, we find, since the inner product (11.28) is symmetric,

$$\frac{\partial x'^i}{\partial x^j} = e'^i \cdot e_j = \eta^i_k \eta_{j\ell} e'_k \cdot e^\ell = \eta^i_k \eta_{j\ell} \frac{\partial x^\ell}{\partial x'^k} \quad (11.43)$$

which is $\pm \partial x^j / \partial x'^i$. So if we use coordinates associated with fixed basis vectors $e_i$ in Minkowski space, then the coefficients for the two kinds of transformation laws differ only by occasional minus signs.

Thus if $A^i$ is a contravariant vector

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j \quad (11.44)$$

then the relation (11.43) between the two kinds of coefficients implies that

$$\eta_{si} A'^i = \eta_{si} \frac{\partial x'^i}{\partial x^j} A^j = \eta_{si} \eta^i_k \eta_{j\ell} \frac{\partial x^\ell}{\partial x'^k} A^j = \eta_{si} \delta^k_s \eta_{j\ell} A^j = \frac{\partial x^\ell}{\partial x'^k} \eta_{lj} A^j \quad (11.45)$$

which shows that $A_\ell = \eta_{lj} A^j$ transforms covariantly

$$A'_s = \frac{\partial x^\ell}{\partial x'^s} A_\ell. \quad (11.46)$$

The metric $\eta$ turns a contravariant vector into a covariant one. It also switches a covariant vector $A_\ell$ back to its contravariant form $A^k$

$$\eta^{kl} A_\ell = \eta^{kl} \eta_{lj} A^j = \delta^k_j A^j = A^k. \quad (11.47)$$

In Minkowski space, one uses $\eta$ to raise and lower indices

$$A_i = \eta_{ij} A^j \text{ and } A^i = \eta^{ij} A_j. \quad (11.48)$$

In general relativity, the spacetime metric $g$ raises and lowers indices.

### 11.8 Lorentz Transformations

In section 11.7, Lorentz transformations arose as linear maps of the coordinates due to a change of basis. They also are linear maps of the basis vectors $e_i$ that preserve the inner products

$$(e_i, e_j) = e_i \cdot e_j = \eta_{ij} = e'_i \cdot e'_j = (e'_i, e'_j). \quad (11.49)$$
The vectors $e_i$ are four linearly independent four-dimensional vectors, and so they span four-dimensional Minkowski space and can represent the vectors $e'_i$ as

$$e'_i = \Lambda_i^k e_k$$  \hspace{1cm} (11.50)

where the coefficients $\Lambda_i^k$ are real numbers. The requirement that the new basis vectors $e'_i$ are Lorentz orthonormal gives

$$\eta_{ij} = e'_i \cdot e'_j = \Lambda_i^k e_k \cdot \Lambda_j^\ell e_\ell = \Lambda_i^k e_k \cdot e_\ell \Lambda_j^\ell = \Lambda_i^k \eta_{k\ell} \Lambda_j^\ell$$  \hspace{1cm} (11.51)

or in matrix notation

$$\eta = \Lambda \eta \Lambda^T$$  \hspace{1cm} (11.52)

where $\Lambda^T$ is the transpose $(\Lambda^T)^\ell_j = \Lambda_j^\ell$. Evidently $\Lambda^T$ satisfies the definition (11.39) of a Lorentz transformation. What Lorentz transformation is it? The point $p$ must remain invariant, so by (11.35 & 11.50) one has

$$p = e'_i x^i = \Lambda_i^k e_k L^i_j x^j = \delta^k_j e_k x^j = e_j x^j$$  \hspace{1cm} (11.53)

whence $\Lambda_i^k L^i_j = \delta^k_j$ or $\Lambda^T L = I$. So $\Lambda^T = L^{-1}$.

By multiplying condition (11.52) by the metric $\eta$ first from the left and then from the right and using the fact that $\eta^2 = I$, we find

$$1 = \eta^2 = \eta \Lambda \eta \Lambda^T = \Lambda \eta \Lambda^T \eta$$  \hspace{1cm} (11.54)

which gives us the inverse matrices

$$\Lambda^{-1} = \eta \Lambda^T \eta = L^T \quad \text{and} \quad (\Lambda^T)^{-1} = \eta \Lambda \eta = L.$$  \hspace{1cm} (11.55)

In special relativity, contravariant vectors transform as

$$dx^i = L^i_j \, dx^j$$  \hspace{1cm} (11.56)

and since $x^j = L^{-1}{}^j_i \, x^i$, the covariant ones transform as

$$\frac{\partial}{\partial x^i} = \frac{\partial x^j}{\partial x^i} \frac{\partial}{\partial x^j} = L^{-1}{}^i_j \frac{\partial}{\partial x^j} = \Lambda^{-1}{}^j_i \frac{\partial}{\partial x^j}.$$  \hspace{1cm} (11.57)

By taking the determinant of both sides of (11.52) and using the transpose (1.194) and product (1.207) rules for determinants, we find that $\det \Lambda = \pm 1$.

**11.9 Special Relativity**

The spacetime of special relativity is flat, four-dimensional Minkowski space. The inner product $(p - q) \cdot (p - q)$ of the interval $p - q$ between two points is physical and independent of the coordinates and therefore invariant. If the
points \( p \) and \( q \) are close neighbors with coordinates \( x^i + dx^i \) for \( p \) and \( x^i \) for \( q \), then that invariant inner product is

\[
(p - q) \cdot (p - q) = e_i \, dx^i \cdot e_j \, dx^j = dx^i \eta_{ij} \, dx^j = dx^2 - (dx^0)^2
\]

(11.58)

with \( dx^0 = c \, dt \). (At some point in what follows, we’ll measure distance in light-seconds so that \( c = 1 \).) If the points \( p \) and \( q \) are on the trajectory of a massive particle moving at velocity \( \mathbf{v} \), then this invariant quantity is the square of the **invariant distance**

\[
ds^2 = dx^2 - c^2 dt^2 = (v^2 - c^2) \, dt^2
\]

(11.59)

which always is negative since \( v < c \). The time in the rest frame of the particle is the **proper time**. The square of its differential element is

\[
d\tau^2 = -ds^2/c^2 = (1 - v^2/c^2) \, dt^2.
\]

(11.60)

A particle of mass zero moves at the speed of light, and so its element \( d\tau \) of proper time is zero. But for a particle of mass \( m > 0 \) moving at speed \( v \), the element of proper time \( d\tau \) is smaller than the corresponding element of laboratory time \( dt \) by the factor \( \sqrt{1 - v^2/c^2} \). The proper time is the time in the rest frame of the particle, \( d\tau = dt \) when \( v = 0 \). So if \( T(0) \) is the lifetime of a particle at rest, then the apparent lifetime \( T(v) \) when the particle is moving at speed \( v \) is

\[
T(v) = dt = \frac{d\tau}{\sqrt{1 - v^2/c^2}} = \frac{T(0)}{\sqrt{1 - v^2/c^2}}
\]

(11.61)

which is longer — an effect known as **time dilation**.

**Example 11.5** (Time Dilation in Muon Decay). A muon at rest has a mean life of \( T(0) = 2.2 \times 10^{-6} \) seconds. Cosmic rays hitting nitrogen and oxygen nuclei make pions high in the Earth’s atmosphere. The pions rapidly decay into muons in \( 2.6 \times 10^{-8} \) s. A muon moving at the speed of light from 10 km takes at least \( t = 10 \, \text{km}/300,000 \, \text{km/sec} = 3.3 \times 10^{-5} \) s to hit the ground. Were it not for time dilation, the probability \( P \) of such a muon reaching the ground as a muon would be

\[
P = e^{-t/T(0)} = \exp(-33/2.2) = e^{-15} = 2.6 \times 10^{-7}.
\]

(11.62)

The (rest) mass of a muon is 105.66 MeV. So a muon of energy \( E = 749 \) MeV has by (11.69) a time-dilation factor of

\[
\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{E}{mc^2} = \frac{749}{105.7} = 7.089 = \frac{1}{\sqrt{1 - (0.99)^2}}.
\]

(11.63)
So a muon moving at a speed of $v = 0.99c$ has an apparent mean life $T(v)$ given by equation (11.61) as

$$T(v) = \frac{E}{mc^2} T(0) = \frac{T(0)}{\sqrt{1 - v^2/c^2}} = \frac{2.2 \times 10^{-6} \text{s}}{\sqrt{1 - (0.99)^2}} = 1.6 \times 10^{-5} \text{s}. \quad (11.64)$$

The probability of survival with time dilation is

$$P = e^{-t/T(v)} = \exp(-33/16) = 0.12 \quad (11.65)$$

so that 12% survive. Time dilation increases the chance of survival by a factor of 460,000 — no small effect.

### 11.10 Kinematics

From the scalar $d\tau$, and the contravariant vector $dx^i$, we can make the 4-vector

$$u^i = \frac{dx^i}{d\tau} = \frac{dt}{d\tau} \left( \frac{dx^0}{dt}, \frac{dx}{dt} \right) = \frac{1}{\sqrt{1 - v^2/c^2}} (c, v) \quad (11.66)$$

in which $u^0 = c dt/d\tau = c/\sqrt{1 - v^2/c^2}$ and $u = u^0 v/c$. The product $mu^i$ is the energy-momentum 4-vector $p^i$

$$p^i = m u^i = m \frac{dx^i}{d\tau} = m \frac{dt}{d\tau} \frac{dx^i}{dt} = \frac{m}{\sqrt{1 - v^2/c^2}} \frac{dx^i}{dt}$$

$$= \frac{m}{\sqrt{1 - v^2/c^2}} (c, v) = \left( \frac{E}{c}, p \right). \quad (11.67)$$

Its invariant inner product is a constant characteristic of the particle and proportional to the square of its mass

$$c^2 p^i p_i = mc u^i mc u_i = -E^2 + c^2 p^2 = -m^2 c^4. \quad (11.68)$$

Note that the time-dilation factor is the ratio of the energy of a particle to its rest energy

$$\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{E}{mc^2} \quad (11.69)$$

and the velocity of the particle is its momentum divided by its equivalent mass $E/c^2$

$$v = \frac{p}{E/c^2}. \quad (11.70)$$
The analog of $\mathbf{F} = m \mathbf{a}$ is
\[ m \frac{d^2 x^i}{d \tau^2} = m \frac{d u^i}{d \tau} = \frac{d p^i}{d \tau} = f^i \] (11.71)
in which $p^0 = E/c$, and $f^i$ is a 4-vector force.

**Example 11.6** (Time Dilation and Proper Time). In the frame of a laboratory, a particle of mass $m$ with 4-momentum $p_{\text{lab}}^i = (E/c, p, 0, 0)$ travels a distance $L$ in a time $t$ for a 4-vector displacement of $x_{\text{lab}}^i = (ct, L, 0, 0)$. In its own rest frame, the particle’s 4-momentum and 4-displacement are $p_{\text{rest}}^i = (mc, 0, 0, 0)$ and $x_{\text{rest}}^i = (c\tau, 0, 0, 0)$. Since the Minkowski inner product of two 4-vectors is Lorentz invariant, we have
\[ (p^i x_i)_{\text{rest}} = (p^i x_i)_{\text{lab}} \quad \text{or} \quad Et - pL = mc^2\tau = mc^2t\sqrt{1 - v^2/c^2} \quad (11.72) \]
so a massive particle’s phase $\exp(-ip^i x_i / \hbar)$ is $\exp(imc^2\tau / \hbar)$. 

**Example 11.7** ($p + \pi \rightarrow \Sigma + K$). What is the minimum energy that a beam of pions must have to produce a sigma hyperon and a kaon by striking a proton at rest? Conservation of the energy-momentum 4-vector gives $p_p + p_{\pi} = p_{\Sigma} + p_K$. We set $c = 1$ and use this equality in the invariant form $(p_p + p_{\pi})^2 = (p_{\Sigma} + p_K)^2$. We compute $(p_p + p_{\pi})^2$ in the the $p_p = (m_p, 0)$ frame and set it equal to $(p_{\Sigma} + p_K)^2$ in the frame in which the spatial momenta of the $\Sigma$ and the $K$ cancel:
\[
(p_p + p_{\pi})^2 = p_p^2 + p_{\pi}^2 + 2p_p \cdot p_{\pi} = -m_p^2 - m_{\pi}^2 - 2m_p E_{\pi} = (p_{\Sigma} + p_K)^2 = -(m_{\Sigma} + m_K)^2. \quad (11.73)
\]
Thus, since the relevant masses (in MeV) are $m_{\Sigma} = 1189.4$, $m_K = 493.7$, $m_p = 938.3$, and $m_{\pi} = 139.6$, the minimum total energy of the pion is
\[
E_{\pi} = \frac{(m_{\Sigma} + m_K)^2 - m_p^2 - m_{\pi}^2}{2m_p} \approx 1030 \quad \text{MeV} \quad (11.74)
\]
of which 890 MeV is kinetic.
11.11 Electrodynamics

In electrodynamics and in MKSA (SI) units, the three-dimensional vector potential \( \mathbf{A} \) and the scalar potential \( \phi \) form a covariant 4-vector potential

\[
A_i = \left( \frac{-\phi}{c}, \mathbf{A} \right).
\]  

(11.75)

The contravariant 4-vector potential is

\[
A^i = \left( \frac{\phi}{c}, \mathbf{A} \right).
\]

(11.76)

in which \( \partial_j = \frac{\partial}{\partial x^j} \), the sum over the repeated indices \( j \) and \( k \) runs from 1 to 3, and \( \epsilon_{ijk} \) is totally antisymmetric with \( \epsilon_{123} = 1 \). The electric field is

\[
E_i = c \left( \frac{\partial A_0}{\partial x^i} - \frac{\partial A_i}{\partial x^0} \right) = -\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t}
\]  

(11.77)

where \( x^0 = ct \). In 3-vector notation, \( \mathbf{E} \) is given by the gradient of \( \phi \) and the time-derivative of \( \mathbf{A} \)

\[
\mathbf{E} = -\nabla \phi - \dot{\mathbf{A}}.
\]  

(11.78)

In terms of the second-rank, antisymmetric Faraday field-strength tensor

\[
F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} = -F_{ji}
\]  

(11.79)

the electric field is

\[
E_i = c F_{i0}
\]

and the magnetic field \( B_i \) is

\[
B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} = \frac{1}{2} \epsilon_{ijk} \left( \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k} \right) = (\nabla \times \mathbf{A})_i
\]  

(11.80)

where the sum over repeated indices runs from 1 to 3. The inverse equation \( F_{jk} = \epsilon_{jki} B_i \) for spatial \( j \) and \( k \) follows from the Levi-Civita identity (1.453)

\[
\epsilon_{jki} B_i = \frac{1}{2} \epsilon_{jki} \epsilon_{nmm} F_{nm} = \frac{1}{2} \epsilon_{ijk} \epsilon_{inm} F_{nm}
\]

\[
= \frac{1}{2} \left( \delta_{jn} \delta_{km} - \delta_{jm} \delta_{kn} \right) F_{nm} = \frac{1}{2} (F_{jk} - F_{kj}) = F_{jk}.
\]  

(11.81)

In 3-vector notation and MKSA = SI units, Maxwell’s equations are a ban on magnetic monopoles and Faraday’s law, both homogeneous,

\[
\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0
\]  

(11.82)

and Gauss’s law and the Maxwell-Ampère law, both inhomogeneous,

\[
\nabla \cdot \mathbf{D} = \rho_f \quad \text{and} \quad \nabla \times \mathbf{H} = j_f + \dot{\mathbf{D}}.
\]  

(11.83)
Here $\rho_f$ is the density of free charge and $j_f$ is the free current density. By free, we understand charges and currents that do not arise from polarization and are not restrained by chemical bonds. The divergence of $\nabla \times \mathbf{H}$ vanishes (like that of any curl), and so the Maxwell-Ampère law and Gauss’s law imply that free charge is conserved

$$0 = \nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot j_f + \nabla \cdot \dot{D} = \nabla \cdot j_f + \dot{\rho}_f. \quad (11.84)$$

If we use this continuity equation to replace $\nabla \cdot j_f$ with $-\dot{\rho}_f$ in its middle form $0 = \nabla \cdot j_f + \nabla \cdot \dot{D}$, then we see that the Maxwell-Ampère law preserves the Gauss-law constraint in time

$$0 = \nabla \cdot j_f + \nabla \cdot \dot{D} = \frac{\partial}{\partial t} (-\rho_f + \nabla \cdot D). \quad (11.85)$$

Similarly, Faraday’s law preserves the constraint $\nabla \cdot \mathbf{B} = 0$

$$0 = -\nabla \cdot (\nabla \times \mathbf{E}) = \frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0. \quad (11.86)$$

In a linear, isotropic medium, the electric displacement $\mathbf{D}$ is related to the electric field $\mathbf{E}$ by the permittivity $\epsilon$, $\mathbf{D} = \epsilon \mathbf{E}$, and the magnetic or magnetizing field $\mathbf{H}$ differs from the magnetic induction $\mathbf{B}$ by the permeability $\mu$, $\mathbf{H} = \mathbf{B}/\mu$.

On a sub-nanometer scale, the microscopic form of Maxwell’s equations applies. On this scale, the homogeneous equations (11.82) are unchanged, but the inhomogeneous ones are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0 \mu_0 \dot{\mathbf{E}} = \mu_0 \mathbf{j} + \frac{\dot{\mathbf{E}}}{c^2} \quad (11.87)$$

in which $\rho$ and $\mathbf{j}$ are the total charge and current densities, and $\epsilon_0 = 8.854 \times 10^{-12}$ F/m and $\mu_0 = 4\pi \times 10^{-7}$ N/A$^2$ are the electric and magnetic constants, whose product is the inverse of the square of the speed of light, $\epsilon_0 \mu_0 = 1/c^2$. Gauss’s law and the Maxwell-Ampère law (11.87) imply (exercise 11.6) that the microscopic (total) current 4-vector $\mathbf{j} = (\epsilon \rho, \mathbf{j})$ obeys the continuity equation $\dot{\rho} + \nabla \cdot \mathbf{j} = 0$. Electric charge is conserved.

In vacuum, $\rho = j = 0$, $\mathbf{D} = \epsilon_0 \mathbf{E}$, and $\mathbf{H} = \mathbf{B}/\mu_0$, and Maxwell’s equations become

$$\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0$$

$$\nabla \cdot \mathbf{E} = 0 \quad \text{and} \quad \nabla \times \mathbf{B} = \frac{1}{c^2} \dot{\mathbf{E}}. \quad (11.88)$$

Two of these equations $\nabla \cdot \mathbf{B} = 0$ and $\nabla \cdot \mathbf{E} = 0$ are constraints. Taking
the curl of the other two equations, we find
\[ \nabla \times (\nabla \times E) = -\frac{1}{c^2} \ddot{E} \quad \text{and} \quad \nabla \times (\nabla \times B) = -\frac{1}{c^2} \ddot{B}. \] (11.89)
One may use the Levi-Civita identity (1.453) to show (exercise 11.8) that
\[ \nabla \times (\nabla \times E) = \nabla (\nabla \cdot E) - \triangle E \quad \text{and} \quad \nabla \times (\nabla \times B) = \nabla (\nabla \cdot B) - \triangle B \] (11.90)
in which \( \triangle \equiv \nabla^2 \). Since in vacuum the divergence of \( E \) vanishes, and since that of \( B \) always vanishes, these identities and the curl-curl equations (11.89) tell us that waves of \( E \) and \( B \) move at the speed of light
\[ \frac{1}{c^2} \ddot{E} - \triangle E = 0 \quad \text{and} \quad \frac{1}{c^2} \ddot{B} - \triangle B = 0. \] (11.91)
We may write the two homogeneous Maxwell equations (11.82) as
\[ \partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki} = \partial_i (\partial_j A_k - \partial_k A_j) + \partial_k (\partial_i A_j - \partial_j A_i) + \partial_j (\partial_k A_i - \partial_i A_k) = 0 \] (11.92)
(exercise 11.9). This relation, known as the **Bianchi identity**, actually is a generally covariant tensor equation
\[ \epsilon^{ijk} \partial_i F_{jk} = 0 \] (11.93)
in which \( \epsilon^{ijk} \) is totally antisymmetric, as explained in Sec. 11.38. There are four versions of this identity (corresponding to the four ways of choosing three different indices \( i, j, k \) from among four and leaving out one, \( \ell \)). The \( \ell = 0 \) case gives the scalar equation \( \nabla \cdot B = 0 \), and the three that have \( \ell \neq 0 \) give the vector equation \( \nabla \times E + \dot{B} = 0 \).
In tensor notation, the microscopic form of the two inhomogeneous equations (11.87)—the laws of Gauss and Ampère—are
\[ \partial_i F^{ki} = \mu_0 j^k \] (11.94)
in which \( j^k \) is the current 4-vector
\[ j^k = (cp, j). \] (11.95)
The **Lorentz force law** for a particle of charge \( q \) is
\[ m \frac{d^2 x^i}{d\tau^2} = m \frac{du^i}{d\tau} = \frac{dp^i}{dt} = f^i = q F^{ij} \frac{dx_j}{d\tau} = q F^{ij} u_j. \] (11.96)
We may cancel a factor of \( dt/d\tau \) from both sides and find for \( i = 1, 2, 3 \)
\[ \frac{dp^i}{dt} = q \left( -F^{i0} + \epsilon_{ijk} B_k v_j \right) \quad \text{or} \quad \frac{dp}{dt} = q \left( E + v \times B \right) \] (11.97)
and for $i = 0$

$$\frac{dE}{dt} = q E \cdot v$$  \hspace{1cm} (11.98)$$

which shows that only the electric field does work. The only special-relativistic correction needed in Maxwell’s electrodynamics is a factor of $1/\sqrt{1 - v^2/c^2}$ in these equations. That is, we use $p = mu = mv/\sqrt{1 - v^2/c^2}$ not $p = mv$ in (11.97), and we use the total energy $E$ not the kinetic energy in (11.98).

The reason why so little of classical electrodynamics was changed by special relativity is that electric and magnetic effects were accessible to measurement during the 1800’s. Classical electrodynamics was almost perfect.

Keeping track of factors of the speed of light is a lot of trouble and a distraction; in what follows, we’ll often use units with $c = 1$.

### 11.12 Tensors

Tensors are structures that transform like products of vectors. A first-rank tensor is a covariant or a contravariant vector. Second-rank tensors also are distinguished by how they transform under changes of coordinates:

contravariant

$$M^{ij} = \frac{\partial x^n}{\partial x^i} \frac{\partial x^j}{\partial x^l} M^{kl}$$

mixed

$$N^i_j = \frac{\partial x^n}{\partial x^i} \frac{\partial x^l}{\partial x^j} N^l_k$$  \hspace{1cm} (11.99)$$

covariant

$$F'_i = \frac{\partial x^n}{\partial x^i} \frac{\partial x^l}{\partial x^j} F_{kl}.$$ 

We can define tensors of higher rank by extending the these definitions to quantities with more indices.

**Example 11.8** (Some Second-Rank Tensors). If $A_k$ and $B_l$ are covariant vectors, and $C^m$ and $D^n$ are contravariant vectors, then the product $C^m D^n$ is a second-rank contravariant tensor, and all four products $A_k C^m$, $A_k D^n$, $B_k C^m$, and $B_k D^n$ are second-rank mixed tensors, while $C^m D^n$ as well as $C^m C^m$ and $D^m D^n$ are second-rank contravariant tensors.

Since the transformation laws that define tensors are linear, any linear combination of tensors of a given rank and kind is a tensor of that rank and kind. Thus if $F_{ij}$ and $G_{ij}$ are both second-rank covariant tensors, then so is their sum

$$H_{ij} = F_{ij} + G_{ij}.$$  \hspace{1cm} (11.100)$$
A covariant tensor is symmetric if it is independent of the order of its indices. That is, if \( S_{ik} = S_{ki} \), then \( S \) is symmetric. Similarly, a contravariant tensor is symmetric if permutations of its indices leave it unchanged. Thus \( A \) is symmetric if \( A^{ik} = A^{ki} \).

A covariant or contravariant tensor is antisymmetric if it changes sign when any two of its indices are interchanged. So \( A_{ik} \), \( B^{ik} \), and \( C_{ijk} \) are antisymmetric if

\[
A_{ik} = -A_{ki} \quad \text{and} \quad B^{ik} = -B^{ki} \quad \text{and} \quad C_{ijk} = C_{jki} = -C_{jik} = -C_{ikj} = -C_{kji}.
\]

\[(11.101)\]

**Example 11.9** (Three Important Tensors). The Maxwell field strength \( F_{kl}(x) \) is a second-rank covariant tensor; so is the metric of spacetime \( g_{ij}(x) \). The Kronecker delta \( \delta^i_j \) is a mixed second-rank tensor; it transforms as

\[
\frac{\partial x^i}{\partial x^k} \frac{\partial x^k}{\partial x^l} \delta^l_j = \frac{\partial x^i}{\partial x^k} \frac{\partial x^k}{\partial x^j} = \delta^i_j.
\]

\[(11.102)\]

So it is invariant under changes of coordinates.

**Example 11.10** (Contractions). Although the product \( A_k C^\ell \) is a mixed second-rank tensor, the product \( A_k C^k \) transforms as a scalar because

\[
A'_k C'^k = \frac{\partial x^\ell}{\partial x^m} A_\ell C^m = \frac{\partial x^\ell}{\partial x^m} A_\ell C^m = \delta^\ell_m A_\ell C^m = A_\ell C^\ell.
\]

\[(11.103)\]

A sum in which an index is repeated once covariantly and once contravariantly is a contraction as in the Kronecker-delta equation (11.26). In general, the rank of a tensor is the number of uncontracted indices.

**11.13 Differential Forms**

By (11.10 & 11.5), a covariant vector field contracted with contravariant coordinate differentials is invariant under arbitrary coordinate transformations

\[
A' = A'_i dx^i = \frac{\partial x^j}{\partial x'^i} A_j \frac{\partial x'^i}{\partial x^k} dx^k = \delta^j_k A_j dx^k = A_k dx^k = A.
\]

\[(11.104)\]

This invariant quantity \( A = A_k dx^k \) is a called a **1-form** in the language of differential forms introduced about a century ago by Élie Cartan (1869–1951, son of a blacksmith).
The wedge product $dx \wedge dy$ of two coordinate differentials is the directed area spanned by the two differentials and is defined to be antisymmetric

$$dx \wedge dy = - dy \wedge dx \quad \text{and} \quad dx \wedge dx = dy \wedge dy = 0 \quad (11.105)$$

so as to transform correctly under a change of coordinates. In terms of the coordinates $u = u(x, y)$ and $v = v(x, y)$, the new element of area is

$$du \wedge dv = \left( \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) \wedge \left( \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right). \quad (11.106)$$

Labeling partial derivatives by subscripts (6.20) and using the antisymmetry (11.105), we see that the new element of area $du \wedge dv$ is the old area $dx \wedge dy$ multiplied by the Jacobian $J(u, v; x, y)$ of the transformation $x, y \to u, v$

$$du \wedge dv = (u_x dx + u_y dy) \wedge (v_x dx + v_y dy)$$

$$= u_x v_x dx \wedge dx + u_x v_y dx \wedge dy + u_y v_x dy \wedge dx + u_y v_y dy \wedge dy$$

$$= (u_x v_y - u_y v_x) dx \wedge dy$$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} dx \wedge dy = J(u, v; x, y) dx \wedge dy. \quad (11.107)$$

A contraction $H = \frac{1}{2} H_{ik} dx^i \wedge dx^k$ of a second-rank covariant tensor with a wedge product of two differentials is a 2-form. A $p$-form is a rank-$p$ covariant tensor contracted with a wedge product of $p$ differentials

$$K = \frac{1}{p!} K_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots dx^{i_p}. \quad (11.108)$$

The exterior derivative $d$ differentiates and adds a differential; it turns a $p$-form into a $(p + 1)$-form. It converts a function or a 0-form $f$ into a 1-form

$$df = \frac{\partial f}{\partial x^i} dx^i \quad (11.109)$$

and a 1-form $A = A_j dx^j$ into a 2-form $dA = d(A_j dx^j) = (\partial_i A_j) dx^i \wedge dx^j$.

**Example 11.11 (The Curl).** The exterior derivative of the 1-form

$$A = A_x dx + A_y dy + A_z dz \quad (11.110)$$
11.13 Differential Forms

is the 2-form

\[
dA = \partial_y A_x \, dy \wedge dx + \partial_z A_x \, dz \wedge dx \\
+ \partial_x A_y \, dx \wedge dy + \partial_z A_y \, dz \wedge dy \\
+ \partial_x A_z \, dx \wedge dz + \partial_y A_z \, dy \wedge dz
\]

\[(11.111)\]

\[
= (\partial_y A_z - \partial_z A_y) \, dy \wedge dz \\
+ (\partial_z A_x - \partial_x A_z) \, dz \wedge dx \\
+ (\partial_x A_y - \partial_y A_x) \, dx \wedge dy
\]

\[
= (\nabla \times A)_x \, dy \wedge dz + (\nabla \times A)_y \, dz \wedge dx + (\nabla \times A)_z \, dx \wedge dy
\]

in which we recognize the curl (6.39) of \(A\).

The exterior derivative of the 1-form \(A = A_j \, dx^j\) is the 2-form

\[
dA = dA_j \wedge dx^j = \partial_i A_j \, dx^i \wedge dx^j = \frac{1}{2} F_{ij} \, dx^i \wedge dx^j = F
\]

\[(11.112)\]

in which \(\partial_i = \partial/\partial x^i\). So \(d\) turns the electromagnetic 1-form \(A\)—the 4-vector potential or gauge field \(A_j\)—into the Faraday 2-form—the tensor \(F_{ij}\). Its square \(dd\) vanishes: \(dd\) applied to any \(p\)-form \(Q\) is zero

\[
\begin{align*}
\quad & dQ_{i_1 \ldots i_n} \wedge \ldots = d(\partial_r Q_{i_1 \ldots}) \wedge dx^r \wedge dx^{i_1} \wedge \ldots = (\partial_s \partial_r Q_{i_1 \ldots}) dx^s \wedge dx^r \wedge dx^{i_1} \wedge \ldots = 0 \\
\quad & (11.113)
\end{align*}
\]

because \(\partial_s \partial_r Q\) is symmetric in \(r\) and \(s\) while \(dx^s \wedge dx^r\) is anti-symmetric.

Some writers drop the wedges and write \(dx^i \wedge dx^j\) as \(dx^i dx^j\) while keeping the rules of antisymmetry \(dx^i dx^j = -dx^j dx^i\) and \((dx^i)^2 = 0\). But this economy prevents one from using invariant quantities like \(S = \frac{1}{2} S_{ik} \, dx^i \wedge dx^k\) in which \(S_{ik}\) is a second-rank covariant symmetric tensor. If \(M_{ik}\) is a covariant second-rank tensor with no particular symmetry, then (exercise 11.7) only its antisymmetric part contributes to the 2-form \(M_{ik} \, dx^i \wedge dx^k\) and only its symmetric part contributes to the quantity \(M_{ik} \, dx^i dx^k\).

The exterior derivative \(d\) applied to the Faraday 2-form \(F = dA\) gives

\[
\quad dF = ddA = 0
\]

\[(11.114)\]

which is the Bianchi identity (11.93). A \(p\)-form \(H\) is closed if \(dH = 0\). By (11.114), the Faraday 2-form is closed, \(dF = 0\).

A \(p\)-form \(H\) is exact if there is a \((p-1)\)-form \(K\) whose differential is \(H = dK\). The identity (11.113) or \(dd = 0\) implies that every exact form is closed. The lemma of Poincaré shows that every closed form is locally exact.

If the \(A_i\) in the 1-form \(A = A_j dx^i\) commute with each other, then the 2-form \(A \wedge A = 0\). But if the \(A_i\) don’t commute because they are matrices or operators or Grassmann variables, then \(A \wedge A\) need not vanish.
Example 11.12 (A Static Electric Field Is Closed and Locally Exact). If  \( \dot{B} = 0 \), then by Faraday’s law (11.82) the curl of the electric field vanishes,  \( \nabla \times E = 0 \). Writing the electrostatic field as the 1-form  \( E = E_i \, dx^i \) for  \( i = 1, 2, 3 \), we may express the vanishing of its curl as
\[
dE = \partial_j E_i \, dx^j \wedge dx^i = \frac{1}{2} (\partial_j E_i - \partial_i E_j) \, dx^j \wedge dx^i = 0 \tag{11.115}
\]
which says that  \( E \) is closed. We can define a quantity  \( V_P(x) \) as a line integral of the 1-form  \( E \) along a path  \( P \) to  \( x \) from some starting point  \( x_0 \)
\[
V_P(x) = - \int_{P, x_0}^x E_i \, dx^i = - \int_P E \tag{11.116}
\]
and so  \( V_P(x) \) will depend on the path  \( P \) as well as on  \( x_0 \) and  \( x \). But if  \( \nabla \times E = 0 \) in some ball (or neighborhood) around  \( x \) and  \( x_0 \), then within that ball the dependence on the path  \( P \) drops out because the difference  \( V_{P'}(x) - V_P(x) \) is the line integral of  \( E \) around a closed loop in the ball which by Stokes’s theorem (6.44) is an integral of the vanishing curl  \( \nabla \times E \) over any surface  \( S \) in the ball whose boundary  \( \partial S \) is the closed curve  \( P' - P \)
\[
V_{P'}(x) - V_P(x) = \int_{P'-P} E_i \, dx^i = \int_S (\nabla \times E) \cdot da = 0 \tag{11.117}
\]
or
\[
V_{P'}(x) - V_P(x) = \int_{\partial S} E = \int_S dE = 0 \tag{11.118}
\]
in the language of forms (George Stokes, 1819–1903). Thus the potential  \( V_P(x) = V(x) \) is independent of the path,  \( E = - \nabla V(x) \), and the 1-form  \( E = E_i \, dx^i = - \partial_i V \, dx^i = -dV \) is locally exact. \( \square \)

The general form of Stokes’s theorem is that the integral of any  \( p \)-form  \( H \) over the boundary  \( \partial R \) of any  \( (p + 1) \)-dimensional, simply connected, orientable region  \( R \) is equal to the integral of the  \( (p + 1) \)-form  \( dH \) over  \( R \)
\[
\int_{\partial R} H = \int_R dH \tag{11.119}
\]
which for  \( p = 1 \) gives (6.44).

Example 11.13 (Stokes’s Theorem for 0-forms). Here  \( p = 0 \), the region  \( R = [a, b] \) is 1-dimensional,  \( H \) is a 0-form, and Stokes’s theorem is
\[
H(b) - H(a) = \int_{\partial R} H = \int_R dH = \int_a^b dH(x) = \int_a^b H'(x) \, dx \tag{11.120}
\]
familiar from elementary calculus. \( \square \)
Example 11.14 (Exterior Derivatives Anticommute with Differentials). The exterior derivative acting on two one-forms $A = A_i dx^i$ and $B = B_j dx^j$ is

$$d(A \wedge B) = d(A_i dx^i \wedge B_j dx^j) = \partial_k (A_i B_j) dx^k \wedge dx^i \wedge dx^j \quad (11.121)$$

$$= (\partial_k A_i) B_j dx^k \wedge dx^i \wedge dx^j + A_i (\partial_k B_j) dx^k \wedge dx^i \wedge dx^j$$

$$= (\partial_k A_i) B_j dx^k \wedge dx^i \wedge dx^j - A_i (\partial_k B_j) dx^k \wedge dx^i \wedge dx^j$$

$$= (\partial_k A_i) dx^k \wedge dx^i \wedge B_j dx^j - A_i dx^i \wedge (\partial_k B_j) dx^k \wedge dx^j$$

$$= dA \wedge B - A \wedge dB.$$ 

If $A$ is a p-form, then $d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB$ (exercise 11.10). □

11.14 Tensor Equations

Maxwell’s homogeneous equations (11.93) relate the derivatives of the field-strength tensor to each other as

$$0 = \partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki}. \quad (11.122)$$

They are generally covariant tensor equations (sections 11.36 & 11.38). In terms of invariant forms, they are the Bianchi identity (11.114)

$$dF = ddA = 0. \quad (11.123)$$

Maxwell’s inhomegeneous equations (11.94) relate the derivatives of the field-strength tensor to the current density $j^i$ and to the square root of the modulus $g$ of the determinant of the metric tensor $g_{ij}$ (section 11.18)

$$\frac{\partial(\sqrt{g} F^{ik})}{\partial x^k} = \mu_0 \sqrt{g} j^i. \quad (11.124)$$

We’ll write them as invariant forms in section 11.30 and derive them from an action principle in section 11.44.

If we can write a physical law in one coordinate system as a tensor equation

$$K^{kl} = 0 \quad (11.125)$$

then in any other coordinate system, the corresponding tensor equation

$$K^{ij} = 0 \quad (11.126)$$

also is valid since

$$K^{ij} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} K^{kl} = 0. \quad (11.127)$$
Similarly, physical laws remain the same when expressed in terms of invariant forms. Thus by writing a theory in terms of tensors or forms, one gets a theory that is true in all coordinate systems if it is true in any. Only such generally covariant theories have a chance at being right in our coordinate system, which is not special. One way to make a generally covariant theory is to start with an action that is invariant under all coordinate transformations.

11.15 The Quotient Theorem

Suppose that the product $BA$ of a quantity $B$ (with unknown transformation properties) with an arbitrary tensor $A$ (of a given rank and kind) is a tensor. Then $B$ is itself a tensor. The simplest example is when $B_i A^i$ is a scalar for all contravariant vectors $A^i$

$$B_i' A'^i = B_j A^j.$$  \hspace{1cm} (11.128)

Then since $A^i$ is a contravariant vector

$$B_i' A'^i = B_i' \frac{\partial x'^i}{\partial x^j} A^j = B_j A^j$$  \hspace{1cm} (11.129)

or

$$\left( B_i' \frac{\partial x'^i}{\partial x^j} - B_j \right) A^j = 0.$$  \hspace{1cm} (11.130)

Since this equation holds for all vectors $A$, we may promote it to the level of a vector equation

$$B_i' \frac{\partial x'^i}{\partial x^j} - B_j = 0.$$  \hspace{1cm} (11.131)

Multiplying both sides by $\partial x^j/\partial x'^k$ and summing over $j$

$$B_i' \frac{\partial x'^i}{\partial x^j} \frac{\partial x^j}{\partial x'^k} = B_j \frac{\partial x^j}{\partial x'^k}$$  \hspace{1cm} (11.132)

we see that the unknown quantity $B_i$ does transform as a covariant vector

$$B_k' = \frac{\partial x^j}{\partial x'^k} B_j.$$  \hspace{1cm} (11.133)

The quotient rule works for unknowns $B$ and tensors $A$ of arbitrary rank and kind. The proof in each case is very similar to the one given here.
11.16 The embedding space

So far we have been considering coordinate systems with constant basis vectors $e_i$ that do not vary with the physical point $p$. Now we shall assume only that we can write the change in the point $p(x)$ due to an infinitesimal change $dx^i(p)$ in its coordinates $x^i(p)$ as

$$dp^\alpha(x) = e_i^\alpha(x) dx^i$$

(11.134)

in which the superscript $\alpha$ labels the $n$ coordinates of the embedding space $\mathbb{R}^n$ and the index $i$ labels the coordinates on the manifold.

**Example 11.15** (A sphere). For the surface of a 2-sphere in $\mathbb{R}^3$, one might have $\alpha = 1, 2, 3$ and $i = \theta, \phi$. One then could use the $3 \times 3$ identity matrix to form inner products

$$e_i \cdot e_j = \sum_{\alpha=1}^{3} e_i^\alpha e_j^\alpha.$$  \hspace{1cm} (11.135)

**Example 11.16** (Spacetime). If we use four coordinates to describe ordinary spacetime as a curved surface in a flat embedding space $\mathbb{R}^8$, then we would have $\alpha = 0, 1, \ldots, 7$ and $i = 0, 1, 2, 3$. We then could use an $8 \times 8$ diagonal matrix with entries $\eta_{\alpha\alpha} = \pm 1$ to form inner products

$$e_i \cdot e_j = \sum_{\alpha, \beta=0}^{7} e_i^\alpha \eta_{\alpha\beta} e_j^\beta.$$  \hspace{1cm} (11.136)

11.17 Tangent vectors

Vectors defined by (11.134) as derivatives

$$e_i^\alpha(x) = \frac{\partial p^\alpha(x)}{\partial x^i}$$

(11.137)

of physical points $p$ with respect to the coordinates $x^i$ are tangent to the manifold at $p$. They are **local coordinate basis vectors**. These tangent vectors span a **tangent space** of as many dimensions as there are coordinates $x^i$. 

In a different system of coordinates $x'$, the same displacement (11.134) is $dp = e'_i(x') dx^i$. The basis vectors $e_i(x)$ and $e'_i(x')$

$$e_i(x) = \frac{\partial p}{\partial x^i} \quad \text{and} \quad e'_i(x') = \frac{\partial p}{\partial x'^i}. \quad (11.138)$$

are linearly related to each other and transform as covariant vectors

$$e'_i(x') = \frac{\partial p}{\partial x'^i} = \frac{\partial p}{\partial x'^i} \frac{\partial x'^j}{\partial x^i} e_j(x). \quad (11.139)$$

### 11.18 The metric tensor

Points are physical, coordinate systems metaphysical. So $p, q, p - q$, and $(p - q) \cdot (p - q)$ are all invariant quantities. When $p$ and $q = p + dp$ lie infinitesimally close to each other on the spacetime manifold, the vector $dp = e_i dx^i$ is the sum of the basis vectors multiplied by the changes in the coordinates $x^i$. Both $dp$ and the inner product $dp \cdot dp$ are physical and so are independent of the coordinates. The squared separation $dp^2$ in one coordinate system

$$dp^2 \equiv dp \cdot dp = (e_i dx^i) \cdot (e_j dx^j) = \sum_{\alpha, \beta=0}^{n-1} e^\alpha_i \eta_{\alpha \beta} e^\beta_j dx^i dx^j \quad (11.140)$$

is the same as in any other

$$dp^2 \equiv dp \cdot dp = (e'_i dx'^i) \cdot (e'_j dx'^j) = \sum_{\alpha, \beta=0}^{n-1} e'^\alpha_i \eta_{\alpha \beta} e'^\beta_j dx'^i dx'^j. \quad (11.141)$$

This invariant squared separation $dp^2$ usually is written as $ds^2$. It defines a $4 \times 4$ metric $g_{ij}$ on spacetime

$$dp^2 \equiv ds^2 \equiv g_{ij} dx^i dx^j \quad (11.142)$$

which, summing implicitly over the $n$ values of $\alpha$ and $\beta$ that label the coordinates of the flat embedding space $\mathbb{R}^n$, we may write as

$$g_{ij} = e_i \cdot e_j = e^\alpha_i \eta_{\alpha \beta} e^\beta_j. \quad (11.143)$$

Since the basis vectors $e_i$ transform (11.139) as covariant vectors, the metric $g_{ij}$ transforms as a covariant tensor

$$g'_{ij} = e'_i \eta_{\alpha \beta} e'_j = e^\alpha_i \eta_{\alpha \beta} e^\beta_j \frac{\partial x^k}{\partial x'^i} \frac{\partial x^\ell}{\partial x'^j} = g_{kl} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^\ell}{\partial x'^j}, \quad (11.144)$$
as it must be since the squared separation $dp^2$ is independent of our coordinates

$$g_{ij}^l dx^l dx^j = g_{ij}^l \frac{\partial x^k}{\partial x^l} dx^k \frac{\partial x^j}{\partial x^\ell} dx^\ell = g_{k\ell} dx^k dx^\ell. \quad (11.145)$$

Thus the **metric tensor** $g_{ij}$ is a rank-2 covariant tensor. By its construction (11.143), it also is a $4 \times 4$ real, symmetric matrix.

**Example 11.17** (Graph paper). Imagine a piece of slightly crumpled graph paper with horizontal and vertical lines. The lines give us a two-dimensional coordinate system $(x^1, x^2)$ that labels each point $p(x)$ on the paper. The vectors $e_1(x) = \partial_1 p(x)$ and $e_2(x) = \partial_2 p(x)$ define how a point moves $dp(x) = e_i(x) dx^i$ when we change its coordinates by $dx^1$ and $dx^2$. The vectors $e_1(x)$ and $e_2(x)$ span a different tangent space at the intersection of every horizontal line with every vertical line. Each tangent space is like the tiny square of the graph paper at that intersection. We can think of the two vectors $e_i(x)$ as three-component vectors in the three-dimensional embedding space we live in. The squared distance between any two nearby points separated by $dp(x)$ is $ds^2 \equiv dp^2(x) = e_i^2(x) (dx^1)^2 + 2e_1(x) \cdot e_2(x) dx^1 dx^2 + e_2^2(x) (dx^2)^2$ in which the inner products $g_{ij} = e_i(x) \cdot e_j(x)$ are defined by the Euclidean metric of the embedding Euclidean space $\mathbb{R}^3$.  

**Example 11.18** (The Sphere $S^2$). Let the point $p$ be a Euclidean 3-vector representing a point on the two-dimensional surface of a sphere of radius $a$. The spherical coordinates $(\theta, \phi)$ label the point $p$, and the basis vectors are

$$e_\theta = \frac{\partial p}{\partial \theta} = a \hat{\theta} \quad \text{and} \quad e_\phi = \frac{\partial p}{\partial \phi} = a \sin \theta \hat{\phi}. \quad (11.146)$$

Their inner products are the components (11.143) of the sphere’s metric tensor which is the matrix

$$\begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} e_\theta \cdot e_\theta & e_\theta \cdot e_\phi \\ e_\phi \cdot e_\theta & e_\phi \cdot e_\phi \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2 \theta \end{pmatrix} \quad (11.147)$$

with determinant $a^4 \sin^2 \theta$ and squared distance $ds^2 = a^2 d\theta^2 + a^2 \sin^2 \theta d\phi^2$.  

If we change coordinates to $r = \sin \theta$, then $dr = \cos \theta d\theta$,

$$ds^2 = \frac{a^2 dr^2}{\cos^2 \theta} + a^2 r^2 d\phi^2 = \frac{a^2 dr^2}{1 - r^2} + a^2 r^2 d\phi^2, \quad (11.148)$$

and the metric is

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{r\phi} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} a^2/(1 - r^2) & 0 \\ 0 & a^2 r^2 \end{pmatrix}. \quad (11.149)$$
Example 11.19 (The hyperboloid $H^2$). The hyperboloid $H^2$ is the surface in $\mathbb{R}^3$ defined by $a^2 + x^2 + y^2 = z^2$ with metric $\eta = \text{diag}(1, 1, -1)$ and squared distance (or line element)

$$ds^2 = dx^2 + dy^2 - dz^2.$$ \hspace{1cm} (11.150)

In terms of the variables $x = ar \cos \phi$ and $y = ar \sin \phi$, a point $p$ on the upper hyperboloid is

$$p = (x, y, \sqrt{x^2 + y^2 + a^2}) = a(r \cos \phi, r \sin \phi, \sqrt{r^2 + 1}).$$ \hspace{1cm} (11.151)

Its derivatives are the basis vectors

$$e_r = \frac{\partial p}{\partial r} = a(\cos \phi, \sin \phi, \frac{r}{\sqrt{r^2 + 1}}) \quad \text{and} \quad e_\phi = \frac{\partial p}{\partial \phi} = a(-r \sin \phi, r \cos \phi, 0).$$ \hspace{1cm} (11.152)

Their inner products with the metric $\eta$ form the metric for the hyperboloid

$$\begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{r\phi} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} e_r \cdot e_r & e_r \cdot e_\phi \\ e_\phi \cdot e_r & e_\phi \cdot e_\phi \end{pmatrix} = \begin{pmatrix} a^2/(r^2 + 1) & 0 \\ 0 & a^2 r^2 \end{pmatrix}. \hspace{1cm} (11.153)$$

We also can derive this metric from the line element $ds^2 = dx^2 + dy^2 - dz^2$ by setting $z^2 = a^2 + a^2 r^2$ and $zdz = a^2 r dr$. In polar coordinates, $ds^2$ is

(exercise 11.12)

$$ds^2 = a^2(\cos \phi dr - r \sin \phi d\phi)^2 + a^2(\sin \phi dr + r \cos \phi d\phi)^2 - \frac{a^4 r^2 dr^2}{z^2} \hspace{1cm} (11.154)$$

and we again get (11.153) as the metric for the hyperboloid $H^2$.

The difference between the squared distances (11.148) and (11.150) is the sign in the denominator $1 \mp r^2$. This sign difference occurs as $k = \pm 1$ in the closed and open Friedmann-Walker-Robinson metrics (11.426).

11.19 The principle of equivalence

The principle of equivalence is that it is always possible to change coordinates to those of an inertial coordinate system in a suitably small neighborhood of any point and that in those coordinates the metric assumes the
diagonal form of the $4 \times 4$ Minkowski metric

$$g_{k\ell}(x) \frac{\partial x^k}{\partial x^\iota} \frac{\partial x^\ell}{\partial x^\jmath} = g'_{ij}(x') = \eta_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ (11.155)

with vanishing first derivatives in a suitably small neighborhood of that point and that the physical laws of gravity-free special relativity apply in that neighborhood. It follows from this principle that the metric $g_{ij}$ of spacetime accounts for all the effects of gravity.

In the $x'$ coordinates, the invariant squared separation $dp^2$ is

$$dp^2 = g'_{ij} dx^i dx^j = e'_i(x') \cdot e'_j(x') \, dx^i dx^j = \eta_{ab} e^a_i(x') \, dx^i_0 \, dx^j_0 = \eta_{ij} \, dx^i_0 \, dx^j_0 = (dx^i)'_0 \, (dx^j)'_0 = ds^2$$ (11.156)

which is the separation of special relativity. Thus we always can change coordinates so that the metric $g_{ij}(x')$ in a suitably small neighborhood of any point $p(x') = p(x)$ is the metric $\eta_{ij}$ of flat spacetime. These coordinates are not unique because every Lorentz transformation by definition leaves the metric of flat spacetime invariant. Coordinate systems in which $g_{ij}(x') = \eta_{ij}$ are called inertial coordinate systems.

11.20 Trace and determinant of the metric tensor

In our statement of the equivalence principle, the relation (11.155) between the spacetime metric $g$ and the metric $\eta$ of special relativity shows that the spacetime metric at any point, viewed as a real, symmetric $4 \times 4$ matrix, is related to the metric $\eta_{ij}$ of flat space, which is a real, symmetric $4 \times 4$ matrix, by an orthogonal similarity transformation (1.308)

$$g_{k\ell} = \frac{\partial x'^i}{\partial x^k} \eta_{ij} \frac{\partial x^j}{\partial x^\ell} = O^i_k \, \eta_{ij} \, O^\ell_j.$$ (11.157)

Since orthogonal transformations leave traces and determinants invariant, it follows that $g$ has trace 2 and determinant $-1$

$$\text{Tr } g = \text{Tr } \eta = 2 \quad \text{and} \quad \det g = \det \eta = -1.$$ (11.158)

Orthogonal transformations preserve eigenvalues, so the eigenvalues of the metric tensor are $(-1, 1, 1, 1)$ which are its signature.
In our formula (11.143) for the metric tensor, we used four \( n \)-vectors \( e_i \) to write it as \( g_{ij} = e_i \cdot e_j = e_i^\top \eta e_j \) in which \( \eta_{\alpha\beta} \) is the \( n \times n \) matrix (11.136). Instead we could use four 4-vectors \( c^a_i(x) \) to write the diagonalization (11.157) of the metric tensor more abstractly as

\[
g_{ij}(x) = c^a_i(x) \eta_{ab} c^b_j(x)
\]

in which \( \eta_{ab} \) is the \( 4 \times 4 \) matrix (11.155). Whether the fundamental variables are the four 4-vectors \( c^a_i(x) \) introduced by Élie Cartan (1869–1951) or the metric tensor \( g_{ij} \) is an open question. Cartan’s four 4-vectors \( c^a_i(x) \) are called a moving frame, a tetrad, or a vierbein.

11.22 The Contravariant Metric Tensor

The inverse \( g^{ik} \) of the covariant metric tensor \( g_{kj} \) satisfies

\[
g^{ik} g_{kj}^\prime = \delta^i_j = g^{ik} g_{kj}
\]

in all coordinate systems. To see how it transforms, we use the transformation law (11.144) of \( g_{kj} \)

\[
\delta^i_j = g^{ik} g_{kj}^\prime = g^{ik} \frac{\partial x^\ell}{\partial x^k} g_{\ell u} \frac{\partial x^u}{\partial x^j}.
\]

Thus in matrix notation, we have \( I = g'^{-1} H g H \) which implies \( g'^{-1} = H^{-1} g^{-1} H^{-1} \) or in tensor notation

\[
g^{i\ell} = \frac{\partial x^i}{\partial x^v} \frac{\partial x^\ell}{\partial x^w} g^{vw}.
\]

Thus the inverse \( g^{ik} \) of the covariant metric tensor is a second-rank contravariant tensor called the contravariant metric tensor.
11.23 Dual vectors and the raising and lowering of indices

The contraction of a contravariant vector $A^i$ with any rank-2 covariant tensor gives a covariant vector. We reserve the symbol $A_i$ for the covariant vector that is the contraction of $A^i$ with the metric tensor

$$A_i = g_{ij} A^j.$$  

(11.163)

This operation is called lowering the index on $A^j$.

Similarly, the contraction of a covariant vector $B_j$ with any rank-2 contravariant tensor is a contravariant vector. But we reserve the symbol $B^i$ for contravariant vector that is the contraction

$$B^i = g^{ij} B_j$$  

(11.164)

of $B_j$ with the inverse of the metric tensor. This is called raising the index on $B_j$.

The vectors $e^i$, for instance, are given by

$$e^i = g^{ij} e_j.$$  

(11.165)

They are therefore orthonormal or dual to the basis vectors $e_i$

$$e_i \cdot e^j = e_i \cdot g^{jk} e_k = g^{jk} e_i \cdot e_k = g^{jk} g_{ik} = g^{jk} g_{ki} = \delta_i^j$$  

(11.166)

with respect to the metric $\eta_{ab}$ of the flat space tangent to the manifold.

11.24 Orthogonal coordinates in $\mathbb{R}^n$

In flat $n$-dimensional euclidean space, it is convenient to use orthogonal basis vectors and orthogonal coordinates. A change $dx^i$ in the coordinates moves the point $p$ by (11.134)

$$dp = e_i dx^i.$$  

(11.167)

The metric $g_{ij}$ is the inner product (11.143)

$$g_{ij} = e_i \cdot e_j.$$  

(11.168)

Since the vectors $e_i$ are orthogonal, the metric is diagonal

$$g_{ij} = e_i \cdot e_j = h_i^2 \delta_{ij}.$$  

(11.169)

The inverse metric

$$g^{ij} = h_i^{-2} \delta_{ij}$$  

(11.170)
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raises indices. For instance, the dual vectors
\[ e^i = g^{ij} e_j = h_i^{-2} e_i \quad \text{satisfy} \quad e^i \cdot e_k = \delta^i_k. \] (11.171)

The invariant squared distance \( dp^2 \) between nearby points (11.140) is
\[ dp^2 = dp \cdot dp = g_{ij} \, dx^i \, dx^j = h_i^2 (dx^i)^2 \] (11.172)
and the invariant volume element is
\[ dV = dp^p = h_1 \ldots h_n \, dx^1 \wedge \ldots \wedge dx^n = g \, dx^1 \wedge \ldots \wedge dx^n = g \, d^n \, x \] (11.173)
in which \( g = \sqrt{\text{det} g_{ij}} \) is the square root of the positive determinant of \( g_{ij} \).

The important special case in which all the scale factors \( h_i \) are unity is cartesian coordinates in euclidean space (section 11.5).

We also can use basis vectors \( \hat{e}_i \) that are orthonormal. By (11.169 & 11.171), these vectors
\[ \hat{e}_i = e_i / h_i = h_i e^i \quad \text{satisfy} \quad \hat{e}_i \cdot \hat{e}_j = \delta_{ij}. \] (11.174)
In terms of them, a physical and invariant vector \( V \) takes the form
\[ V = e_i \, V^i = h_i \, \hat{e}_i \, V^i = e^i \, V_i = h_i^{-1} \, \hat{e}_i \, V_i = \hat{e}_i \, \nabla_i \] (11.175)
where
\[ \nabla_i \equiv h_i \, V^i = h_i^{-1} \, V_i \quad \text{(no sum).} \] (11.176)
The dot-product is then
\[ V \cdot U = g_{ij} \, V^i \, U^j = \overline{\nabla}_i \, \overline{U}_i. \] (11.177)

In euclidian \( n \)-space, we even can choose coordinates \( x^i \) so that the vectors \( e_i \) defined by \( dp = e_i \, dx^i \) are orthonormal. The metric tensor is then the \( n \times n \) identity matrix \( g_{ik} = e_i \cdot e_k = I_{ik} = \delta_{ik} \). But since this is euclidian \( n \)-space, we also can expand the \( n \) fixed orthonormal cartesian unit vectors \( \hat{\ell} \) in terms of the \( e_i(x) \) which vary with the coordinates as \( \hat{\ell} = e_i(x) (e_i(x) \cdot \hat{\ell}) \).

11.25 Polar Coordinates

In polar coordinates in flat 2-space, the change \( dp \) in a point \( p \) due to a change in its coordinates is \( dp = \hat{r} \, dr + \hat{\theta} \, r \, d\theta \) so \( dp = e_r \, dr + e_\theta \, d\theta \) with \( e_r = \hat{e}_r = \hat{r} \) and \( e_\theta = r \, \hat{e}_\theta = r \, \hat{\theta} \). The metric tensor for polar coordinates is
\[ (g_{ij}) = (e_i \cdot e_j) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \] (11.178)
The contravariant basis vectors are \( e^r = \hat{r} \) and \( e^\theta = \hat{\theta} / r \). A physical vector \( V \) is \( V = V^i \, e_i = V_\hat{r} \, \hat{r} + V_\hat{\theta} \, \hat{\theta} \).
11.26 Cylindrical Coordinates

For cylindrical coordinates in flat 3-space, the change $dp$ in a point $p$ due to a change in its coordinates is

$$dp = \hat{\rho} d\rho + \hat{\phi} \rho d\phi + \hat{z} dz = e_\rho d\rho + e_\phi d\phi + e_z dz$$  \hspace{1cm} (11.179)

with $e_\rho = \hat{\rho}$, $e_\phi = \rho \hat{\phi}$, and $e_z = \hat{z}$. The metric tensor for cylindrical coordinates is

$$(g_{ij}) = (e_i \cdot e_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$  \hspace{1cm} (11.180)

with determinant $\det g_{ij} = \rho^2$. The invariant volume element is

$$dV = \rho dz \, dx^1 \wedge dx^2 \wedge dx^3 = \sqrt{\rho} d\rho d\phi dz = \rho d\rho d\phi dz.$$  \hspace{1cm} (11.181)

The contravariant basis vectors are $e^\rho = \hat{\rho}$, $e^\phi = \hat{\phi}/\rho$, and $e^z = \hat{z}$. A physical vector $V$ is

$$V = V^i e_i = V^i e_i = V_\rho \hat{\rho} + V_\phi \hat{\phi} + V_z \hat{z}.$$  \hspace{1cm} (11.182)

Incidentally, since

$$p = (\rho \cos \phi, \rho \sin \phi, z),$$  \hspace{1cm} (11.183)

the formulas for the basis vectors of cylindrical coordinates in terms of those of rectangular coordinates are (exercise 11.14)

$$\hat{\rho} = \cos \phi \hat{x} + \sin \phi \hat{y}$$

$$\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}$$

$$\hat{z} = \hat{z}.$$  \hspace{1cm} (11.184)

11.27 Spherical Coordinates

For spherical coordinates in flat 3-space, the change $dp$ in a point $p$ due to a change in its coordinates is

$$dp = \hat{r} dr + \hat{\theta} r d\theta + \hat{\phi} r \sin \theta d\phi = e_r dr + e_\theta d\theta + e_\phi d\phi$$  \hspace{1cm} (11.185)

so $e_r = \hat{r}$, $e_\theta = r \hat{\theta}$, and $e_\phi = r \sin \theta \hat{\phi}$ or

$$e_r = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

$$e_\theta = r (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta)$$  \hspace{1cm} (11.186)

$$e_\phi = r \sin \theta (-\sin \phi, \cos \phi, 0).$$
The metric tensor for spherical coordinates is

\[ (g_{ij}) = (e_i \cdot e_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \]  \hspace{1cm} (11.187) 

with determinant \( \det g_{ij} = g = r^4 \sin^2 \theta \). The invariant volume element is

\[ dV = r^2 \sin^2 \theta \, dx^1 \wedge dx^2 \wedge dx^3 = \sqrt{g} \, dr \, d\theta \, d\phi = r^2 \sin \theta \, dr \, d\theta \, d\phi. \]  \hspace{1cm} (11.188) 

The orthonormal basis vectors are \( \hat{e}_r = \hat{r} \), \( \hat{e}_\theta = \hat{\theta} \), and \( \hat{e}_\phi = \hat{\phi} \). The contravariant basis vectors are \( e^r = \hat{r} \), \( e^\theta = \hat{\theta} / r \), \( e^\phi = \hat{\phi} / r \sin \theta \). A physical vector \( V \) is

\[ V = V^i e_i = V_r \hat{r} + V_\theta \hat{\theta} + V_\phi \hat{\phi}. \]  \hspace{1cm} (11.189) 

Incidentally, since

\[ p = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta), \]  \hspace{1cm} (11.190) 

the formulas for the basis vectors of spherical coordinates in terms of those of rectangular coordinates are (exercise 11.15)

\[ e_r = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \]
\[ e_\theta = r \left( \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z} \right) \]  \hspace{1cm} (11.191) 
\[ e_\phi = r \sin \theta \left( -\sin \phi \hat{x} + \cos \phi \hat{y} \right). \]

### 11.28 The Gradient of a Scalar Field

If \( f(x) \) is a scalar field, then the difference between it and \( f(x + dx) \) defines the \textbf{gradient} \( \nabla f \) as (6.26)

\[ df(x) = f(x + dx) - f(x) = \frac{\partial f(x)}{\partial x^i} \, dx^i = \nabla f(x) \cdot dp. \]  \hspace{1cm} (11.192) 

Since \( dp = e_j dx^j \), the \textbf{invariant} form

\[ \nabla f = e^i \frac{\partial f}{\partial x^i} = \hat{e}_i \frac{\partial f}{\partial x^i} \]  \hspace{1cm} (11.193) 

satisfies this definition (11.192) of the gradient

\[ \nabla f \cdot dp = \frac{\partial f}{\partial x^i} e^i \cdot e_j dx^j = \frac{\partial f}{\partial x^i} \delta^i_j dx^j = \frac{\partial f}{\partial x^i} dx^i = df. \]  \hspace{1cm} (11.194)
In two polar coordinates, the gradient is
\[ \nabla f = e^i \frac{\partial f}{\partial x^i} = \frac{\hat{e}_i}{h_i} \frac{\partial f}{\partial r} = \hat{\theta} \frac{\partial f}{\partial \theta} + \frac{1}{r} \frac{\partial f}{\partial r}. \] (11.195)

In three cylindrical coordinates, it is (6.27)
\[ \nabla f = e^i \frac{\partial f}{\partial x^i} = \frac{\hat{e}_i}{h_i} \frac{\partial f}{\partial \rho} + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\phi} + \frac{\partial f}{\partial z} \hat{z} \] (11.196)

and in three spherical coordinates it is (6.28)
\[ \nabla f = \frac{\partial f}{\partial x^i} e^i = \frac{\hat{e}_i}{h_i} \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}. \] (11.197)

11.29 Levi-Civita’s Tensor

In 3 dimensions, Levi-Civita’s symbol \( \epsilon_{ijk} \) is totally antisymmetric with \( \epsilon_{123} = 1 \) in all coordinate systems.

We can turn his symbol into something that transforms as a tensor by multiplying it by the square root of the determinant of a rank-2 covariant tensor. A natural choice is the metric tensor. Thus the Levi-Civita tensor \( \epsilon_{ijk} \) is the totally antisymmetric rank-3 covariant (pseudo)tensor
\[ \eta_{ijk} = \sqrt{g} \epsilon_{ijk} \] (11.198)
in which \( g = |\det g_{mn}| \) is the absolute value of the determinant of the metric tensor \( g_{mn} \). The determinant’s definition (1.184) and product rule (1.207) imply that Levi-Civita’s tensor \( \eta_{ijk} \) transforms as
\[ \eta'_{ijk} = \sqrt{g'} \epsilon'_{ijk} = \sqrt{g'} \epsilon_{ijk} = \sqrt{\det \left( \frac{\partial x^t}{\partial x^m} \frac{\partial x^u}{\partial x^n} g_{tu} \right)} \epsilon_{ijk} \]
\[ = \sqrt{\det \left( \frac{\partial x^t}{\partial x^m} \right) \det \left( \frac{\partial x^u}{\partial x^n} \right) \det (g_{tu})} \epsilon_{ijk} \]
\[ = \det \left( \frac{\partial x}{\partial x'} \right) \sqrt{g} \epsilon_{ijk} = \sigma \det \left( \frac{\partial x}{\partial x'} \right) \sqrt{g} \epsilon_{ijk} \]
\[ = \sigma \frac{\partial x^t}{\partial x'^i} \frac{\partial x^u}{\partial x'^j} \frac{\partial x^v}{\partial x'^k} \sqrt{g} \epsilon_{uvw} = \sigma \frac{\partial x^t}{\partial x'^i} \frac{\partial x^u}{\partial x'^j} \frac{\partial x^v}{\partial x'^k} \eta_{uvw} \] (11.199)
in which \( \sigma \) is the sign of the Jacobian \( \det(\partial x/\partial x') \). Levi-Civita’s tensor is a pseudotensor because it doesn’t change sign under the parity transformation \( x'^i = -x^i \).
We get $\eta_{ijk}$ with upper indices by using the inverse $g^{nm}$ of the metric tensor

\[ \eta_{ijk} = g^{it} g^{ju} g^{kv} \eta_{tuv} = g^{it} g^{ju} g^{kv} \sqrt{g} \epsilon_{tuv} = \sqrt{g} \epsilon_{ijk} \det(g^{mn}) \]

in which $s$ is the sign of the determinant $\det g_{ij} = sg$.

Similarly in 4 dimensions, Levi-Civita’s symbol $\epsilon_{ijkl}$ is totally antisymmetric with $\epsilon_{0123} = 1$ in all coordinate systems. No meaning attaches to whether the indices of the Levi-Civita symbol are up or down; some authors even use the notation $\epsilon(ijk\ell)$ or $\epsilon[ijk\ell]$ to emphasize this fact.

In 4 dimensions, the Levi-Civita pseudo tensor is

\[ \epsilon_{ijkl} = p g^{ijkl} \]

(11.201)

It transforms as

\[ \eta_{ijkl} = \sqrt{g} \epsilon_{ijkl} = \left| \det \left( \frac{\partial x}{\partial x'} \right) \right| \sqrt{g} \epsilon_{ijkl} = \sigma \det \left( \frac{\partial x}{\partial x'} \right) \sqrt{g} \epsilon_{ijkl} \]

\[ = \sigma \frac{\partial x^t}{\partial x'^i} \frac{\partial x^u}{\partial x'^j} \frac{\partial x^w}{\partial x'^k} \sqrt{g} \epsilon_{tuvw} = \sigma \frac{\partial x^t}{\partial x'^i} \frac{\partial x^u}{\partial x'^j} \frac{\partial x^w}{\partial x'^k} \sqrt{g} \epsilon_{tuvw} \]

(11.202)

where $\sigma$ is the sign of the Jacobian $\det(\partial x/\partial x')$.

Raising the indices on $\eta$ with det $g_{ij} = sg$ we have

\[ \eta^{ijkl} = g^{it} g^{ju} g^{kv} g^{lw} \eta_{tuvw} = g^{it} g^{ju} g^{kv} g^{lw} \sqrt{g} \epsilon_{tuvw} = \sqrt{g} \epsilon_{ijkl} \det(g^{mn}) \]

\[ = \sqrt{g} \epsilon_{ijkl} / \det(g_{mn}) = s \epsilon_{ijkl} / \sqrt{g} \epsilon / \sqrt{g}. \]

(11.203)

In $n$ dimensions, one may define Levi-Civita’s symbol $\epsilon(i_1 \ldots i_n)$ as totally antisymmetric with $\epsilon(1 \ldots n) = 1$ and his tensor as $\eta_{i_1 \ldots i_n} = \sqrt{g} \epsilon(i_1 \ldots i_n)$.

11.30 The Hodge Star

In 3 cartesian coordinates, the Hodge dual turns 1-forms into 2-forms

\[ *dx = dy \wedge dz \quad *dy = dz \wedge dx \quad *dz = dx \wedge dy \]

(11.204)

and 2-forms into 1-forms

\[ *(dx \wedge dy) = dz \quad *(dy \wedge dz) = dx \quad *(dz \wedge dx) = dy. \]

(11.205)

It also maps the 0-form 1 and the volume 3-form into each other

\[ *1 = dx \wedge dy \wedge dz \quad *(dx \wedge dy \wedge dz) = 1 \]

(11.206)
More generally in 3-space, we define the Hodge dual, also called the Hodge star, as

\[ *1 = \frac{1}{3!} \eta_{ijk} dx^i \wedge dx^j \wedge dx^k \quad * (dx^i \wedge dx^j \wedge dx^k) = g^{\ell i} g^{j \nu} g^{k \mu} \eta_{\nu \mu \ell} \]  

(11.207)

and so if the sign of \( \det g_{ij} \) is \( s = +1 \), then \( * * 1 = 1 \), \( * * dx^i = dx^i \), \( * * (dx^i \wedge dx^j) = dx^i \wedge dx^j \), and \( * * (dx^i \wedge dx^j \wedge dx^k) = dx^i \wedge dx^j \wedge dx^k \).

**Example 11.20 (Divergence and Laplacian).** The dual of the 1-form

\[ df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \]  

is the 2-form

\[ * df = \frac{\partial f}{\partial x} dy \wedge dz + \frac{\partial f}{\partial y} dz \wedge dx + \frac{\partial f}{\partial z} dx \wedge dy \]  

(11.209)

and its exterior derivative is the laplacian

\[ d * df = \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) dx \wedge dy \wedge dz \]  

(11.210)

multiplied by the volume 3-form.

Similarly, the dual of the one form

\[ A = A_x dx + A_y dy + A_z dz \]  

(11.211)

is the 2-form

\[ * A = A_x dy \wedge dz + A_y dz \wedge dx + A_z dx \wedge dy \]  

(11.212)

and its exterior derivative is the divergence

\[ d * A = \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dx \wedge dy \wedge dz \]  

(11.213)

times \( dx \wedge dy \wedge dz \).

In flat Minkowski 4-space with \( c = 1 \), the Hodge dual turns 1-forms into 3-forms

\[ * dt = - dx \wedge dy \wedge dz \quad * dx = - dy \wedge dz \wedge dt \]

\[ * dy = - dz \wedge dx \wedge dt \quad * dz = - dx \wedge dy \wedge dt \]  

(11.214)
2-forms into 2-forms
\[ *(dx \wedge dt) = dy \wedge dz \quad *(dx \wedge dy) = -dz \wedge dt \]
\[ *(dy \wedge dt) = dz \wedge dx \quad *(dy \wedge dz) = -dx \wedge dt \]
\[ *(dz \wedge dt) = dx \wedge dy \quad *(dz \wedge dx) = -dy \wedge dt \] (11.215)

3-forms into 1-forms
\[ *(dx \wedge dy \wedge dz) = -dt \quad *(dy \wedge dz \wedge dt) = -dx \]
\[ *(dz \wedge dx \wedge dt) = -dy \quad *(dx \wedge dy \wedge dt) = -dz \] (11.216)

and interchanges 0-forms and 4-forms
\[ *1 = dt \wedge dx \wedge dy \wedge dz \quad * (dt \wedge dx \wedge dy \wedge dz) = -1. \] (11.217)

More generally in 4 dimensions, we define the Hodge star as
\[ *1 = \frac{1}{4!} \eta_{k\ell mn} dx^k \wedge dx^\ell \wedge dx^m \wedge dx^n \]
\[ *dx^i = \frac{1}{3!} g^{ik} \eta_{k\ell mn} dx^\ell \wedge dx^m \wedge dx^n \]
\[ *(dx^i \wedge dx^j) = \frac{1}{2} g^{ik} g^{j\ell} \eta_{k\ell mn} dx^m \wedge dx^n \] (11.218)
\[ *(dx^i \wedge dx^j \wedge dx^k) = g^{it} g^{ju} g^{kv} \eta_{tuvw} dx^w \]
\[ * \left( dx^i \wedge dx^j \wedge dx^k \wedge dx^\ell \right) = g^{it} g^{ju} g^{kv} g^{lw} \eta_{tuvw} = \eta^{ijkl}. \]

Thus (exercise 11.17) if the determinant \( \det g_{ij} \) of the metric is negative, then
\[ **dx^i = dx^i \quad **(dx^i \wedge dx^j) = -dx^i \wedge dx^j \]
\[ *(dx^i \wedge dx^j \wedge dx^k) = dx^i \wedge dx^j \wedge dx^k \quad *1 = -1. \] (11.219)

In \( n \) dimensions, the Hodge star turns \( p \)-forms into \( n-p \)-forms
\[ *(dx^{i_1} \wedge \ldots \wedge dx^{i_p}) = g^{i_1 k_1} \ldots g^{i_p k_p} \eta_{k_1 \ldots k_p \ell_1 \ldots \ell_{n-p}} (n-p)! \ dx^{\ell_1} \wedge \ldots \wedge dx^{\ell_{n-p}}. \] (11.220)

**Example 11.21** (The Inhomogeneous Maxwell Equations). Since the homogeneous Maxwell equations are
\[ dF = ddA = 0 \] (11.221)
we first form the dual \( *F = *dA \)
\[ *F = \frac{1}{2} F_{ij} * (dx^i \wedge dx^j) = \frac{1}{4} F_{ij} g^{ik} g^{j\ell} \eta_{k\ell mn} dx^m \wedge dx^n = \frac{1}{4} F^{k\ell} \eta_{k\ell mn} dx^m \wedge dx^n \]
and then apply the exterior derivative

\[ d * F = \frac{1}{4} d \left( F^{k \ell} \eta_{k \ell mn} dx^m \wedge dx^n \right) = \frac{1}{4} \partial_p \left( F^{k \ell} \eta_{k \ell mn} \right) dx^p \wedge dx^m \wedge dx^n. \]

To get back to a 1-form like \( j = j_k \, dx^k \), we apply a second Hodge star

\[ * d * F = \frac{1}{4} \partial_p \left( F^{k \ell} \eta_{k \ell mn} \right) * (dx^p \wedge dx^m \wedge dx^n) \]

\[ = \frac{1}{4} \partial_p \left( F^{k \ell} \eta_{k \ell mn} \right) g^{ps} g^{mt} g^{nu} \eta_{stuv} \, dx^v \]

\[ = \frac{1}{4} \partial_p \left( \sqrt{g} F^{k \ell} \right) \epsilon_{k \ell mn} g^{ps} g^{mt} g^{nu} \sqrt{g} \epsilon_{stuv} \, dx^v \]

\[ = \frac{1}{4} \partial_p \left( \sqrt{g} F^{k \ell} \right) \epsilon_{k \ell mn} g^{ps} g^{mt} g^{nu} g^{aw} \epsilon_{stuv} \sqrt{g} \, dx^w \]

\[ = \frac{s}{4} \partial_p \left( \sqrt{g} F^{k \ell} \right) \epsilon_{k \ell mn} \epsilon_{pmnw} \, dx^w \]

in which we used the definition (1.184) of the determinant. Levi-Civita’s 4-symbol obeys the identity (exercise 11.18)

\[ \epsilon_{k \ell mn} \epsilon_{pmnw} = 2 \left( \delta^p_k \delta^w_\ell - \delta^w_k \delta^p_\ell \right). \]

Applying it to \(* d * F\), we get

\[ * d * F = \frac{s}{2 \sqrt{g}} \partial_p \left( \sqrt{g} F^{k \ell} \right) (\delta^p_k \delta^w_\ell - \delta^w_k \delta^p_\ell) \, dx^w = -\frac{s}{\sqrt{g}} \partial_p \left( \sqrt{g} F^{k p} \right) \, dx_k. \]

In our spacetime \( s = -1 \). Setting \(* d * F\) equal to \( j = j_k \, dx^k = j^k \, dx_k \) multiplied by the permeability \( \mu_0 \) of the vacuum, we arrive at expressions for the microscopic inhomogeneous Maxwell equations in terms of both tensors and forms

\[ \frac{1}{\sqrt{g}} \partial_p \left( \sqrt{g} F^{k \ell} \right) = \mu_0 \, j^k \quad \text{and} \quad * d * F = \mu_0 \, j. \]

They and the homogeneous Bianchi identity (11.93, 11.114, & 11.283)

\[ \epsilon^{ij k \ell} \partial_c F_{j k} = dF = d dA = 0 \]

are invariant under general coordinate transformations.
11.31 Notations for Derivatives

We have various notations for derivatives. We can use the variables $x$, $y$, and so forth as subscripts to label derivatives

$$f_x = \frac{\partial f}{\partial x} \quad \text{and} \quad f_y = \frac{\partial f}{\partial y}. \quad (11.226)$$

If we use indices to label variables, then we can use commas

$$f_{,i} = \frac{\partial f}{\partial x^i} \quad \text{and} \quad f_{,ik} = \frac{\partial^2 f}{\partial x^k \partial x^i}. \quad (11.227)$$

and $f_{,k'} = \partial f/\partial x^{nk}$. For instance, we may write part of (11.236) as $e_{i,\ell} = e_{\ell,i}$.

In the next section, we will use a semicolon to mean a covariant derivative.

11.32 Covariant derivatives and affine connections

If $F(x)$ is a vector field, then its invariant description in terms of spacetime-dependent basis vectors $e_i(x)$ is

$$F(x) = F^i(x) e_i(x). \quad (11.228)$$

Since the basis vectors $e_i(x)$ vary with $x$, the derivative of $F(x)$ contains two terms

$$\frac{\partial F}{\partial x^\ell} = \frac{\partial F^i}{\partial x^\ell} e_i + F^i \frac{\partial e_i}{\partial x^\ell}. \quad (11.229)$$

In general, the derivative of a vector $e_i$ is not a linear combination of the basis vectors $e_k$. For instance, on the 2-dimensional surface of a sphere in 3-dimensions, the derivative

$$\frac{\partial e_\theta}{\partial \theta} = -\hat{r} \quad (11.230)$$

points to the sphere’s center and isn’t a linear combination of $e_\theta$ and $e_\phi$.

The inner product of a derivative $\partial e_i/\partial x^\ell$ with a dual basis vector $e^k$ is the **Levi-Civita affine connection**

$$\Gamma^k_{\ell i} = e^k \cdot \frac{\partial e_i}{\partial x^\ell} \quad (11.231)$$

which relates spaces that are tangent to the manifold at infinitesimally separated points. It is called an **affine connection** because the different tangent spaces lack a common origin.

In terms of the affine connection (11.231), the inner product of the derivative (11.229) with $e^k$ is

$$e^k \cdot \frac{\partial F}{\partial x^\ell} = e^k \cdot \frac{\partial F^i}{\partial x^\ell} e_i + F^i e^k \cdot \frac{\partial e_i}{\partial x^\ell} = \frac{\partial F^k}{\partial x^\ell} + \Gamma^k_{\ell i} F^i \quad (11.232)$$
a combination that is called a **covariant derivative** (section 11.35)

\[
D_\ell F^k = \nabla_\ell F^k \equiv \frac{\partial F^k}{\partial x^\ell} + \Gamma^k_{\ell i} F^i \equiv F^k_{;\ell} = e^k \cdot \frac{\partial F}{\partial x^\ell}.
\]

(11.233)

It is a second-rank mixed tensor. The covariant derivative of the scalar \(F\) is

\[
D_\ell F = e_k e^k \cdot \partial_\ell F = \left( \frac{\partial F^k}{\partial x^\ell} + \Gamma^k_{\ell i} F^i \right) e_k = F^k_{;\ell} e_k.
\]

(11.234)

Some physicists write the **affine connection** \(\Gamma^k_{i\ell}\) as

\[
\left\{ \begin{array}{c} k \\ i \ell \end{array} \right\} = \Gamma^k_{i\ell}
\]

(11.235)

and call it a **Christoffel symbol of the second kind**.

The vectors \(e_i\) are the spacetime derivatives (11.138) of the point \(p\), and so the affine connection (11.231) is a double derivative of \(p\)

\[
\Gamma^k_{i\ell} = e^k \cdot \frac{\partial e_i}{\partial x^\ell} = e^k \cdot \frac{\partial^2 p}{\partial x^i \partial x^\ell} = e^k \cdot \frac{\partial^2 p}{\partial x^i \partial x^\ell} = e^k \cdot \frac{\partial e_i}{\partial x^\ell} = \Gamma^k_{i\ell}
\]

(11.236)

and thus is symmetric in its two lower indices

\[
\Gamma^k_{i\ell} = \Gamma^k_{\ell i}.
\]

(11.237)

Affine connections are **not** tensors. Tensors transform homogeneously; connections transform inhomogeneously. The connection \(\Gamma^k_{i\ell}\) transforms as

\[
\Gamma'^{k}_{i\ell} = e'^k \cdot \frac{\partial e'_i}{\partial x'^\ell} = e^k \cdot \frac{\partial x'^k}{\partial x^i} \cdot \frac{\partial x^m}{\partial x'^i} \frac{\partial}{\partial x'^m} \left( \frac{\partial x^n}{\partial x'^i} e_n \right)
\]

\[
= \frac{\partial x'^k}{\partial x^p} \frac{\partial x'^m}{\partial x'^n} \frac{\partial x^n}{\partial x'^m} \frac{\partial x'^p}{\partial x^i} e_p \cdot \frac{\partial e'_i}{\partial x'^\ell} + \frac{\partial x'^k}{\partial x^p} \frac{\partial^2 x^p}{\partial x'^n \partial x'^i} e_n \Gamma'^m_{\ell n} + \frac{\partial x'^k}{\partial x^p} \frac{\partial^2 x^p}{\partial x'^n \partial x'^\ell} e_n \Gamma'^m_{\ell n}.
\]

(11.238)

The electromagnetic field \(A_i(x)\) and other gauge fields are connections.

Since the Levi-Civita connection \(\Gamma^k_{i\ell}\) is symmetric in \(i\) and \(\ell\), in four-dimensional spacetime, there are 10 of them for each \(k\), or 40 in all. The 10 correspond to 3 rotations, 3 boosts, and 4 translations.

### 11.33 Torsion

We can use Cartan’s tetrads to define a more general covariant derivative. The quantity \(F = F^k c_k\) is a scalar under general coordinate transformations.
It has one flat-space index $F^a = F^k e_k^a$ which participates under Lorentz transformations, but we’ll ignore that for the moment. Its $k$-derivative is

$$F^a_{;i} = F^k_{;i} e_k^a F^a_{;i} + F^k_{;i} e_k^a F^a_{;i}.$$  (11.239)

The four 4-vectors $e^a_k$ have four dual vectors $e^k_a$ that obey the rules

$$e^a_k e^a_a = \delta_k^a \quad \text{and} \quad e^a_k e^b_a = \delta_b^a.$$  (11.240)

The inner product of this equation with the dual vector $e^k$

$$F^a_{;i} e^k = F^k_{;i} e_k^a F^a_{;i} + F^k_{;i} e_k^a F^a_{;i} = F^k_{;i} e_k^a e_k^a F^a_{;i} + F^k_{;i} e_k^a e_k^a F^a_{;i} = F^k_{;i} e_k^a e_k^a F^a_{;i}$$

is Cartan’s covariant derivative $F^a_{;i}$ of the contravariant vector $F^k$. It is a mixed tensor $F^a_{;i}$ defined in terms of a more general connection $\omega^a_{ki}$

$$\omega^a_{ki} = e^k_a e^a_{ki}$$  (11.241)

that, unlike the Levi-Civita connection $\Gamma^a_{ki} = \Gamma^a_{ik}$, is not necessarily symmetric in its two lower indices $k$ and $i$.

The antisymmetric part of the Cartan connection is the torsion tensor

$$T^a_{ki} \equiv \frac{1}{2} \left( \omega^a_{ki} - \omega^a_{ik} \right)$$  (11.242)

which is a tensor.

### 11.34 Parallel Transport

The movement of a vector along a curve on a manifold so that its length and direction in successive tangent spaces do not change is called parallel transport. If the vector is $F = F^i e_i$, then we want the inner product $e^k \cdot dF$ of $dF$ with all dual tangent vectors $e^k$ to vanish along the curve. But this is just the condition that the covariant derivative (11.233) of $F$ should vanish along the curve

$$e^k \cdot \frac{\partial F}{\partial x^k} = \frac{\partial F^k}{\partial x^k} + \Gamma^k_{li} F^l = D_k F^k = 0.$$  (11.244)

**Example 11.22** (Parallel Transport on a Sphere). We parallel-transport the vector $v = (0, 1, 0)$ up from the equator along the line of longitude $\phi = 0$. Along this path, the vector $v = (0, 1, 0)$ is constant, so $\partial_\theta v = 0$ and $D_\theta v^k = 0$. Then we parallel-transport it down from the north pole along the line of longitude $\phi = \pi/2$ to the equator. Along this path, $\phi = \pi/2$, and the
vector \( \mathbf{v} = e_\theta / r = (0, \cos \theta, -\sin \theta) \) obeys the parallel-transport condition (11.244) because its \( \theta \)-derivative
\[
\frac{\partial \mathbf{v}}{\partial \theta} = \frac{1}{r} \frac{\partial e_\theta}{\partial \theta} = \frac{\partial}{\partial \theta} (0, \cos \theta, -\sin \theta) = (0, -\sin \theta, -\cos \theta) = -\mathbf{e}_\phi|_{\phi = \pi/2}
\]
is perpendicular to the tangent vectors \( e_\theta \) and \( e_\phi \) along the curve \( \phi = \pi/2 \). We then parallel-transport \( \mathbf{v} \) along the equator back to the starting point \( \phi = 0 \). Along this path, the vector \( \mathbf{v} = (0, 0, -1) \) is constant, so \( \partial_{\phi} \mathbf{v} = 0 \) and \( D_\phi \mathbf{v}^k = 0 \). The change from \( \mathbf{v} = (0, 1, 0) \) to \( \mathbf{v} = (0, 0, -1) \) is due to the curvature of the sphere.

**11.35 Covariant Derivatives**

In comma notation, the derivative of a contravariant vector field \( F = F^i e_i \) is
\[
F_{,\ell} = F^i_{,\ell} e_i + F^i e_{i,\ell}
\]
which in general lies outside the space spanned by the basis vectors \( e_i \). So we use the affine connections (11.231) to form the inner product
\[
e^k \cdot F_{,\ell} = e^k \cdot (F^i_{,\ell} e_i + F^i e_{i,\ell}) = F^i_{,\ell} \delta^k_i + F^i \Gamma^k_{i\ell} = F^k_{,\ell} + \Gamma^k_{\ell i} F^i.
\]
This **covariant derivative of a contravariant vector field** often is written with a semicolon
\[
F^k_{,\ell} = e^k \cdot F_{,\ell} = F^k_{,\ell} + \Gamma^k_{\ell i} F^i.
\]
It transforms as a mixed second-rank tensor. The invariant change \( dF \) projected onto \( e^k \) is
\[
e^k \cdot dF = e^k \cdot F_{,\ell} dx^\ell = F^k_{,\ell} dx^\ell.
\]
In terms of its covariant components, the derivative of a vector \( V \) is
\[
V_{,\ell} = (V_k e^k),_{\ell} = V_{k,\ell} e^k + V_k e^k_{,\ell}.
\]
To relate the derivatives of the vectors \( e^i \) to the affine connections \( \Gamma^k_{i\ell} \), we differentiate the orthonormality relation
\[
\delta^k_i = e^k \cdot e_i
\]
which gives us
\[
0 = e^k_{,\ell} \cdot e_i + e^k \cdot e_{i,\ell} \quad \text{or} \quad e^k_{,\ell} \cdot e_i = -e^k \cdot e_{i,\ell} = -\Gamma^k_{i\ell}.
\]
Since \( e_i \cdot e^k = -\Gamma^k_{i\ell} \), the inner product of \( e_i \) with the derivative of \( V \) is
\[
e_i \cdot V,_{\ell} = e_i \cdot \left( V_{k,\ell} e^k + V_k e^k,_{\ell} \right) = V_{i,\ell} - V_k \Gamma^k_{i\ell}.
\] (11.253)

This **covariant derivative of a covariant vector field** also is often written with a semicolon
\[
V_{i,\ell} = e_i \cdot V,_{\ell} = V_{i,\ell} - V_k \Gamma^k_{i\ell}.
\] (11.254)

It transforms as a rank-2 covariant tensor. Note the minus sign in \( V_{i,\ell} \) and the plus sign in \( F^k_{i,\ell} \). The change \( e_i \cdot dV \) is
\[
e_i \cdot dV = e_i \cdot V,_{\ell} \, dx^\ell = V_{i,\ell} \, dx^\ell.
\] (11.255)

Since \( dV \) is invariant, \( e_i \) covariant, and \( dx^\ell \) contravariant, the quotient rule (section 11.15) confirms that the covariant derivative \( V_{i,\ell} \) of a covariant vector \( V \) is a rank-2 covariant tensor.

We used similar logic to derive Cartan’s covariant derivative (11.241) of a contravariant vector
\[
F^k_{i,\ell} = F^k_{i,\ell} + F^i \omega^k_{i\ell}.
\] (11.256)

To find his formula for the covariant derivative of a covariant vector, we first differentiate the scalar \( V = V_k e^k \)
\[
V_{a,\ell} = V_{k,\ell} c^k_a + V_k c^k_{a,\ell}
\] (11.257)
and then use (11.240) to contract it with the dual vector \( c^a_i \)
\[
c^a_i \, V_{a,\ell} = V_{k,\ell} \, c^a_i \, c^k_a + V_k \, c^a_i \, c^k_{a,\ell} = V_{k,\ell} \, \delta^k_i + V_k \, c^a_i \, c^k_{a,\ell} = V_{i,\ell} + V_k \, c^a_i \, c^k_{a,\ell}.
\] (11.258)

To evaluate \( c^a_i \, c^k_{a,\ell} \), we differentiate the first of the dual-vector relations (11.240) to show that
\[
c^a_i \, c^k_{a,\ell} = -c^k_a \, c^a_i,_{\ell}.
\] (11.259)
Cartan’s covariant derivative of a covariant vector then is
\[
V_{i,\ell} = V_{i,\ell} - V_k \, c^k_a \, c^a_{i,\ell} = V_{i,\ell} - V_k \, \omega^k_{i,\ell}.
\] (11.260)

### 11.36 The Covariant Curl

Because the connection \( \Gamma^k_{i\ell} \) is symmetric (11.237) in its lower indices, the covariant curl of a covariant vector \( V \) is simply its ordinary curl
\[
V_{\ell,\iota} - V_{i,\ell} = V_{\ell,\iota} - V_k \Gamma^k_{\ell\iota} - V_{i,\ell} + V_k \Gamma^k_{i\ell} = V_{\ell,\iota} - V_{i,\ell}.
\] (11.261)
Thus the Faraday field-strength tensor $F_{\ell \ell}$, which is defined as the curl of the covariant vector field $A_i$

$$F_{\ell \ell} = A_{\ell,i} - A_{i,\ell}$$

is a generally covariant second-rank tensor.

If however we must use Cartan’s covariant derivative (11.241), then we would define the Faraday field-strength tensor $F_{\ell \ell}$ as

$$F_{\ell \ell} = A_{\ell,i} - A_{i,\ell} + A_\ell \left( \omega^k_{\ell \ell} - \omega^k_{i i} \right) = A_{\ell,i} - A_{i,\ell} + 2A_kT^k_{i \ell}$$

in which $T^k_{i \ell}$ is the torsion tensor (11.243).

**Example 11.23** (Orthogonal coordinates). In orthogonal coordinates, the curl is defined (6.39, 11.111) in terms of the totally antisymmetric Levi-Civita symbol $\epsilon^{ijk}$ (with $\epsilon_{123} = \epsilon^{123} = 1$), as

$$\nabla \times \mathbf{V} = \sum_{i=1}^{3} (\nabla \times \mathbf{V})_i \hat{e}_i = \frac{1}{h_1 h_2 h_3} \sum_{ijk=1}^{3} e_i \epsilon^{ijk} V_{k,j}$$

which, in view of (11.261) and the antisymmetry of $\epsilon^{ijk}$, is

$$\nabla \times \mathbf{V} = \sum_{ijk=1}^{3} (\nabla \times \mathbf{V})_i \hat{e}_i = \sum_{ijk=1}^{3} \frac{1}{h_i h_j h_k} e_i \epsilon^{ijk} V_{k,j}$$

or by (11.174 & 11.176)

$$\nabla \times \mathbf{V} = \sum_{ijk=1}^{3} \frac{1}{h_i h_j h_k} h_i \hat{e}_i \epsilon^{ijk} V_{k,j} = \sum_{ijk=1}^{3} \frac{1}{h_i h_j h_k} h_i \hat{e}_i \epsilon^{ijk} (h_k \mathbf{V}_k)_{,j}.$$  

(11.266)

Often one writes this as a determinant

$$\nabla \times \mathbf{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ V_1 & V_2 & V_3 \end{vmatrix} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ h_1 \mathbf{V}_1 & h_2 \mathbf{V}_2 & h_3 \mathbf{V}_3 \end{vmatrix}.$$  

(11.267)

In cylindrical coordinates, the curl is

$$\nabla \times \mathbf{V} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho \hat{\phi} & \hat{z} \\ \rho \phi \partial_\rho & \partial_\phi & \partial_z \\ V_\rho & \rho V_\phi & V_z \end{vmatrix}.$$  

(11.268)
In spherical coordinates, it is
\[
\nabla \times \hat{V} = \frac{1}{r^2 \sin \theta} \begin{vmatrix}
\hat{r} & \hat{\theta} & r \sin \theta \hat{\phi} \\
\partial_r & \partial_\theta & \partial_\phi \\
r \hat{V}_r & r \hat{V}_\theta & r \sin \theta \hat{V}_\phi
\end{vmatrix}.
\]
(11.269)

In more formal language, the curl is
\[
dV = d \left( V_k dx^k \right) = V_{k,i} dx^i \wedge dx^k = \frac{1}{2} (V_{k,i} - V_{i,k}) \ dx^i \wedge dx^k.
\]
(11.270)

### 11.37 Covariant derivative of a tensor

Let’s write a second-rank contravariant tensor in a coordinate-invariant way as a sum of direct products of tangent vectors \( e_i \)
\[
T = T^{jm} e_j \otimes e_m.
\]
(11.271)

Its derivative is
\[
T_{;k} = T^{jm}_{;k} e_j \otimes e_m + T^{jm} e_{j,k} \otimes e_m + T^{jm} e_j \otimes e_{m,k}.
\]
(11.272)

The covariant derivative \( T^{i\ell}_{;k} \) of the tensor is the inner product of this derivative with the contravariant or dual tangent vectors \( e^i \otimes e^\ell \)
\[
T^{i\ell}_{;k} = e^i \otimes e^\ell \cdot T_{;k}
\]
\[
= T^{jm}_{;k} e^i \cdot e_j e^\ell \cdot e_m + T^{jm} e^i \cdot e_{j,k} e^\ell \cdot e_m + T^{jm} e^i \cdot e_j e^\ell \cdot e_{m,k}
\]
\[
= T^{jm}_{;k} \delta_j^i \delta^\ell_m + T^{jm} e^i \cdot e_{j,k} e^\ell \cdot e_m + T^{jm} \delta^i_j e^\ell \cdot e_{m,k}
\]
\[
= T^{i\ell}_{;k} + T^{j\ell} e^i \cdot e_{j,k} + T^{im} e^\ell \cdot e_{m,k}.
\]
(11.273)

In it we recognize two instances of our formula \( \Gamma_j^i = e^i \cdot e_{j,k} \) for the affine connection (11.231). Replacing these inner products with affine connections, we may write the covariant derivative \( T^{i\ell}_{;k} \) of the tensor as
\[
T^{i\ell}_{;k} = T^{i\ell}_{;k} + T^{j\ell} \Gamma_j^i + T^{im} \Gamma^\ell_{mk}.
\]
(11.274)

A similar argument leads to a similar formula for the covariant derivative of a covariant tensor. A coordinate-independent form of a covariant tensor \( T^{jm} \) is
\[
T = T^{jm} e^j \otimes e^m.
\]
(11.275)
Its derivative is

\[ T_{,k} = T_{jm,k} \, e^j \otimes e^m + T_{jm} \, e^j \cdot e^m + T_{jm} \, e^j \otimes e^m. \tag{11.276} \]

The covariant derivative \( T_{ij,k} \) of the tensor is the inner product of this derivative with the covariant tangent vectors \( e_i \otimes e_j \)

\[
T_{ij,k} = e_i \otimes e_j \cdot T_{,k} = T_{ij,k} + T_{ij} \, e_i \cdot e^j + T_{im} \, e^i \cdot e^m. \tag{11.277}
\]

In it we see two instances of our other formula \( e^j \cdot e_i = -\Gamma^j_{ik} \) for the affine connection (11.252). Replacing them, we may write we may write the covariant derivative \( T_{ij,k} \) of the tensor as

\[
T_{ij,k} = T_{ij,k} - T_{ij} \, \Gamma^j_{ik} - T_{im} \, \Gamma^m_{ik}. \tag{11.278}
\]

The rule for a general tensor is to treat every contravariant index as in (11.274) and every covariant index as in (11.278). The covariant derivative of a mixed rank-4 tensor, for instance, is

\[
T^{ab}_{xy,k} = T^{ab}_{xy,k} + T^{ab}_{xy} \, \Gamma^{a}_{jk} + T^{am}_{xy} \, \Gamma^{m}_{jk} - T^{ab}_{xy} \, \Gamma^{j}_{zk} - T^{ab}_{ym} \, \Gamma^{m}_{yk}. \tag{11.279}
\]

11.38 Covariant Derivatives and Antisymmetry

Let us apply our rule (11.278) for the covariant derivative of a second-rank tensor \( A_{i\ell} \)

\[
A_{i\ell,k} = A_{i\ell,k} - \Gamma^m_{ik} \, A_{m\ell} - \Gamma^m_{ik} \, A_{hm} \tag{11.280}
\]

to an antisymmetric tensor

\[
A_{i\ell} = -A_{\ell i}. \tag{11.281}
\]

Then by adding together the three cyclic permutations of the indices \( i\ell k \) we find that the antisymmetry of the tensor and the symmetry (11.237) of the affine connection \( \Gamma^m_{ik} = \Gamma^m_{ki} \) conspire to cancel the quadratic terms leaving us with

\[
A_{i\ell,k} + A_{ki,\ell} + A_{k\ell,i} = A_{i\ell,k} - \Gamma^m_{ik} \, A_{m\ell} - \Gamma^m_{ik} \, A_{hm} + A_{ki,\ell} - \Gamma^m_{k\ell} \, A_{mi} - \Gamma^m_{k\ell} \, A_{km} + A_{k\ell,i} - \Gamma^m_{\ell i} \, A_{mk} - \Gamma^m_{\ell i} \, A_{km} = A_{i\ell,k} + A_{ki,\ell} + A_{k\ell,i} \tag{11.282}
\]

an identity named after Luigi Bianchi (1856–1928).
The Maxwell field-strength tensor $F_{\ell\ell}$ is antisymmetric by construction ($F_{\ell\ell} = A_{\ell,i} - A_{i,\ell}$), and so the homogeneous Maxwell equations

$$\varepsilon^{ijkl} F_{jk,\ell} = F_{jk,\ell} + F_{k\ell,j} + F_{\ell j,k} = A_{k,j\ell} - A_{j,k\ell} + A_{\ell,kj} - A_{k,\ell j} + A_{j,\ell k} - A_{\ell,jk} = 0$$  \hspace{1cm} (11.283)

are tensor equations valid in all coordinate systems. This is another example of how amazingly right Maxwell was in the middle of the nineteenth century.

If however we use Cartan’s covariant derivative (11.241) to define the Cartan-Faraday tensor (11.263)

**11.39 Affine Connection and Metric Tensor**

To relate the affine connection $\Gamma^m_{\ell i}$ to the derivatives of the metric tensor $g_{k\ell}$, we lower the contravariant index $m$ to get

$$\Gamma_{k\ell i} = g_{km} \Gamma^m_{\ell i} = g_{km} \Gamma^m_{\ell i} = \Gamma_{ki\ell}$$  \hspace{1cm} (11.284)

which is symmetric in its last two indices and which some call a Christoffel symbol of the first kind, written $[\ell i, k]$. One can raise the index $k$ back up by using the inverse of the metric tensor

$$g^{mk} \Gamma_{k\ell i} = g^{mk} g_{kn} \Gamma^n_{\ell i} = \delta^m_n \Gamma^n_{\ell i} = \Gamma^m_{\ell i}.$$  \hspace{1cm} (11.285)

Although we can raise and lower these indices, the connections $\Gamma^m_{\ell i}$ and $\Gamma_{k\ell i}$ are not tensors.

The definition (11.231) of the affine connection tells us that

$$\Gamma_{k\ell i} = g_{km} \Gamma^m_{\ell i} = g_{km} e^m \cdot e_{\ell,i} = e_k \cdot e_{\ell,i} = \Gamma_{ki\ell} = e_k \cdot e_{i,\ell}.$$  \hspace{1cm} (11.286)

By differentiating the definition $g_{i\ell} = e_i \cdot e_{\ell}$ of the metric tensor, we find

$$g_{i,\ell,k} = e_{i,k} \cdot e_{\ell} + e_i \cdot e_{\ell,k} = e_{\ell} \cdot e_{i,k} + e_i \cdot e_{\ell,k} = \Gamma_{\ell ik} + \Gamma_{i k\ell}.$$  \hspace{1cm} (11.287)

Permuting the indices cyclicly, we have

$$g_{ki,\ell} = \Gamma_{ik\ell} + \Gamma_{ki\ell}$$
$$g_{lki} = \Gamma_{kli} + \Gamma_{kli}.$$  \hspace{1cm} (11.288)

If we now subtract relation (11.287) from the sum of the two formulas (11.288) keeping in mind the symmetry $\Gamma_{abc} = \Gamma_{acb}$, then we find that four of the six terms cancel

$$g_{ki,\ell} + g_{lki} - g_{i,\ell,k} = \Gamma_{ik\ell} + \Gamma_{ki\ell} + \Gamma_{kli} + \Gamma_{i k\ell} - \Gamma_{\ell ik} - \Gamma_{i,\ell k} = 2\Gamma_{k\ell i}$$  \hspace{1cm} (11.289)

leaving a formula for $\Gamma_{k\ell i}$

$$\Gamma_{k\ell i} = \frac{1}{2} (g_{ki,\ell} + g_{lki} - g_{i,\ell,k}).$$  \hspace{1cm} (11.290)
Thus the connection is three derivatives of the metric tensor
\[ \Gamma^s_{it} = g^{sk} \Gamma_{kli} = \frac{1}{2} g^{sk} (g_{ki,t} + g_{tk,i} - g_{it,k}). \] (11.291)

11.40 Covariant Derivative of the Metric Tensor

Let us apply our formula (11.278) for the covariant derivative of a covariant tensor to the metric tensor \( g_{it,k} \equiv g_{it,k} - \Gamma^m_{ik} g_{mt} - \Gamma^n_{k\ell} g_{in} \). (11.292)

If we now substitute our formula (11.291) for the connections \( \Gamma^l_{ik} \) and \( \Gamma^n_{k\ell} \)
\[ g_{it,k} = g_{it,k} - \frac{1}{2} g^{ns} (g_{is,k} + g_{sk,i} - g_{ik,s}) g_{mt} - \frac{1}{2} g^{ns} (g_{is,k} + g_{sk,\ell} - g_{ik,s}) g_{in} \] (11.293)
and use the fact (11.160) that the metric tensors \( g_{it} \) and \( g^{\ell k} \) are mutually inverse, then we find
\[
\begin{align*}
g_{it,k} &= g_{it,k} - \frac{1}{2} \delta^s_i (g_{is,k} + g_{sk,i} - g_{ik,s}) - \frac{1}{2} \delta^s_i (g_{is,k} + g_{sk,\ell} - g_{ik,s}) \\
&= g_{it,k} - \frac{1}{2} (g_{it,k} + g_{tk,i} - g_{ik,\ell}) - \frac{1}{2} (g_{ti,k} + g_{ik,\ell} - g_{tk,i}) \\
&= 0. \quad (11.294)
\end{align*}
\]

The covariant derivative of the metric tensor vanishes. This result follows from our choice of the Levi-Civita connection (11.231); it is not true for some other connections.

11.41 Divergence of a contravariant vector

The contracted covariant derivative of a contravariant vector is a scalar known as the divergence,
\[ \nabla \cdot V = V^i_{j,i} = V^i_{j,i} + \Gamma^i_{jk} V^k. \] (11.295)

Because two indices in the connection
\[ \Gamma^i_{jk} = \frac{1}{2} g^{i\ell} (g_{it,k} + g_{k\ell,i} - g_{ik,\ell}) \] (11.296)
are contracted, its last two terms cancel because they differ only by the interchange of the dummy indices \( i \) and \( \ell \)
\[ g^{i\ell} g_{k\ell,i} = g^{ks} g_{k\ell,i} = g^{i\ell} g_{ki,\ell} = g^{i\ell} g_{ik,\ell}. \] (11.297)
So the contracted connection collapses to
\[ \Gamma^i_{jk} = \frac{1}{2} g^{i\ell} g_{it,k}. \] (11.298)
There is a nice formula for this last expression. To derive it, let \( g \equiv g_{i\ell} \) be the 4 \( \times \) 4 matrix whose elements are those of the covariant metric tensor \( g_{i\ell} \). Its determinant, like that of any matrix, is the cofactor sum (1.195) along any row or column, that is, over \( \ell \) for fixed \( i \) or over \( i \) for fixed \( \ell \)

\[
\det(g) = \sum_{i \text{ or } \ell} g_{i\ell} C_{i\ell}
\]

in which the cofactor \( C_{i\ell} \) is \((-1)^{i+\ell}\) times the determinant of the reduced matrix consisting of the matrix \( g \) with row \( i \) and column \( \ell \) omitted. Thus the partial derivative of \( \det g \) with respect to the \( i\ell \)th element \( g_{i\ell} \) is

\[
\frac{\partial \det(g)}{\partial g_{i\ell}} = C_{i\ell}
\]

in which we allow \( g_{i\ell} \) and \( g_{\ell i} \) to be independent variables for the purposes of this differentiation. The inverse \( g^{i\ell} \) of the metric tensor \( g \), like the inverse (1.197) of any matrix, is the transpose of the cofactor matrix divided by its determinant \( \det(g) \)

\[
g^{i\ell} = \frac{C_{\ell i}}{\det(g)} = \frac{1}{\det(g)} \frac{\partial \det(g)}{\partial g_{i\ell}}.
\]

Using this formula and the chain rule, we may write the derivative of the determinant \( \det(g) \) as

\[
\det(g)_{,k} = \frac{\partial \det(g)}{\partial g_{i\ell}} g_{i\ell,k} = \det(g) g^{\ell i}_{,k} g_{i\ell,k}
\]

and so since \( g_{i\ell} = g_{\ell i} \), the contracted connection (11.298) is

\[
\Gamma^i_{ik} = \frac{1}{2} g^{i\ell} g_{i\ell,k} = \frac{\det(g)_{,k}}{2 \det(g)} = \frac{|\det(g)|_{,k}}{2 |\det(g)|} = \frac{g_{,k}}{2 \sqrt{g}} = \frac{(\sqrt{g})_{,k}}{\sqrt{g}}
\]

in which \( g \equiv |\det(g)| \) is the absolute value of the determinant of the metric tensor.

Thus from (11.295), we arrive at our formula for the covariant divergence of a contravariant vector:

\[
\nabla \cdot V = V^i_{,i} = V^i_{,i} + \Gamma^i_{ik} V^k = V^k_{,k} + (\sqrt{g})_{,k} V^k = \frac{(\sqrt{g})_{,k} V^k}{\sqrt{g}}.
\]

An important application of this formula is the generally covariant form (11.224) of Maxwell’s inhomogeneous equations

\[
\frac{1}{\sqrt{g}} \left( \sqrt{g} F^{k\ell} \right)_{,\ell} = \mu_0 j^k.
\]
11.41 Divergence of a contravariant vector

More formally, the Hodge dual (11.218) of the 1-form \( V = V_i \, dx^i \) is

\[
*V = V_i \ast dx^i = V_i \frac{1}{3!} g^{ik} \eta_{klmn} \, dx^\ell \wedge dx^m \wedge dx^n
\]

\[
= \frac{1}{3!} \sqrt{g} V^k \epsilon_{klmn} \, dx^\ell \wedge dx^m \wedge dx^n \quad (11.306)
\]

in which \( g \) is the absolute value of the determinant of the metric tensor \( g_{ij} \).

The exterior derivative now gives

\[
d \ast V = \frac{1}{3!} \left( \sqrt{g} V^k \right)_p \epsilon_{klmn} \, dx^p \wedge dx^\ell \wedge dx^m \wedge dx^n.
\]

(11.307)

So using (11.218) to apply a second Hodge star, we get (exercise 11.20)

\[
* \ast \ast V = \frac{1}{3!} \left( \sqrt{g} V^k \right)_p \epsilon_{klmn} \ast \left( dx^p \wedge dx^\ell \wedge dx^m \wedge dx^n \right)
\]

\[
= \frac{1}{3!} \left( \sqrt{g} V^k \right)_p \epsilon_{klmn} g^{pt} g^{uq} g^{m} \eta_{tvuw}
\]

\[
= \frac{1}{3!} \left( \sqrt{g} V^k \right)_p \epsilon_{klmn} g^{pt} g^{uq} g^{m} \epsilon_{tvuw} \sqrt{g}
\]

\[
= \frac{1}{3!} \left( \sqrt{g} V^k \right)_p \epsilon_{klmn} \frac{1}{\det g_{ij}} \, e^{plmn}
\]

\[
= \frac{s}{\sqrt{g}} \left( \sqrt{g} V^k \right)_p \delta^p_k = \frac{s}{\sqrt{g}} \left( \sqrt{g} V^k \right)_k \quad (11.308)
\]

So in our spacetime with \( \det g_{ij} = -g \)

\[
- \ast \ast V = \frac{1}{\sqrt{g}} \left( \sqrt{g} V^k \right)_k \quad (11.309)
\]

In 3-space the Hodge star (11.207) of a 1-form \( V = V_i \, dx^i \) is

\[
*V = V_i \ast dx^i = V_i \frac{1}{2} g^{ij} \eta_{ijk} \, dx^j \wedge dx^k = \frac{1}{2} \sqrt{g} V^\ell \epsilon_{\ell jk} \, dx^j \wedge dx^k \quad (11.310)
\]

Applying the exterior derivative, we get the invariant form

\[
d \ast V = \frac{1}{2} \left( \sqrt{g} V^\ell \right)_p \epsilon_{\ell jk} \, dx^p \wedge dx^j \wedge dx^k \quad (11.311)
\]

We add a star by using the definition (11.207) of the Hodge dual in a 3-space in which the determinant \( \det g_{ij} \) is positive and the identity (exercise 11.19)

\[
\epsilon_{\ell jk} \epsilon^{ijk} = 2 \delta^p_\ell \quad (11.312)
\]
as well as the definition (1.184) of the determinant

\[
\ast d \ast V = \frac{1}{2} \left( \sqrt{g} V^\ell \right)_p \epsilon_{\ell j k} \ast \left( dx^p \wedge dx^j \wedge dx^k \right)
\]

\[
= \frac{1}{2} \left( \sqrt{g} V^\ell \right)_p \epsilon_{\ell j k} g^{p't} g^{j'u} g^{k'v} \eta_{tuv}
\]

\[
= \frac{1}{2} \left( \sqrt{g} V^\ell \right)_p \epsilon_{\ell j k} g^{p't} g^{j'u} g^{k'v} \epsilon_{tuv} \sqrt{g}
\]

\[
= \frac{1}{2} \left( \sqrt{g} V^\ell \right)_p \epsilon_{\ell j k} \epsilon_{\rho j k} \frac{\sqrt{g}}{\det g_{ij}}
\]

\[
= \frac{1}{\sqrt{g}} \left( \sqrt{g} V^\ell \right)_p \delta_p^\ell = \frac{1}{\sqrt{g}} (\sqrt{g} V^p)_p.
\] (11.313)

**Example 11.24** (Divergence in Orthogonal Coordinates). In two orthogonal coordinates, equations (11.169 & 11.176) imply that \( \sqrt{g} = h_1 h_2 \) and \( V^k = \nabla_k/h_k \), and so the divergence (11.304) of a vector \( \nabla \) is

\[
\nabla \cdot \nabla = \frac{1}{h_1 h_2} \sum_{k=1}^2 \left( \frac{h_1 h_2}{h_k} \nabla_k \right)_{,k}
\] (11.314)

which in polar coordinates (section 11.25) with \( h_r = 1 \) and \( h_\theta = r \), is

\[
\nabla \cdot \nabla = \frac{1}{r} \left[ (r \nabla_r)_{,r} + (\nabla_\theta)_{,\theta} \right] = \frac{1}{r} \left[ (r \nabla_r)_{,r} + \nabla_{\theta,\theta} \right].
\] (11.315)

In three orthogonal coordinates, equations (11.169 & 11.176) give \( \sqrt{g} = h_1 h_2 h_3 \) and \( V^k = \nabla_k/h_k \), and so the divergence (11.304) of a vector \( V \) is (6.29)

\[
\nabla \cdot V = \frac{1}{h_1 h_2 h_3} \sum_{k=1}^3 \left( \frac{h_1 h_2 h_3}{h_k} \nabla_k \right)_{,k}.
\] (11.316)

In cylindrical coordinates (section 11.26), \( h_\rho = 1 \), \( h_\phi = \rho \), and \( h_z = 1 \); so

\[
\nabla \cdot V = \frac{1}{\rho} \left[ (\rho \nabla_\rho)_{,\rho} + (\nabla_\phi)_{,\phi} + (\rho \nabla_z)_{,z} \right]
\]

\[
= \frac{1}{\rho} \left[ (\rho \nabla_\rho)_{,\rho} + \nabla_{\phi,\phi} + \rho \nabla_{z,z} \right].
\] (11.317)

In spherical coordinates (section 11.27), \( h_r = 1 \), \( h_\theta = r \), \( h_\phi = r \sin \theta \), \( g = |\det g| = r^4 \sin^2 \theta \) and the inverse \( g^{ij} \) of the metric tensor is

\[
(g^{ij}) = \begin{pmatrix}
1 & 0 & 0 \\
0 & r^{-2} & 0 \\
0 & 0 & r^{-2} \sin^{-2} \theta
\end{pmatrix}.
\] (11.318)
So our formula (11.314) gives us
\[
\nabla \cdot V = \frac{1}{r^2 \sin \theta} \left[ (r^2 \sin \theta V_r)_r + (r \sin \theta V_\theta)_\theta + (r \nabla_\phi)_\phi \right] \\
= \frac{1}{r^2 \sin \theta} \left[ \sin \theta (r^2 V_r)_r + r (\sin \theta V_\theta)_\theta + r \nabla_\phi \phi \right] 
\]
(11.319)
as the divergence \( \nabla \cdot V \).

### 11.42 The Covariant Laplacian

In flat 3-space, we write the laplacian as \( \nabla \cdot \nabla = \nabla^2 \) or as \( \Delta \). In euclidian coordinates, both mean \( \partial_x^2 + \partial_y^2 + \partial_z^2 \). In flat minkowski space, one often turns the triangle into a square and writes the 4-laplacian as \( \partial^2 \).

The gradient \( f_{,k} \) of a scalar field \( f \) is a covariant vector, and \( f^{,i} = g^{ik} f_{,k} \) is its contravariant form. The **invariant laplacian** \( \Box f \) of a scalar field \( f \) is the covariant divergence \( f^{,i} ; i \). We may use our formula (11.304) for the divergence of a contravariant vector to write it in these equivalent ways
\[
\Box f = f^{,i} = (g^{ik} f_{,k})_{;i} = \left( \frac{\sqrt{g}}{\sqrt{g}} f^{,i} \right)_{;i} = \left( \frac{\sqrt{g}}{\sqrt{g}} g^{ik} f_{,k} \right)_{;i}.
\]
(11.320)

To find the laplacian \( \Box f \) in terms of forms, we apply the exterior derivative to the Hodge dual (11.218) of the 1-form \( df = f_{,i} dx^i \)
\[
d \ast df = d \left( f_{,i} \ast dx^i \right) = d \left( \frac{1}{3!} f_{,i} g^{ik} \eta_{kmn} dx^k \wedge dx^m \wedge dx^n \right) \\
= \frac{1}{3!} \left( f^{,k} \sqrt{g} \right)_{,p} \epsilon_{kmn} dx^p \wedge dx^k \wedge dx^m \wedge dx^n
\]
(11.321)
and then add a star using (11.218)
\[
\ast d \ast df = \frac{1}{3!} \left( f^{,k} \sqrt{g} \right)_{,p} \epsilon_{kmn} \ast \left( dx^p \wedge dx^k \wedge dx^m \wedge dx^n \right) \\
= \frac{1}{3!} \left( f^{,k} \sqrt{g} \right)_{,p} \epsilon_{kmn} g^{pl} g^{fu} g^{mv} g^{nv} \sqrt{g} \epsilon_{tuw}.
\]
(11.322)
The definition (1.184) of the determinant now gives (exercise 11.20)
\[
\ast d \ast df = \frac{1}{3!} \left( f^{,k} \sqrt{g} \right)_{,p} \epsilon_{kmn} \epsilon^{plmn} \frac{\sqrt{g}}{\det g} \\
= \left( f^{,k} \sqrt{g} \right)_{,p} \delta^p_k \frac{s}{\sqrt{g}} = \frac{s}{\sqrt{g}} \left( f^{,k} \sqrt{g} \right)_{,k}.
\]
(11.323)
In our spacetime \( \det g_{ij} = sg = -g \), and so the laplacian is
\[
\Box f = -g \ast df = \frac{1}{\sqrt{g}} \left( f^k \sqrt{g} \right)_k .
\] (11.324)

**Example 11.25** (Invariant Laplacians). In two orthogonal coordinates, equations (11.169 \& 11.170) imply that \( \sqrt{\gamma} = \sqrt{\det(g_{ij})} = h_1 h_2 \) and that \( f^i = g^{ik} f_{,k} = h_i^{-2} f_{,i} \), and so the laplacian (11.320) of a scalar \( f \) is
\[
\Delta f = \frac{1}{h_1 h_2} \left( \sum_{i=1}^{2} \frac{h_1 h_2}{h_i^2} f_{,i} \right) .
\] (11.325)

In polar coordinates, where \( h_1 = 1 \), \( h_2 = r \), and \( g = r^2 \), the laplacian is
\[
\Delta f = \frac{1}{r} \left[ (rf_{,r}),_r + (r^{-1} f_{,\theta}),_\theta \right] = f_{,rr} + r^{-1} f_{,r} + r^{-2} f_{,\theta \theta} .
\] (11.326)

In three orthogonal coordinates, equations (11.169 \& 11.170) imply that \( \sqrt{\gamma} = \sqrt{\det(g_{ij})} = h_1 h_2 h_3 \) and that \( f^i = g^{ik} f_{,k} = h_i^{-2} f_{,i} \), and so the laplacian (11.320) of a scalar \( f \) is (6.33)
\[
\Delta f = \frac{1}{h_1 h_2 h_3} \left( \sum_{i=1}^{3} \frac{h_1 h_2 h_3}{h_i^2} f_{,i} \right) .
\] (11.327)

In cylindrical coordinates (section 11.26), \( h_\rho = 1 \), \( h_\phi = \rho \), \( h_z = 1 \), \( g = \rho^2 \), and the laplacian is
\[
\Delta f = \frac{1}{\rho} \left[ (\rho f_{,\rho}),_\rho + \frac{1}{\rho} f_{,\phi \phi} + \rho f_{,zz} \right] = f_{,\rho \rho} + \frac{1}{\rho} f_{,\rho} + \frac{1}{\rho^2} f_{,\phi \phi} + f_{,zz} .
\] (11.328)

In spherical coordinates (section 11.27), \( h_r = 1 \), \( h_\theta = r \), \( h_\phi = r \sin \theta \), and \( g = \det g = r^4 \sin^2 \theta \). So (11.327) gives us the laplacian of \( f \) as (6.35)
\[
\Delta f = \frac{(r^2 \sin \theta f_{,r}),_r + (\sin \theta f_{,\theta}),_\theta + (f_{,\phi / \sin \theta}),_\phi}{r^2 \sin \theta} = \frac{(r^2 f_{,r}),_r + (\sin \theta f_{,\theta}),_\theta + \frac{f_{,\phi \phi}}{r^2 \sin^2 \theta}}{r^2 \sin \theta} .
\] (11.329)

If the function \( f \) is a function only of the radial variable \( r \), then the laplacian is simply
\[
\Delta f(r) = \frac{1}{r^2} \left[ r^2 f^\prime(r) \right]^\prime = \frac{1}{r} [rf(r)]'' = f''(r) + \frac{2}{r} f'(r)
\] (11.330)
in which the primes denote \( r \)-derivatives. \(\square\)
11.43 The Principle of Stationary Action

It follows from a path-integral formulation of quantum mechanics that the classical motion of a particle is given by the principle of stationary action \( S = 0 \). In the simplest case of a free non-relativistic particle, the lagrangian is \( L = m\dot{x}^2/2 \) and the action is

\[
S = \int_{t_1}^{t_2} \frac{m}{2} \dot{x}^2 \, dt.
\]  

(11.331)

The classical trajectory is the one that when varied slightly by \( \delta x \) (with \( \delta x(t_1) = \delta x(t_2) = 0 \) does not change the action to first order in \( \delta x \). We first note that the change \( \delta \dot{x} \) in the velocity is the time derivative of the change in the path

\[
\delta \dot{x} = \dot{x}' - \dot{x} = \frac{d}{dt} (x' - x) = \frac{d}{dt} \delta x.
\]  

(11.332)

So since \( \delta x(t_1) = \delta x(t_2) = 0 \), the stationary path satisfies

\[
0 = \delta S = \int_{t_1}^{t_2} m \dot{x} \cdot \delta \dot{x} \, dt = \int_{t_1}^{t_2} m \dot{x} \cdot \frac{d\delta x}{dt} \, dt
\]

\[
= \int_{t_1}^{t_2} \left[ m \frac{d}{dt} (\dot{x} \cdot \delta x) - m \ddot{x} \cdot \delta x \right] dt
\]

\[
= m [\dot{x} \cdot \delta x]_{t_1}^{t_2} - m \int_{t_1}^{t_2} \ddot{x} \cdot \delta x \, dt = -m \int_{t_1}^{t_2} \ddot{x} \cdot \delta x \, dt.
\]  

(11.333)

If the first-order change in the action is to vanish for arbitrary small variations \( \delta x \) in the path, then the acceleration must vanish

\[
\ddot{x} = 0
\]  

(11.334)

which is the classical equation of motion for a free particle.

If the particle is moving under the influence of a potential \( V(x) \), then the action is

\[
S = \int_{t_1}^{t_2} \left( \frac{m}{2} \dot{x}^2 - V(x) \right) \, dt.
\]  

(11.335)

Since \( \delta V(x) = \nabla V(x) \cdot \delta x \), the principle of stationary action requires that

\[
0 = \delta S = \int_{t_1}^{t_2} (-m\ddot{x} - \nabla V) \cdot \delta x \, dt
\]  

(11.336)

or

\[
m\ddot{x} = -\nabla V
\]  

(11.337)
which is the classical equation of motion for a particle of mass $m$ in a potential $V$.

The action for a free particle of mass $m$ in special relativity is

$$S = -m \int_{\tau_1}^{\tau_2} dt = - \int_{t_1}^{t_2} m \sqrt{1 - \dot{x}^2} dt$$

(11.338)

where $c = 1$ and $\dot{x} = dx/dt$. The requirement of stationary action is

$$0 = \delta S = - \delta \int_{t_1}^{t_2} m \sqrt{1 - \dot{x}^2} dt = m \int_{t_1}^{t_2} \frac{\dot{x} \cdot \delta \dot{x}}{\sqrt{1 - \dot{x}^2}} dt$$

(11.339)

But $1/\sqrt{1 - \dot{x}^2} = dt/\tau$ and so

$$0 = \delta S = m \int_{t_1}^{t_2} \frac{dx}{dt} \cdot \frac{d\delta x}{dt} dt = m \int_{\tau_1}^{\tau_2} \frac{dx}{d\tau} \cdot \frac{d\delta x}{d\tau} d\tau$$

$$= m \int_{\tau_1}^{\tau_2} \frac{dx}{d\tau} \cdot \frac{d\delta x}{d\tau} d\tau. \quad (11.340)$$

So integrating by parts, keeping in mind that $\delta x(\tau_2) = \delta x(\tau_1) = 0$, we have

$$0 = \delta S = m \int_{\tau_1}^{\tau_2} \left[ \frac{d}{d\tau} \left( \frac{dx}{d\tau} \cdot \delta \dot{x} \right) - \frac{d^2 x}{d\tau^2} \cdot \delta x \right] d\tau = -m \int_{\tau_1}^{\tau_2} \frac{d^2 x}{d\tau^2} \cdot \delta x d\tau.$$  

(11.341)

To have this hold for arbitrary $\delta x$, we need

$$\frac{d^2 x}{d\tau^2} = 0 \quad (11.342)$$

which is the equation of motion for a free particle in special relativity.

What about a charged particle in an electromagnetic field $A_i$? Its action is

$$S = -m \int_{\tau_1}^{\tau_2} d\tau + q \int_{x_1}^{x_2} A_i(x) \, dx^i = \int_{\tau_1}^{\tau_2} \left( -m + q A_i \frac{dx^i}{d\tau} \right) d\tau. \quad (11.343)$$

We now treat the first term in a four-dimensional manner

$$\delta d\tau = \delta \sqrt{-\eta_{ik} dx^i dx^k} = -\frac{\eta_{ik} dx^i \delta dx^k}{\sqrt{-\eta_{ik} dx^i dx^k}} = -u_k \delta dx^k = -u_k d\delta x^k \quad (11.344)$$

in which $u_k = dx_k/d\tau$ is the 4-velocity (11.66) and $\eta$ is the Minkowski metric (11.27) of flat spacetime. The variation of the other term is

$$\delta (A_i dx^i) = (\delta A_i) \, dx^i + A_i \, \delta dx^i = A_{i,k} \delta x^k \, dx^i + A_i \, d\delta x^i \quad (11.345)$$
Putting them together, we get for $\delta S$

$$\delta S = \int_{\tau_1}^{\tau_2} \left( m u_k \frac{d\delta x^k}{d\tau} + q A_{i,k} \delta x^k \frac{dx^i}{d\tau} + q A_i \frac{d\delta x^i}{d\tau} \right) d\tau. \quad (11.346)$$

After integrating by parts the last term, dropping the boundary terms, and changing a dummy index, we get

$$\delta S = \int_{\tau_1}^{\tau_2} \left( - m \frac{du_k}{d\tau} \delta x^k + q A_{i,k} \delta x^k \frac{dx^i}{d\tau} - q \frac{dA_k}{d\tau} \delta x^k \right) d\tau$$

$$= \int_{\tau_1}^{\tau_2} \left[ - m \frac{du_k}{d\tau} + q (A_{i,k} - A_{k,i}) \frac{dx^i}{d\tau} \right] \delta x^k d\tau. \quad (11.347)$$

If this first-order variation of the action is to vanish for arbitrary $\delta x^k$, then the particle must follow the path

$$0 = -m \frac{du_k}{d\tau} + q (A_{i,k} - A_{k,i}) \frac{dx^i}{d\tau} \quad \text{or} \quad \frac{dp^k}{d\tau} = q F_{ki} u_i \quad (11.348)$$

which is the Lorentz force law (11.96).

### 11.44 A Particle in a Gravitational Field

The invariant action for a particle of mass $m$ moving along a path $x^i(t)$ is

$$S = -m \int_{\tau_1}^{\tau_2} d\tau = -m \int \left( - g_{\ell\ell} dx^\ell dx^\ell \right)^{\frac{1}{2}}. \quad (11.349)$$

Proceeding as in Eq.(11.344), we compute the variation $\delta d\tau$ as

$$\delta d\tau = \delta \sqrt{-g_{\ell\ell} dx^\ell dx^\ell} = \frac{\delta (g_{\ell\ell}) dx^\ell dx^\ell - 2 g_{\ell\ell} dx^\ell \delta dx^\ell}{2\sqrt{-g_{\ell\ell} dx^\ell dx^\ell}}$$

$$= -\frac{1}{2} g_{\ell\ell,k} \delta x^k u^\ell d\tau - g_{\ell\ell} \delta dx^\ell d\tau$$

$$= -\frac{1}{2} g_{\ell\ell,k} \delta x^k u^\ell d\tau - g_{\ell\ell} \frac{d\delta x^\ell}{d\tau} \quad (11.350)$$

in which $u^\ell = dx^\ell/d\tau$ is the 4-velocity (11.66). The condition of stationary action then is

$$0 = \delta S = -m \int_{\tau_1}^{\tau_2} \delta d\tau = m \int_{\tau_1}^{\tau_2} \left( \frac{1}{2} g_{\ell\ell,k} \delta x^k u^\ell u^\ell + g_{\ell\ell} u^\ell \frac{d\delta x^\ell}{d\tau} \right) d\tau \quad (11.351)$$
which we integrate by parts keeping in mind that \( \delta x^\ell(\tau_2) = \delta x^\ell(\tau_1) = 0 \)

\[
0 = m \int_{\tau_1}^{\tau_2} \left( \frac{1}{2} g_{i\ell,k} \delta x^k u^i u^\ell - \frac{d(g_{i\ell u^i})}{d\tau} \delta x^\ell \right) d\tau \\
= m \int_{\tau_1}^{\tau_2} \left( \frac{1}{2} g_{i\ell,k} \delta x^k u^i u^\ell - g_{i\ell,k} u^i u^k \delta x^\ell - g_{i\ell} \frac{d u^i}{d\tau} \delta x^\ell \right) d\tau. \tag{11.352}
\]

Now interchanging the dummy indices \( \ell \) and \( k \) on the second and third terms, we have

\[
0 = m \int_{\tau_1}^{\tau_2} \left( \frac{1}{2} g_{i\ell,k} u^i u^\ell - g_{ik,\ell} u^i u^\ell - g_{ik} \frac{d u^i}{d\tau} \right) \delta x^k d\tau \tag{11.353}
\]

or since \( \delta x^k \) is arbitrary

\[
0 = \frac{1}{2} g_{i\ell,k} u^i u^\ell - g_{ik,\ell} u^i u^\ell - g_{ik} \frac{d u^i}{d\tau}. \tag{11.354}
\]

If we multiply this equation of motion by \( g^{rk} \) and note that \( g^{ik,\ell} u^i u^\ell = g_{lk,i} u^i u^\ell \), then we find

\[
0 = \frac{d u^r}{d\tau} + \frac{1}{2} g^{rk} (g_{lk,\ell} + g_{lk,i} - g_{il,k}) u^i u^\ell. \tag{11.355}
\]

So using the symmetry \( g_{i\ell} = g_{\ell i} \) and the formula (11.291) for \( \Gamma^r_{i\ell} \), we get

\[
0 = \frac{d u^r}{d\tau} + \Gamma^r_{i\ell} u^i u^\ell \quad \text{or} \quad 0 = \frac{d^2 x^r}{d\tau^2} + \Gamma^r_{i\ell} \frac{d x^i}{d\tau} \frac{d x^\ell}{d\tau} \tag{11.356}
\]

which is the geodesic equation. In empty space, particles fall along geodesics **independently of their masses**.

The right-hand side of the geodesic equation (11.356) is a contravariant vector because \( \langle 11.347 \& 11.353 \rangle \) under general coordinate transformations, the inhomogeneous terms arising from \( \ddot{x}^r \) cancel those from \( \Gamma^r_{i\ell} \dot{x}^i \dot{x}^\ell \). Here and often in what follows we’ll use \( \ldots \) to mean proper-time derivatives.

The action for a particle of mass \( m \) and charge \( q \) in a gravitational field \( \Gamma^r_{i\ell} \) and an electromagnetic field \( A_i \) is

\[
S = -m \int - g_{i\ell} dx^i dx^\ell \right]^\frac{1}{2} + q \int_{\tau_1}^{\tau_2} A_i(x) dx^i \tag{11.357}
\]

because the interaction \( q \int A_i dx^i \) is invariant under general coordinate transformations. By (11.347 \& 11.353), the first-order change in \( S \) is

\[
\delta S = m \int_{\tau_1}^{\tau_2} \left[ \frac{1}{2} g_{i\ell,k} u^i u^\ell - g_{ik,\ell} u^i u^\ell - g_{ik} \frac{d u^i}{d\tau} + q (A_{i,k} - A_k,i) u^i \right] \delta x^k d\tau \tag{11.358}
\]
and so by combining the Lorentz force law (11.348) and the geodesic equation (11.356) and by writing $F^r{}_{i} \dot{x}^i$ as $F^r{}_{i} \dot{x}^i$, we have

$$0 = \frac{d^2 x^r}{d \tau^2} + \Gamma^r_{i\ell} \frac{dx^i}{d \tau} \frac{dx^\ell}{d \tau} - \frac{q}{m} F^r_{\ell} \frac{dx^\ell}{d \tau}$$

(11.359)

as the equation of motion of a particle of mass $m$ and change $q$. It is striking how nearly perfect the electromagnetism of Faraday and Maxwell is.

The action of the electromagnetic field interacting with an electric current $j^k$ in a gravitational field is

$$S = \int \left[ -\frac{1}{4} F_{k\ell} F^{k\ell} + \mu_0 A_k j^k \right] \sqrt{g} \, d^4x$$

(11.360)

in which $\sqrt{g} \, d^4x$ is the invariant volume element. After an integration by parts, the first-order change in the action is

$$\delta S = \int \left[ -\frac{\partial}{\partial x^\ell} \left( F^{k\ell} \sqrt{g} \right) + \mu_0 j^k \sqrt{g} \right] \delta A_k \, d^4x,$$

(11.361)

and so the inhomogeneous Maxwell equations in a gravitational field are

$$\frac{\partial}{\partial x^\ell} \left( \sqrt{g} F^{k\ell} \right) = \mu_0 \sqrt{g} j^k.$$  

(11.362)

11.45 The Principle of Equivalence

The principle of equivalence says that in any gravitational field, one may choose free-fall coordinates in which all physical laws take the same form as in special relativity without acceleration or gravitation—at least over a suitably small volume of spacetime. Within this volume and in these coordinates, things behave as they would at rest deep in empty space far from any matter or energy. The volume must be small enough so that the gravitational field is constant throughout it.

Example 11.26 (Elevators). When a modern elevator starts going down from a high floor, it accelerates downward at something less than the local acceleration of gravity. One feels less pressure on one’s feet; one feels lighter. After accelerating downward for a few seconds, the elevator assumes a constant downward speed, and then one feels the normal pressure of one’s weight on one’s feet. The elevator seems to be slowing down for a stop, but actually it has just stopped accelerating downward.

What if the cable snapped, and a frightened passenger dropped his laptop?
He could catch it very easily as it would not seem to fall because the elevator, the passenger, and the laptop would all fall at the same rate. The physics in the falling elevator would be the same as if the elevator were at rest in empty space far from any gravitational field. The laptop’s clock would tick as fast as it would at rest in the absence of gravity.

The transformation from arbitrary coordinates $x^k$ to free-fall coordinates $y^i$ changes the metric $g_{j\ell}$ to the diagonal metric $\eta_{ik}$ of flat spacetime $\eta = \text{diag}(-1,1,1,1)$, which has two indices and is not a Levi-Civita tensor. Algebraically, this transformation is a congruence (1.312)

$$\eta_{ik} = \frac{\partial x^j}{\partial y^i} g_{j\ell} \frac{\partial x^\ell}{\partial y^k}.$$ (11.363)

The geodesic equation (11.356) follows from the principle of equivalence (Weinberg, 1972; Hobson et al., 2006). Suppose a particle is moving under the influence of gravitation alone. Then one may choose free-fall coordinates $y(x)$ so that the particle obeys the force-free equation of motion

$$\frac{d^2 y^i}{d\tau^2} = 0,$$ (11.364)

with $d\tau$ the proper time $d\tau^2 = -\eta_{ik} dy^i dy^k$. The chain rule applied to $y^i(x)$ in (11.364) gives

$$0 = \frac{d}{d\tau} \left( \frac{\partial y^i}{\partial x^k} \frac{dx^k}{d\tau} \right)$$

$$= \frac{\partial y^i}{\partial x^k} \frac{d^2 x^k}{d\tau^2} + \frac{\partial^2 y^i}{\partial x^k \partial x^\ell} \frac{dx^k}{d\tau} \frac{dx^\ell}{d\tau}.$$ (11.365)

We multiply by $\partial x^m/\partial y^i$ and use the identity

$$\frac{\partial x^m}{\partial y^i} \frac{\partial y^i}{\partial x^k} = \delta^m_k$$ (11.366)

to get the equation of motion (11.364) in the $x$-coordinates

$$\frac{d^2 x^m}{d\tau^2} + \Gamma^m_{k\ell} \frac{dx^k}{d\tau} \frac{dx^\ell}{d\tau} = 0$$ (11.367)

in which the affine connection is

$$\Gamma^m_{k\ell} = \frac{\partial x^m}{\partial y^i} \frac{\partial^2 y^i}{\partial x^k \partial x^\ell}.$$ (11.368)

So the principle of equivalence tells us that a particle in a gravitational field obeys the geodesic equation (11.356).
11.46 Weak, Static Gravitational Fields

Slow motion in a weak, static gravitational field is an important example. Because the motion is slow, we neglect $u^i$ compared to $u^0$ and simplify the geodesic equation (11.356) to

$$0 = \frac{du^r}{d\tau} + \Gamma^r_{00} (u^0)^2. \quad (11.369)$$

Because the gravitational field is static, we neglect the time derivatives $g^r_{k0,0}$ and $g^r_{0k,0}$ in the connection formula (11.291) and find for $\Gamma^r_{00}$

$$\Gamma^r_{00} = \frac{1}{2} g^{r^k} (g_{k0,0} + g_{0k,0} - g_{00,k}) = -\frac{1}{2} g^{r^k} g_{00,k} \quad (11.370)$$

with $\Gamma^0_{00} = 0$. Because the field is weak, the metric can differ from $\eta_{ij}$ by only a tiny tensor $g_{ij} = \eta_{ij} + h_{ij}$ so that to first order in $|h_{ij}| \ll 1$ we have $\Gamma^r_{00} = -\frac{1}{2} h_{00,r}$ for $r = 1, 2, 3$. With these simplifications, the geodesic equation (11.356) reduces to

$$\frac{d^2x^r}{d\tau^2} = \frac{1}{2} (u^0)^2 h_{00,r} \quad \text{or} \quad \frac{d^2x^r}{d\tau^2} = \frac{1}{2} \left( \frac{dx^0}{d\tau} \right)^2 h_{00,r}. \quad (11.371)$$

So for slow motion, the ordinary acceleration is described by Newton’s law

$$\frac{d^2x}{dt^2} = \frac{c^2}{2} \nabla h_{00}. \quad (11.372)$$

If $\phi$ is his potential, then for slow motion in weak static fields

$$g_{00} = -1 + h_{00} = -1 - 2\phi/c^2 \quad \text{and so} \quad h_{00} = -2\phi/c^2. \quad (11.373)$$

Thus, if the particle is at a distance $r$ from a mass $M$, then $\phi = -GM/r$ and $h_{00} = -2\phi/c^2 = 2GM/rc^2$ and so

$$\frac{d^2x}{dt^2} = -\nabla \phi = \nabla \frac{GM}{r} = -GM \frac{r}{r^3}. \quad (11.374)$$

How weak are the static gravitational fields we know about? The dimensionless ratio $\phi/c^2$ is $10^{-39}$ on the surface of a proton, $10^{-9}$ on the Earth, $10^{-6}$ on the surface of the sun, and $10^{-4}$ on the surface of a white dwarf.

11.47 Gravitational Time Dilation

Suppose we have a system of coordinates $x^i$ with a metric $g_{ik}$ and a clock at rest in this system. Then the proper time $d\tau$ between ticks of the clock is

$$d\tau = (1/c)\sqrt{-g_{ij} dx^i dx^j} = \sqrt{-g_{00}} dt \quad (11.375)$$
where $dt$ is the time between ticks in the $x^i$ coordinates, which is the laboratory frame in the gravitational field $g_{00}$. By the principle of equivalence (section 11.45), the proper time $d\tau$ between ticks is the same as the time between ticks when the same clock is at rest deep in empty space.

If the clock is in a weak static gravitational field due to a mass $M$ at a distance $r$, then

$$-g_{00} = 1 + 2\phi/c^2 = 1 - 2GM/c^2r$$

(11.376)

is a little less than unity, and the interval of proper time between ticks

$$d\tau = \sqrt{-g_{00}} dt = \sqrt{1 - 2GM/c^2r} dt$$

(11.377)

is slightly less than the interval $dt$ between ticks in the coordinate system of an observer at $x$ in the rest frame of the clock and the mass, and in its gravitational field. Since $dt > d\tau$, the laboratory time $dt$ between ticks is greater than the proper or intrinsic time $d\tau$ between ticks of the clock unaffected by any gravitational field. Clocks near big masses run slow.

Now suppose we have two identical clocks at different heights above sea level. The time $T_\ell$ for the lower clock to make $N$ ticks will be longer than the time $T_u$ for the upper clock to make $N$ ticks. The ratio of the clock times will be

$$\frac{T_\ell}{T_u} = \frac{\sqrt{1 - 2GM/c^2(r + h)}}{\sqrt{1 - 2GM/c^2r}} \approx 1 + \frac{gh}{c^2}.$$  

(11.378)

Now imagine that a photon going down passes the upper clock which measures its frequency as $\nu_u$ and then passes the lower clock which measures its frequency as $\nu_\ell$. The slower clock will measure a higher frequency. The ratio of the two frequencies will be the same as the ratio of the clock times

$$\frac{\nu_\ell}{\nu_u} = 1 + \frac{gh}{c^2}.$$  

(11.379)

As measured by the lower, slower clock, the photon is blue shifted.

**Example 11.27** (Pound, Rebka, and Mössbauer). Pound and Rebka in 1960 used the Mössbauer effect to measure the blue shift of light falling down a 22.6 m shaft. They found

$$\frac{\nu_\ell - \nu_u}{\nu} = \frac{gh}{c^2} = 2.46 \times 10^{-15}$$

(11.380)

Example 11.28 (Redshift of the Sun). A photon emitted with frequency $\nu_0$ at a distance $r$ from a mass $M$ would be observed at spatial infinity to have frequency $\nu$

$$\nu = \nu_0 \sqrt{-g_{00}} = \nu_0 \sqrt{1 - 2MG/c^2r} \quad (11.381)$$

for a redshift of $\Delta\nu = \nu_0 - \nu$. Since the Sun’s dimensionless potential $\phi/\Omega/c^2$ is $-MG/c^2r = -2.12 \times 10^{-6}$ at its surface, sunlight is shifted to the red by 2 parts per million.

### 11.48 Curvature

The curvature tensor or Riemann tensor is

$$R^{i}_{mnk} = \Gamma^{i}_{mn,k} - \Gamma^{i}_{mk,n} + \Gamma^{j}_{kj} \Gamma^{i}_{nm} - \Gamma^{i}_{nj} \Gamma^{j}_{km} \quad (11.382)$$

which we may write as the commutator

$$R^{i}_{mnk} = (R_{nk})^{i}_{m} = [\partial_{k} + \Gamma_{k,n} + \Gamma_{n}^{i}]^{i}_{m} \quad (11.383)$$

in which the $\Gamma$’s are treated as matrices

$$\Gamma^{i}_{k} = \begin{pmatrix}
\Gamma^{0}_{0} & \Gamma^{1}_{0} & \Gamma^{2}_{0} & \Gamma^{3}_{0} \\
\Gamma^{0}_{1} & \Gamma^{1}_{1} & \Gamma^{2}_{1} & \Gamma^{3}_{1} \\
\Gamma^{0}_{2} & \Gamma^{1}_{2} & \Gamma^{2}_{2} & \Gamma^{3}_{2} \\
\Gamma^{0}_{3} & \Gamma^{1}_{3} & \Gamma^{2}_{3} & \Gamma^{3}_{3}
\end{pmatrix} \quad (11.384)$$

with $(\Gamma^{i}_{k} \Gamma^{j}_{n})^{m}_{i} = \Gamma^{i}_{kj} \Gamma^{j}_{nm}$ and so forth. Just as there are two conventions for the Faraday tensor $F_{ik}$ which differ by a minus sign, so too there are two conventions for the curvature tensor $R^{i}_{mnk}$. Weinberg (Weinberg, 1972) uses the definition (11.382); Carroll (Carroll, 2003), Schutz (Schutz, 2009), and Zee (Zee, 2013) use an extra minus sign.

The Ricci tensor is a contraction of the curvature tensor

$$R_{mk} = R_{mnk} \quad (11.385)$$

and the scalar curvature is a further contraction

$$R = g^{mk} R_{mk} \quad (11.386)$$

Example 11.29 (Curvature of a Sphere). While in four-dimensional space-time indices run from 0 to 3, on the sphere they are just $\theta$ and $\phi$. There are
only eight possible affine connections, and because of the symmetry (11.237)
in their lower indices $\Gamma^i_{\theta\phi} = \Gamma^i_{\phi\theta}$, only six are independent.

The point $p$ on a sphere of radius $r$ has cartesian coordinates

$$p = r \ (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad (11.387)$$

so the two 3-vectors are

$$e_\theta = \frac{\partial p}{\partial \theta} = r \ (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) = r \hat{\theta} \quad (11.388)$$

$$e_\phi = \frac{\partial p}{\partial \phi} = r \sin \theta \ (-\sin \phi, \cos \phi, 0) = r \sin \theta \hat{\phi}$$

and the metric $g_{ij} = e_i \cdot e_j$ is

$$(g_{ij}) = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \sin^2 \theta \end{pmatrix}. \quad (11.389)$$

Differentiating the vectors $e_\theta$ and $e_\phi$, we find

$$e_{\theta,\theta} = -r \ (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = -r \hat{r} \quad (11.390)$$

$$e_{\theta,\phi} = r \cos \theta \ (-\sin \phi, \cos \phi, 0) = r \cos \theta \hat{\phi} \quad (11.391)$$

$$e_{\phi,\theta} = e_{\theta,\phi} \quad (11.392)$$

$$e_{\phi,\phi} = -r \sin \theta \ (\cos \phi, \sin \phi, 0). \quad (11.393)$$

The metric with upper indices $(g^{ij})$ is the inverse of the metric $(g_{ij})$

$$(g^{ij}) = \begin{pmatrix} r^{-2} & 0 \\ 0 & r^{-2} \sin^{-2} \theta \end{pmatrix} \quad (11.394)$$

so the dual vectors $e^i$ are

$$e^\theta = r^{-1} \ (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) = r^{-1} \hat{\theta}$$

$$e^\phi = \frac{1}{r \sin \theta} \ (-\sin \phi, \cos \phi, 0) = \frac{1}{r \sin \theta} \hat{\phi}. \quad (11.395)$$

The affine connections are given by (11.231) as

$$\Gamma^i_{jk} = \Gamma^i_{kj} = e^i \cdot e_{j,k}. \quad (11.396)$$

Since both $e^\theta$ and $e^\phi$ are perpendicular to $\hat{r}$, the affine connections $\Gamma^\theta_{\theta\theta}$ and $\Gamma^\phi_{\phi\phi}$ both vanish. Also, $e_{\phi,\phi}$ is orthogonal to $\hat{\phi}$, so $\Gamma^\phi_{\phi\phi} = 0$ as well. Similarly, $e_{\theta,\phi}$ is perpendicular to $\hat{\phi}$, so $\Gamma^\theta_{\theta\phi} = \Gamma^\phi_{\phi\theta}$ also vanishes.

The two nonzero affine connections are

$$\Gamma^\phi_{\theta\phi} = e^\phi \cdot e_{\theta,\phi} = r^{-1} \sin^{-1} \theta \hat{\phi} \cdot r \cos \theta \hat{\theta} = \cot \theta \quad (11.397)$$
and

\[
\Gamma^\theta_{\phi\phi} = e^\theta \cdot e_{\phi,\phi} \\
= -\sin \theta (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) \cdot (\cos \phi, \sin \phi, 0) \\
= -\sin \theta \cos \theta.
\] (11.398)

In terms of the two non-zero affine connections \( \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot \theta \) and \( \Gamma^\theta_{\phi\phi} = -\sin \theta \cos \theta \), the two Christoffel matrices (11.384) are

\[
\Gamma_\theta = \begin{pmatrix} 0 & 0 \\ 0 & \Gamma^\phi_{\theta\phi} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \cot \theta \end{pmatrix}
\] (11.399)

and

\[
\Gamma_\phi = \begin{pmatrix} 0 & \Gamma^\theta_{\phi\phi} \\ \Gamma^\phi_{\phi\theta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sin \theta \cos \theta \\ \cot \theta & 0 \end{pmatrix}.
\] (11.400)

Their commutator is

\[
[\Gamma_\theta, \Gamma_\phi] = \begin{pmatrix} 0 & \cos^2 \theta \\ \cot^2 \theta & 0 \end{pmatrix} = -[\Gamma_\phi, \Gamma_\theta]
\] (11.401)

and both \([\Gamma_\theta, \Gamma_\theta]\) and \([\Gamma_\phi, \Gamma_\phi]\) vanish.

So the commutator formula (11.383) gives for Riemann’s curvature tensor

\[
R^\theta_{\theta\theta\theta} = [\partial_\theta + \Gamma_\theta, \partial_\theta + \Gamma_\theta]_\theta^\theta = 0 \\
R^\phi_{\phi\theta\theta} = [\partial_\theta + \Gamma_\theta, \partial_\phi + \Gamma_\phi]_\theta^\phi = (\Gamma_\phi, \theta)_\theta^\phi + [\Gamma_\phi, \Gamma_\theta]_\theta^\phi \\
= (\cot \theta, \theta) + \cot^2 \theta = -1 \\
R^\phi_{\phi\theta\phi} = [\partial_\phi + \Gamma_\phi, \partial_\phi + \Gamma_\phi]_\phi^\phi = - (\Gamma_\phi, \phi)_{\phi}^\phi + [\Gamma_\phi, \Gamma_\phi]_{\phi}^\phi \\
= \cos^2 \theta - \sin^2 \theta - \cos^2 \theta = -\sin^2 \theta \\
R^\phi_{\phi\phi\phi} = [\partial_\phi + \Gamma_\phi, \partial_\phi + \Gamma_\phi]_{\phi}^\phi = 0.
\] (11.402)

The Ricci tensor (11.385) is the contraction \( R_{mk} = R^m_{mnk} \), and so

\[
R_{\theta\theta} = R^\theta_{\theta\theta\theta} + R^\phi_{\theta\phi\theta} = -1 \\
R_{\phi\phi} = R^\theta_{\phi\phi\theta} + R^\phi_{\phi\phi\phi} = -\sin^2 \theta.
\] (11.403)

The curvature scalar (11.386) is the contraction \( R = g^{km}R_{mk} \), and so since \( g^{\theta\theta} = r^{-2} \) and \( g^{\phi\phi} = r^{-2} \sin^{-2} \theta \), it is

\[
R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = -r^{-2} - \sin^2 \theta r^{-2} \sin^{-2} \theta = -\frac{2}{r^2}
\] (11.404)

for a 2-sphere of radius \( r \).
Gauss invented a formula for the curvature \( K \) of a surface; for all two-dimensional surfaces, his \( K = -R/2 \).

\section*{11.49 Einstein’s Equations}

The source of the gravitational field is the \textbf{energy-momentum tensor} \( T_{ij} \). In many astrophysical and most cosmological models, the energy-momentum tensor is assumed to be that of a \textbf{perfect fluid}, which is isotropic in its rest frame, does not conduct heat, and has zero viscosity. For a perfect fluid of pressure \( p \) and density \( \rho \) with 4-velocity \( u^i \) (defined by (11.66)), the energy-momentum or \textbf{stress-energy} tensor \( T_{ij} \) is

\[
T_{ij} = p g_{ij} + (p + \rho) u_i u_j \quad (11.405)
\]
in which \( g_{ij} \) is the spacetime metric.

An important special case is the energy-momentum tensor due to a nonzero value of the energy density of the vacuum. In this case \( p = -\rho \) and the energy-momentum tensor is

\[
T_{ij} = -\rho g_{ij} \quad (11.406)
\]
in which \( \rho \) is the (presumably constant) value of the energy density of the ground state of the theory. This energy density \( \rho \) is a plausible candidate for the \textbf{dark-energy} density. It is equivalent to a \textbf{cosmological constant} \( \Lambda = 8\pi G \rho \).

Whatever its nature, the energy-momentum tensor usually is defined so as to satisfy the conservation law

\[
0 = (T^i)_;i = \partial_i T^i_j + \Gamma^i_{ic} T^c_j - T^i_c \Gamma^c_{ji}. \quad (11.407)
\]

Einstein’s equations relate the Ricci tensor (11.385) and the scalar curvature (11.386) to the energy-momentum tensor

\[
R_{ij} - \frac{1}{2} g_{ij} R = -\frac{8\pi G}{c^2} T_{ij} \quad (11.408)
\]
in which \( G = 6.7087 \times 10^{-39} \text{hc (GeV/c}^2)^{-2} = 6.6742 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2} \) is Newton’s constant. Taking the trace and using \( g^{ji} g_{ij} = \delta^j_i = 4 \), we relate the scalar curvature to the trace \( T = T^i_i \) of the energy-momentum tensor

\[
R = \frac{8\pi G}{c^2} T. \quad (11.409)
\]
So another form of Einstein’s equations (11.408) is
\[ R_{ij} = -\frac{8\pi G}{c^2} \left( T_{ij} - \frac{T}{2} g_{ij} \right). \] (11.410)

On small scales, such as that of our solar system, one may neglect dark energy. So in empty space and on small scales, the energy-momentum tensor vanishes \( T_{ij} = 0 \) along with its trace and the scalar curvature \( T = 0 = R \), and Einstein’s equations are
\[ R_{ij} = 0. \] (11.411)

11.50 The Action of General Relativity

If we make an action that is a scalar, invariant under general coordinate transformations, and then apply to it the principle of stationary action, we will get tensor field equations that are invariant under general coordinate transformations. If the metric of spacetime is among the fields of the action, then the resulting theory will be a possible theory of gravity. If we make the action as simple as possible, it will be Einstein’s theory.

To make the action of the gravitational field, we need a scalar. Apart from the volume 4-form \( *1 = \sqrt{g} \, d^4x = \sqrt{g} \, c \, dt \, d^3x \), the simplest scalar we can form from the metric tensor and its first and second derivatives is the scalar curvature \( R \) which gives us the Einstein-Hilbert action
\[ S_{EH} = -\frac{c^3}{16\pi G} \int R \sqrt{g} \, d^4x = -\frac{c^3}{16\pi G} \int g^{ik} R_{ik} \sqrt{g} \, d^4x. \] (11.412)

If \( \delta g^{ik}(x) \) is a tiny change in the inverse metric, then we may write the first-order change in the action \( S_{EH} \) as (exercise 11.46)
\[ \delta S_{EH} = -\frac{c^3}{16\pi G} \int \left( R_{ik} - \frac{1}{2} g_{ik} R \right) \sqrt{g} \delta g^{ik} \, d^4x. \] (11.413)

Thus the principle of least action \( \delta S_{EH} = 0 \) leads to Einstein’s equations
\[ G_{ik} \equiv R_{ik} - \frac{1}{2} g_{ik} R = 0 \] (11.414)
for empty space in which \( G_{ik} \) is Einstein’s tensor.

The stress-energy tensor \( T_{ik} \) is defined so that the change in the action of the matter fields due to a tiny change \( \delta g^{ik}(x) \) (vanishing at infinity) in the metric is
\[ \delta S_m = -\frac{c}{2} \int T_{ik} \sqrt{g} \delta g^{ik} \, d^4x = \frac{c}{2} \int T^{ik} \sqrt{g} \delta g_{ik} \, d^4x \] (11.415)
in which the sign change arises from the identity
\[ 0 = \delta g^{ik} g_{kl} + g^{ik} \delta g_{kl}. \]
So the principle of least action \( \delta S = \delta S_{EH} + \delta S_m = 0 \) implies Einstein’s equations (11.408, 11.410) in the presence of matter and energy
\[
R_{ik} - \frac{1}{2} g_{ik} R = -\frac{8\pi G}{c^2} T_{ij} \quad \text{or} \quad R_{ij} = -\frac{8\pi G}{c^2} \left( T_{ij} - \frac{T}{2} g_{ij} \right). \quad (11.416)
\]

### 11.51 Standard Form

Tensor equations are independent of the choice of coordinates, so it’s wise to choose coordinates that simplify one’s work. For a static and isotropic gravitational field, this choice is the standard form (Weinberg, 1972, ch. 8)

\[
ds^2 = -B(r) c^2 dt^2 + A(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \quad (11.417)
\]
in which \( B(r) \) and \( A(r) \) are functions that one may find by solving the field equations (11.408). Since \( ds^2 = -c^2 d\tau^2 = g_{ij} dx^i dx^j \), the nonzero components of the metric tensor are \( g_{rr} = A(r) \), \( g_{\theta\theta} = r^2 \), \( g_{\phi\phi} = r^2 \sin^2 \theta \), and \( g_{00} = -B(r) \), and those of its inverse are \( g^{rr} = A^{-1}(r) \), \( g^{\theta\theta} = r^{-2} \), \( g^{\phi\phi} = r^{-2} \sin^{-2} \theta \), and \( g^{00} = -B^{-1}(r) \). By differentiating the metric tensor and using (11.291), one gets the components of the connection \( \Gamma^k_{ij} \), such as \( \Gamma^\theta_{\phi\phi} = -\sin \theta \cos \theta \), and the components (11.385) of the Ricci tensor \( R_{ij} \), such as (Weinberg, 1972, ch. 8)

\[
R_{rr} = \frac{B''(r)}{2B(r)} - \frac{1}{4} \left( \frac{B'(r)}{B(r)} \right)^2 \left( \frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)} \right) - \frac{1}{r} \left( \frac{A'(r)}{A(r)} \right) \quad (11.418)
\]
in which the primes mean \( d/dr \).

### 11.52 Schwarzschild’s Solution

If one ignores the small dark-energy parameter \( \Lambda \), one may solve Einstein’s field equations (11.411) in empty space

\[
R_{ij} = 0 \quad (11.419)
\]
outside a mass \( M \) for the standard form of the Ricci tensor. One finds (Weinberg, 1972) that \( A(r) \, B(r) = 1 \) and that \( r \, B(r) = r \) plus a constant, and
one determines the constant by invoking the Newtonian limit $g_{00} = -B \to -1 + 2MG/c^2 r$ as $r \to \infty$. In 1916, Schwarzschild found the solution

$$ds^2 = -\left(1 - \frac{2MG}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2MG}{c^2 r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(11.420)

which one can use to analyze orbits around a star. The singularity in

$$g_{rr} = \left(1 - \frac{2MG}{c^2 r}\right)^{-1}$$

(11.421)

at the Schwarzschild radius $r_s = 2MG/c^2$ is an artifact of the coordinates; the scalar curvature $R$ and other invariant curvatures are not singular at the Schwarzschild radius. Moreover, for the Sun, the Schwarzschild radius $r_s = 2M_\odot G/c^2$ is only 2.95 km, far less than the radius of the Sun, which is $6.955 \times 10^5$ km. So the surface at $r_s = 2M_\odot G/c^2$ is far from the empty space in which Schwarzschild’s metric applies (Karl Schwarzschild, 1873–1916).

### 11.53 Black Holes

Suppose an uncharged, spherically symmetric star of mass $M$ has collapsed within a sphere of radius $r_b$ less than its Schwarzschild radius $r_s = 2MG/c^2$. Then for $r > r_b$, the Schwarzschild metric (11.420) is correct. By Eq. (11.375), the apparent time $dt$ of a process of proper time $d\tau$ at $r \geq 2MG/c^2$ is

$$dt = d\tau/\sqrt{-g_{00}} = d\tau/\sqrt{1 - \frac{2MG}{c^2 r}}.$$  

(11.422)

The apparent time $dt$ becomes infinite as $r \to 2MG/c^2$. To outside observers, the star seems frozen in time.

Due to the gravitational redshift (11.381), light of frequency $\nu_p$ emitted at $r \geq 2MG/c^2$ will have frequency $\nu$

$$\nu = \nu_p \sqrt{-g_{00}} = \nu_p \sqrt{1 - \frac{2MG}{c^2 r}}.$$  

(11.423)

when observed at great distances. Light coming from the surface at $r_s = 2MG/c^2$ is redshifted to zero frequency $\nu = 0$. The star is black. It is a black hole with a surface or horizon at its Schwarzschild radius $r_s = 2MG/c^2$, although there is no singularity there. If the radius of the Sun were less than its Schwarzschild radius of 2.95 km, then the Sun would be a black hole. The radius of the Sun is $6.955 \times 10^5$ km.

Black holes are not really black. Stephen Hawking (1942–) has shown
that the intense gravitational field of a black hole of mass $M$ radiates at
temperature

$$T = \frac{\hbar c^3}{8\pi k G M}$$

in which $k = 8.617343 \times 10^{-5}$ eV K$^{-1}$ is Boltzmann’s constant, and $\hbar$ is
Planck’s constant $\hbar = 6.6260693 \times 10^{-34}$ Js divided by $2\pi$, $\hbar = \hbar/(2\pi)$.

The black hole is entirely converted into radiation after a time

$$t = \frac{5120 \pi G^2}{\hbar c^4} M^3$$

proportional to the cube of its mass.

11.54 Cosmology

Astrophysical observations tell us that on the largest observable scales, space is flat or very nearly flat; that the visible universe contains at least $10^{90}$ particles; and that the cosmic microwave background radiation is isotropic to one part in $10^5$ apart from a Doppler shift due the motion of the Earth. These and other observations suggest that potential energy expanded our universe by $\exp(60) = 10^{26}$ during a brief period of inflation that could have been as short as $10^{-35}$ s. The potential energy that powered inflation almost immediately became the radiation of the Big Bang. During and after inflation, negative gravitational potential energy kept the total energy constant.

Within the first three minutes, some of that radiation became hydrogen, helium, neutrinos, and dark matter Weinberg (1988, 2010). But the era of radiation, during which most of the energy of the visible universe was radiation, lasted for 50,952 years.

Because the momentum of a particle but not its mass falls with the expansion of the universe, the era of radiation gradually gave way to an era of matter. The universe changed from radiation dominated to matter dominated as its temperature $kT$ dropped below 0.81 eV. After 380,000 years, the universe had cooled to 3000 K or $kT = 0.26$ eV, and less than 1% of the atoms were ionized. Photons no longer scattered off a plasma of electrons and ions. The universe became transparent. The photons that last scattered just before this initial transparency became the cosmic microwave background radiation or CMBR that now surrounds us, redshifted to $T_0 = 2.7255 \pm 0.0006$ K.

Between 10 and 17 million years after the Big Bang, the temperature of
the known universe fell from 373 to 273 K. If by then the supernovas of very early, very heavy stars had produced carbon, nitrogen, and oxygen, biochemistry may have started during this period of 7 million years.

The era of matter lasted for 10.19 billion years.

The era of dark energy began about 3.6 billion years ago as matter ceased to be the dominant form of energy. Since then, most of the energy of the visible universe has been of an unknown kind called dark energy. Dark energy may be the energy of the vacuum; classically, it seems to be equivalent to a cosmological constant. Dark energy has been accelerating the expansion of the universe for the past 3.6 billion years and may continue to do so forever.

It is now 13.799 ± 0.021 billion years after the Big Bang or the time of infinite redshift and zero scale factor. The present value Ade et al. (2015) of the Hubble frequency $H$ is the Hubble constant $H_0 = 67.74 ± 0.46$ km (s Mpc)$^{-1} = 2.1953 \times 10^{-18}$ s$^{-1}$, one parsec being 3.085 677 581 49 × $10^{16}$ m or about 3.262 light-years. The present critical mass density $\rho_0 = 3H_0^2/8\pi G = 8.6197 \times 10^{-27}$ kg m$^{-3}$ is the present density of a universe that is isotropic, homogeneous, and flat. The ratio $\Omega_0 = \rho_0/\rho_0$ of the present density $\rho_0$ of the universe to the present critical mass density $\rho_0$ is $\Omega_0 = 1.0000±0.0088$. The ratio $\Omega_{A0} = \rho_{A0}/\rho_0$ of the dark-energy density
\( \rho_{\Lambda 0} \) (taken to be a constant) to the critical density is \( \Omega_{\Lambda 0} = 0.6911 \pm 0.0062 \). The ratio \( \Omega_{m0} = \rho_{m0}/\rho_{c0} \) of the present total matter density \( \rho_{m0} \) to the present critical density \( \rho_{c0} = \Omega_{m0} = 0.3089 \pm 0.0062 \). The ratio \( \Omega_{b0} = \rho_{b0}/\rho_{c0} \) of the present density of baryons \( \rho_{b0} \) to the present critical density \( \rho_{c0} = \Omega_{b0} = 0.0486 \pm 0.0007 \). Baryons account for 4.9% of the density of the universe and only 15.7% of the matter density, the rest being dark matter, which interacts very little with light.

Einstein’s equations (11.408) are second-order, non-linear partial differential equations for 10 unknown functions \( g_{ij}(x) \) in terms of the energy-momentum tensor \( T_{ij}(x) \) throughout the universe, which of course we don’t know. The problem is not quite hopeless, however. The ability to choose arbitrary coordinates, the appeal to symmetry, and the choice of a reasonable form for \( T_{ij} \) all help.

Hubble showed us that the universe is expanding. The cosmic microwave background radiation looks the same in all spatial directions (apart from a Doppler shift due to the motion of the Earth relative to the local supercluster of galaxies). Observations of clusters of galaxies reveal a universe that is homogeneous on suitably large scales of distance. So it is plausible that the universe is homogeneous and isotropic in space, but not in time. One may show (Carroll, 2003) that for a universe of such symmetry, the line element in comoving coordinates (in which the Doppler shift is isotropic) is

\[
\frac{ds^2}{c^2} = \frac{dt^2}{1 - \frac{k}{r^2}} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \]

(11.426)

in which the dimensionless function of time \( a(t) \) is the scale factor and \( k \) is a constant whose dimension is inverse squared length.

Whitney’s embedding theorem tells us that any smooth four-dimensional manifold can be embedded in a flat space of eight dimensions with a suitable signature. We need only four or five dimensions to embed the spacetime described by the line element (11.426). If the universe is closed, then the signature is \((-1,1,1,1,1)\), and our three-dimensional space is the 3-sphere which is the surface of a four-dimensional sphere in four space dimensions. The points of the universe then are

\[
p = (ct, a \sin \chi \sin \theta \cos \phi, a \sin \chi \sin \theta \sin \phi, a \sin \chi \cos \theta, a \cos \chi)
\]

(11.427)

in which \( 0 \leq \chi \leq \pi \), \( 0 \leq \theta \leq \pi \), and \( 0 \leq \phi \leq 2\pi \). If the universe is flat, then the embedding space is flat, four-dimensional Minkowski space with points

\[
p = (ct, ar \sin \theta \cos \phi, ar \sin \theta \sin \phi, ar \cos \theta)
\]

(11.428)
in which $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. If the universe is open, then the embedding space is a flat five-dimensional space with signature $(-1,1,1,1,-1)$, and our three-dimensional space is a hyperboloid in a flat Minkowski space of four dimensions. The points of the universe then are

$$p = (ct, a \sinh \chi \sin \theta \cos \phi, a \sinh \chi \sin \theta \sin \phi, a \sinh \chi \cos \theta, a \cosh \chi)$$

(11.429)

in which $0 \leq \chi \leq \infty$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$.

In all three cases, the corresponding **Robertson-Walker metric** is

$$g_{ij} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & a^2/(1 - kr^2) & 0 & 0 \\
0 & 0 & a^2 r^2 & 0 \\
0 & 0 & 0 & a^2 r^2 \sin^2 \theta
\end{pmatrix}
$$

(11.430)

in which the coordinates $(ct, r, \theta, \phi)$ are numbered $(0,1,2,3)$, and $k$ is a constant with units of $1/\text{length}^2$. One always may choose coordinates (exercise 11.31) such that $k$ numerically is 0 or $\pm 1$. This constant determines whether the spatial universe is **open** $k < 0$, **flat** $k = 0$, or **closed** $k > 0$.

The **scale factor** $a$, which is a function of time $a(t)$, tells us how space expands and contracts. These coordinates are called **comoving** because a point at rest (fixed $r, \theta, \phi$) sees the same Doppler shift in all directions.

The metric (11.430) is diagonal; its inverse $g^{ij}$ also is diagonal; and so we may use our formula (11.291) to compute the affine connections $\Gamma^k_{i\ell}$, such as

$$\Gamma^0_{i\ell} = \frac{1}{2} g^{0k} (g_{\ell k, \ell} + g_{\ell k, \ell} - g_{\ell \ell, k}) = \frac{1}{2} g^{00} (g_{\ell 0, \ell} + g_{\ell 0, \ell} - g_{\ell 0, 0}) = \frac{1}{2} g_{\ell \ell, 0}$$

(11.431)

so that

$$\Gamma^0_{11} = \frac{a \dot{a}}{c} \frac{1}{1 - kr^2} \quad \Gamma^0_{22} = a \dot{a} r^2 / c \quad \text{and} \quad \Gamma^0_{22} = a \dot{a} r^2 \sin^2 \theta / c.$$

(11.432)

in which a dot means a time derivative. The other $\Gamma^0_{ij}$’s vanish. Similarly, for fixed $\ell = 1, 2, \text{or} 3$

$$\Gamma^\ell_{0\ell} = \frac{1}{2} g^{\ell k} (g_{0 k, \ell} + g_{0 k, \ell} - g_{0 \ell, k})$$

$$= \frac{1}{2} g^{\ell \ell} (g_{0 \ell, \ell} + g_{0 \ell, \ell} - g_{0 \ell, 0})$$

$$= \frac{1}{2} g^{\ell \ell} g_{\ell \ell, 0} = \frac{\dot{a}}{ca} = \Gamma^\ell_{00} \quad \text{no sum over} \ \ell.$$

(11.433)
The other nonzero $\Gamma$’s are
\[
\Gamma^1_{22} = -r \left(1 - kr^2\right) \quad \Gamma^1_{33} = -r \left(1 - kr^2\right) \sin^2 \theta 
\]
\[
\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r} = \Gamma^2_{21} = \Gamma^3_{31} \tag{11.435}
\]
\[
\Gamma^2_{33} = -\sin \theta \cos \theta \quad \Gamma^3_{23} = \cot \theta = \Gamma^3_{32}. \tag{11.436}
\]

Our formulas (11.385 & 11.383) for the Ricci and curvature tensors give
\[
R_{00} = R_{0n0} = \left[\partial_0 + \Gamma_0, \partial_n + \Gamma_n\right]_0^n. \tag{11.437}
\]
Clearly the commutator of $\Gamma_0$ with itself vanishes, and one may use the formulas (11.432–11.436) for the other connections to check that
\[
\left[\Gamma_0, \Gamma_n\right]_0^n = \Gamma^n_{0k} \Gamma^k_{n0} - \Gamma^n_{n,k} \Gamma^k_{00} = 3 \left(\frac{\dot{a}}{ca}\right)^2 \tag{11.438}
\]
and that
\[
\partial_0 \Gamma^n_{n0} = 3 \partial_0 \left(\frac{\dot{a}}{ca}\right) = 3 \frac{\ddot{a}}{c^2 a} - 3 \left(\frac{\dot{a}}{ca}\right)^2 \tag{11.439}
\]
while $\partial_n \Gamma^n_{00} = 0$. So the 00-component of the Ricci tensor is
\[
R_{00} = 3 \frac{\ddot{a}}{c^2 a}. \tag{11.440}
\]

Similarly, one may show that the other non-zero components of Ricci’s tensor are
\[
R_{11} = -\frac{A}{1 - kr^2} \quad R_{22} = -r^2 A \quad \text{and} \quad R_{33} = -r^2 A \sin^2 \theta \tag{11.441}
\]
in which $A = a\ddot{a}/c^2 + 2\dot{a}^2/c^2 + 2k$. The scalar curvature (11.386) is
\[
R = g^{ab} R_{ab} = -\frac{6}{a^2} \left(\frac{a\ddot{a}}{c^2} + \frac{\dot{a}^2}{c^2} + k\right). \tag{11.442}
\]

In comoving coordinates such as those of the Robertson-Walker metric (11.430), the 4-velocity (11.66) is $u^i = (1, 0, 0, 0)$, and so the energy-momentum tensor (11.405) is
\[
T_{ij} = \begin{pmatrix}
\rho & 0 & 0 & 0 \\
0 & p g_{11} & 0 & 0 \\
0 & 0 & p g_{22} & 0 \\
0 & 0 & 0 & p g_{33}
\end{pmatrix}. \tag{11.443}
\]
Its trace is
\[
T = g^{ij} T_{ij} = -\rho + 3p. \tag{11.444}
\]
Thus successively using our formulas (11.430) for $g_{00} = -1$, (11.440) for $R_{00}$, (11.443) for $T_{ij}$, and (11.444) for $T$, we can write the 00 Einstein equation (11.410) as the second-order equation

$$
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p)
$$

(11.445)

which is nonlinear because $\rho$ and $3p$ depend upon $a$. The sum $\rho + 3p$ determines the acceleration $\ddot{a}$ of the scale factor $a(t)$. The apparently positive sum $\rho + 3p$ can be negative, and when it is, it accelerates the expansion of the universe.

Because of the isotropy of the metric, the three nonzero spatial Einstein equations (11.410) give us only one relation

$$
\frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{c^2 k}{a^2} = 4\pi G \rho
$$

(11.446)

Using the 00-equation (11.445) to eliminate the second derivative $\ddot{a}$, we have

$$
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{c^2 k}{a^2}
$$

(11.447)

which is a first-order nonlinear equation. It and the second-order equation (11.445) are known as the **Friedmann equations**.

The LHS of the first-order Friedmann equation (11.469) is the square of the **Hubble rate**

$$
H = \frac{\dot{a}}{a}
$$

(11.448)

which is an inverse time or a frequency. Its present value $H_0$ is the **Hubble constant**. In terms of $H$, Friedmann’s first-order equation (11.469) is

$$
H^2 = \frac{8\pi G}{3} \rho - \frac{c^2 k}{a^2}.
$$

(11.449)

The energy density of a flat universe with $k = 0$ is the **critical energy density**

$$
\rho_c = \frac{3H^2}{8\pi G}.
$$

(11.450)

The ratio of the present energy density $\rho_0$ to the present critical energy density $\rho_{c0}$ is $\Omega_0$

$$
\Omega_0 = \frac{\rho_0}{\rho_{c0}} = \frac{8\pi G}{3H_0^2} \rho_0.
$$

(11.451)
From (11.449), we see that $\Omega$ is
\[
\Omega = 1 + \frac{c^2k}{(aH)^2} = 1 + \frac{c^2k}{\dot{a}^2}.
\] (11.452)

Thus $\Omega = 1$ both in a flat universe ($k = 0$) and as $aH \to \infty$. One use of inflation is to expand $a$ by $10^{26}$ so as to force $\Omega$ almost exactly to unity.

Something like inflation is needed because in a universe in which the energy density is due to matter and/or radiation, the present value of $\Omega$
\[
\Omega_0 = 1.0000 \pm 0.0062
\] (11.453)
is unlikely. To see why, we note that conservation of energy ensures that $a^3$ times the matter density $\rho_m$ is constant. Radiation redshifts by $a$, so energy conservation implies that $a^4$ times the radiation density $\rho_r$ is constant. So with $n = 3$ for matter and 4 for radiation, $\rho a^n \equiv 3 F^2/8\pi G$ is a constant. In terms of $F$ and $n$, Friedmann’s first-order equation (11.469) is
\[
\dot{a}^2 = \frac{8\pi G}{3} \rho a^2 - c^2 k = \frac{F^2}{a^{n-2}} - c^2 k
\] (11.454)
In the small-$a$ limit of the early Universe, we have
\[
\dot{a} = F/a^{(n-2)/2} \quad \text{or} \quad a^{(n-2)/2}da = F dt
\] (11.455)
which we integrate to $a \sim t^{2/n}$ so that $\dot{a} \sim t^{2/n-1}$. Now (11.452) says that
\[
|\Omega - 1| = \frac{c^2|k|}{\dot{a}^2} \propto t^{2-4/n} = \begin{cases} t & \text{radiation} \\ t^{2/3} & \text{matter} \end{cases}
\] (11.456)
Thus, $\Omega$ deviated from unity at least as fast as $t^{2/3}$ during the early Universe. At this rate, the inequality $|\Omega_0 - 1| < 0.009$ could last 13.8 billion years only if $\Omega$ at $t = 1$ second had been unity to within two parts in $10^{14}$. The only known explanation for such early flatness is inflation.

The relation (11.452) between $\Omega$ and $aH$ shows that $k = 0$ is equivalent to $\Omega = 1$, that $\Omega > 1 \iff k > 0$, and that $\Omega < 1 \iff k < 0$. And writing the same relation (11.452) as $(aH)^2 = c^2 k/(\Omega - 1)$, we see that as $\Omega \to 1$ the product $aH \to \infty$, which is the essence of flatness since curvature vanishes as the scale factor $a \to \infty$. Imagine blowing up a balloon.

Staying for the moment with a universe without inflation and with an energy density composed of radiation and/or matter, we note that the first-order equation (11.454) in the form $\dot{a}^2 = F^2/a^{n-2} - c^2 k$ tells us that for a closed ($k > 0$) universe, in the limit $a \to \infty$ we’d have $\dot{a}^2 \to -1$ which is impossible. Thus a closed universe cannot expand indefinitely.

The first-order equation Friedmann (11.469) says that $\rho a^2 \geq 3c^2 k/8\pi G$. 
So in a closed universe \((k > 0)\), the energy density \(\rho\) is positive and increases without limit as \(a \to 0\) as in a collapse. In open \((k < 0)\) and flat \((k = 0)\) universes, the same Friedmann equation (11.469) in the form \(\dot{a}^2 = 8\pi G \rho a^2/3 - c^2 k\) tells us that if \(\rho\) is positive, then \(\dot{a}^2 > 0\), which means that \(\dot{a}\) never vanishes. Hubble told us that \(\dot{a} > 0\) now. So if our universe is open or flat, then it always expands.

Due to the expansion of the universe, the wavelength of radiation grows with the scale factor \(a(t)\). A photon emitted at time \(t\) and scale factor \(a(t)\) with wavelength \(\lambda(t)\) will be seen now at time \(t_0\) and scale factor \(a(t_0)\) to have a longer wavelength \(\lambda(t_0)\)

\[
\frac{\lambda(t_0)}{\lambda(t)} = \frac{a(t_0)}{a(t)} = z + 1 \quad (11.457)
\]

in which the redshift \(z\) is the ratio

\[
z = \frac{\lambda(t_0) - \lambda(t)}{\lambda(t)} = \frac{\Delta \lambda}{\lambda}. \quad (11.458)
\]

Now \(H = \dot{a}/a = da/(a dt)\) implies \(dt = da/(a H)\), and \(z = a_0/a - 1\) implies \(dz = -a_0 da/a^2\), so we find

\[
dt = -\frac{dz}{(1 + z) H(z)} \quad (11.459)
\]

which relates time intervals to redshift intervals. An on-line calculator is available for macroscopic intervals (Wright, 2006).

### 11.55 Density and pressure

The 0-component of the energy-momentum conservation law (11.407) is

\[
0 = (T^a_0)_a = \partial_a T^a_0 + \Gamma^a_{ac} T^c_0 - T^a_c \Gamma^c_0 a
\]

\[
= -\partial_0 T_{00} - \Gamma^a_{00} T_{00} - g^{cc} T_{cc} \Gamma^c_0 a
\]

\[
= -\dot{\rho} - 3\frac{\dot{a}}{a} \rho - 3p = -\dot{\rho} - 3\frac{\dot{a}}{a} (\rho + p). \quad (11.460)
\]

or

\[
\frac{d\rho}{da} = -\frac{3}{a} (\rho + p). \quad (11.461)
\]

The energy density \(\rho\) is composed of fractions \(\rho_i\) each contributing its own partial pressure \(p_i\) according to its own equation of state

\[
p_i = w_i \rho_i \quad (11.462)
\]
in which \( w_i \) is a constant. The rate of change (11.462) of the density \( \rho_i \) is then
\[
\frac{d\rho_i}{da} = -\frac{3}{a} (1 + w_i) \rho_i. \tag{11.463}
\]
In terms of the present density \( \rho_{i0} \) and scale factor \( a_0 \), the solution is
\[
\rho_i = \rho_{i0} \left( \frac{a_0}{a} \right)^{3(1+w_i)}. \tag{11.464}
\]

There are three important kinds of density. The dark-energy density \( \rho_\Lambda \) is assumed to be like a cosmological constant \( \Lambda \) or like the energy density of the vacuum, so it is independent of the scale factor \( a \)
\[
\rho_\Lambda = \rho_{\Lambda 0} \tag{11.465}
\]
and has \( w_\Lambda = -1 \). The matter density \( \rho_m \) is assumed to have no pressure, \( w_m = 0 \), and so the matter density falls inversely with the volume
\[
\rho_m = \rho_{m0} \left( \frac{a_0}{a} \right)^3. \tag{11.466}
\]
The density of radiation \( \rho_r \) has \( w_r = 1/3 \) because wavelengths scale with the scale factor, and so there’s an extra factor of \( a \)
\[
\rho_r = \rho_{r0} \left( \frac{a_0}{a} \right)^4. \tag{11.467}
\]
The total density \( \rho \) varies with \( a \) as
\[
\rho = \rho_{\Lambda 0} + \rho_{m0} \left( \frac{a_0}{a} \right)^3 + \rho_{r0} \left( \frac{a_0}{a} \right)^4. \tag{11.468}
\]

In simple cosmological models, only the dominant component of the density is considered.

### 11.56 How the scale factor evolves with time

The first-order Friedmann equation (11.469) expresses the square of the instantaneous Hubble rate \( H = \dot{a}/a \) in terms of the density \( \rho \) and the scale factor \( a(t) \)
\[
H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{c^2 k}{a^2} \tag{11.469}
\]
in which \( k = \pm 1 \) or 0. The critical density \( \rho_c \) is the one that satisfies this equation for a flat \( (k = 0) \) universe. Its present value is
\[
\rho_{0c} = \frac{3H_0^2}{8\pi G}. \tag{11.470}
\]
11.56 How the scale factor evolves with time

Dividing Friedmann’s equation by the square of the present Hubble rate $H_0^2$, we get

\[ \frac{H^2}{H_0^2} = \frac{1}{H_0^2} \left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{H_0^2} \left( \frac{8\pi G}{3} \rho - \frac{c^2 k}{a^2} \right) = \frac{\rho}{\rho_0 c} - \frac{c^2 k}{a^2 H_0^2}, \]  

(11.471)

in which $\rho$ is the total density (11.468)

\[ \frac{H^2}{H_0^2} = \frac{\rho_X}{\rho_0 c} + \frac{\rho_r}{\rho_0 c} + \frac{\rho_m}{\rho_0 c} - \frac{c^2 k}{a^2 H_0^2} \]

(11.472)

These density ratios are called $\Omega_{\Lambda 0}$, $\Omega_{r0}$, $\Omega_{m0}$, and $\Omega_{k0} \equiv -c^2 k/a_0^2 H_0^2$, and in terms of them this formula (11.472) for $H^2/H_0^2$ is

\[ \frac{H^2}{H_0^2} = \Omega_{\Lambda 0} + \Omega_{k0} \frac{a_0^2}{a^2} + \Omega_{m0} \frac{a_0^3}{a^3} + \Omega_{r0} \frac{a_0^4}{a^4}. \]

(11.473)

Since $H = \dot{a}/a$, we have $dt = H_0^{-1} (da/a) (H_0/H)$, and so with $x = a/a_0$, the time interval $dt$ is

\[ dt = \frac{1}{H_0} \frac{dx}{x} \frac{1}{\sqrt{\Omega_{\Lambda 0} + \Omega_{k0} x^{-2} + \Omega_{m0} x^{-3} + \Omega_{r0} x^{-4}}}. \]

(11.474)

Integrating and setting the origin of time $t(0) = 0$ at $x = a/a_0 = 0$, we find that the time $t(a/a_0)$ during which the ratio $a(t)/a_0$ grew from 0 to $a(t)/a_0$ is

\[ t(a/a_0) = \frac{1}{H_0} \int_0^{a/a_0} \frac{dx}{\sqrt{\Omega_{\Lambda 0} x^2 + \Omega_{k0} x^{-2} + \Omega_{m0} x^{-1} + \Omega_{r0} x^{-4}}}. \]

(11.475)

The definition (11.457) of the redshift gives $a/a_0 = 1/(z+1)$. So the formula (11.475) for $t(a/a_0)$ also says that a photon emitted at time $t(a/a_0)$ will be seen now with redshift $z(t) = a_0/a - 1$.

The Planck Collaboration’s values Ade et al. (2015) for the density ratios $\Omega_{\Lambda 0}$, $\Omega_{k0}$, and $\Omega_{m0}$ are

\[ \Omega_{\Lambda 0} = 0.6911 \pm 0.0062 \]
\[ \Omega_{k0} = 0.0008 \pm 0.004 \]
\[ \Omega_{m0} = 0.3089 \pm 0.0062. \]

(11.476)

We use may use the present temperature $T_0 = 2.7255$ K of the cosmic microwave background radiation and our formula (4.98) for the energy density
of photons to estimate the mass density of photons as
\[ \rho_\gamma = \frac{8\pi^5 (k_B T_0)^4}{15h^3c^5} = 4.6451 \times 10^{-31} \text{ kg m}^{-3}. \] (11.477)

Adding in three kinds of neutrinos and antineutrinos at \( T_0 = (4/11)^{1/3} T_0 \), we get for the present density of massless and nearly massless particles (Weinberg, 2010, section 2.1)
\[ \rho_r = \left[ 1 + 3 \left( \frac{7}{8} \right) \left( \frac{4}{11} \right)^{4/3} \right] \rho_\gamma = 7.8099 \times 10^{-31} \text{ kg m}^{-3}. \] (11.478)

The fraction \( \Omega_{r0} \) the present energy density that is due to radiation is then
\[ \Omega_{r0} = \frac{\rho_r}{\rho_{cb}} = 9.0606 \times 10^{-5}. \] (11.479)

By putting the \( \Omega \) values (11.476 & 11.479) into the integral (11.475) and integrating, we may get the time as a function of the scale factor. Figure 11.2 plots the reduced scale factor \( a(t)/a_0 \) (solid) and the redshift \( z(t) \) (….) as functions of the time \( t \) in Gyr since the time of infinite redshift. The age of the universe is 13.8 Gyr (vertical line). A photon emitted with wavelength \( \lambda \) at time \( t \) now has wavelength \( \lambda_0 = (a_0/a(t)) \lambda \). The change is its wavelength is \( \Delta \lambda = \lambda z(t) \).

**Example 11.30** (\( w = -1/3 \), No Acceleration). If \( w = -1/3 \), then \( p = w \rho = -\rho/3 \) and \( \rho + 3p = 0 \). The second-order Friedmann equation (11.445) then tells us that \( \ddot{a} = 0 \). The scale factor does not accelerate.

To find its constant speed, we use its equation of state (11.464)
\[ \rho = \rho_0 \left( \frac{a_0}{a} \right)^{3(1+w)} = \rho_0 \left( \frac{a_0}{a} \right)^2. \] (11.480)

Now all the terms in Friedmann's first-order equation (11.469) have a common factor of \( 1/a^2 \) which cancels leaving us with the square of the constant speed
\[ \dot{a}^2 = \frac{8\pi G}{3} \rho_0 a_0^2 - c^2 k \] (11.481)
in which \( \rho_0 a_0^2 \) must exceed \( 3c^2 k / 8\pi G \). Since \( \dot{a} = aH \) is constant, the scale factor grows linearly with time like the dashed line \( a(t) = a_0 H_0 t \) in figure 11.2 if we set \( a(0) = 0 \). \( \square \)
11.57 Inflation ($w = -1$)

Inflation occurs when the ground state of the theory has a positive and constant potential-energy density $\rho > 0$ that dwarfs the energy densities of the matter and radiation. The internal energy of the universe then is proportional to its volume $U = \rho V$, and the pressure $p$ as given by the thermodynamic relation

$$ p = -\frac{\partial U}{\partial V} = -\rho $$

is negative. The equation of state (11.462) tells us that in this case $w = -1$. The second-order Friedmann equation (11.445) becomes

$$ \frac{\ddot{a}}{a} = -4\pi G \frac{1}{3} (\rho + 3p) = \frac{8\pi G}{3} \rho = g^2 $$
By it and the first-order Friedmann equation (11.469) and by choosing \( t = 0 \) as the time at which the scale factor \( a \) is zero or minimal, one may show (exercise 11.38) that in a flat \( (k = 0) \), closed \( (k > 0) \), and open \( (k < 0) \), universes, the scale factor varies as

\[
a(t) = a(0) e^{qt} \quad \text{flat, } k = 0 \tag{11.484}
\]

\[
a(t) = \frac{\cosh gt}{g} \quad \text{closed, } k > 0 \tag{11.485}
\]

\[
a(t) = \frac{\sinh gt}{g} \quad \text{open, } k < 0. \tag{11.486}
\]

A de Sitter universe is flat, has \( a(t) = a(0) \exp(gt) \), and can be infinitely old.

Studies of the cosmic microwave background radiation suggest that inflation did occur in the very early universe—possibly on a time scale as short as \( 10^{-35} \) s. The origin of the vacuum energy density \( \rho \) that drove inflation is unknown. In chaotic inflation, a scalar field \( \phi \) fluctuated to a mean value \( \langle \phi \rangle \) very different from the one \( \langle 0 | \phi | 0 \rangle \) that minimizes the energy density of the vacuum. When \( \langle \phi \rangle \) settled to \( \langle 0 | \phi | 0 \rangle \), the potential energy of the vacuum was released as radiation in a Big Bang.

Anti-de Sitter models have \( w = -1 \) and a negative potential energy \( \rho = -p < 0 \).

11.58 The era of radiation \( (w = 1/3) \)

Until a redshift of \( z = 3408.3 \) or 50,953 years after the time of infinite redshift our universe was dominated by radiation. During The First Three Minutes (Weinberg, 1988) of the era of radiation, the quarks and gluons formed hadrons, which decayed into protons and neutrons. As the neutrons decayed \( (\tau = 885.7 \) s), they and the protons formed the light elements—principally hydrogen, deuterium, and helium in a process called big-bang nucleosynthesis.

We can guess the value of \( w \) for radiation by noticing that the energy-momentum tensor of the electromagnetic field (in suitable units)

\[
T^{ab} = F_a^c F^{bc} - \frac{1}{4} g^{ab} F_{cd} F^{cd} \tag{11.487}
\]

is traceless

\[
T = T^a_a = F_a^c F_c^a - \frac{1}{4} g^a_a F_{cd} F^{cd} = 0. \tag{11.488}
\]
But by (11.444) its trace must be $T = 3p - \rho$. So for radiation $p = \rho/3$ and $w = 1/3$. The relation (11.467) between the density and the scale factor for radiation then is

$$\rho = \rho_0 \left( \frac{a_0}{a} \right)^4.$$  \hfill (11.489)

The density drops both with the volume $a^3$ and with the scale factor $a$ due to a redshift; so it drops as $1/a^4$.

Early in the era of radiation, we can ignore the matter and vacuum densities and focus on the radiation density so that the quantity

$$f^2 = \frac{8\pi G \rho a^4}{3}$$ \hfill (11.490)

is a constant. The Friedmann equations (11.445 & 11.446) then are

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) = -\frac{8\pi G \rho}{3} \quad \text{or} \quad \ddot{a} = -\frac{f^2}{a^3}$$ \hfill (11.491)

and

$$\dot{a}^2 + k = \frac{f^2}{a^2}.$$ \hfill (11.492)

With calendars chosen so that $a(0) = 0$, this last equation (11.492) tells us that for flat $(k = 0)$, closed $(k = 1)$, and open $(k = -1)$ universes

$$a(t) = \sqrt{2f} t \quad \text{flat, } k = 0$$ 
$$a(t) = \sqrt{f^2 - (t - f)^2} \quad \text{closed, } k = 1$$ 
$$a(t) = \sqrt{(t + f)^2 - f^2} \quad \text{open, } k = -1,$$ \hfill (11.493, 11.494, 11.495)

as we saw in (6.488). The scale factor (11.494) of a closed universe of radiation has a maximum $a = f$ at $t = f$ and falls back to zero at $t = 2f$.

For a flat, $k = 0$, universe, our formula (11.493) for the time dependence $a(t) \sim \sqrt{t}$ of the scale factor $a(t)$ together with the quartic dependence (11.489) of the energy density $\rho \sim 1/a^4$ upon the scale factor tell us that the time $t$ varied as the inverse of the square root of the energy density $t \sim 1/\sqrt{\rho}$. During this era of radiation, the energy of highly relativistic particles dominated the energy density, which therefore was proportional to the fourth power of the temperature, $\rho \sim T^4$. It follows that the time varied as the inverse square of the temperature, $t \sim 1/T^2$.

Weinberg gives more accurate estimates. When the temperature was in the range $10^{12} > T > 10^{10}$ K or $m_\mu c^2 > kT > m_e c^2$, where $m_\mu$ is the mass of the muon and $m_e$ that of the electron, the radiation was mostly electrons,
positrons, photons, and neutrinos, and the relation between the time $t$ and the temperature $T$ was (Weinberg, 2010, ch. 3)

$$ t = 0.994 \sec \times \left[ \frac{10^{10} \text{K}}{T} \right]^2 + \text{constant}. \quad (11.496) $$

By $10^9$ K, the positrons had annihilated with electrons, and the neutrinos fallen out of equilibrium. Between $10^9$ K and $10^6$ K, when the energy density of nonrelativistic particles became relevant, the time-temperature relation was (Weinberg, 2010, ch. 3)

$$ t = 1.78 \sec \times \left[ \frac{10^{10} \text{K}}{T} \right]^2 + \text{constant}'. \quad (11.497) $$

At times of tens of thousands of years, the matter density $\rho_m$ became important because it drops with the cube of the scale factor as $\rho_m(t) = \Omega_{m0} a_0^3/a^3(t)$ while the radiation density $\rho_r(t)$ drops with the fourth power of the scale factor as $\rho_r(t) = \Omega_{r0} a_0^4/a^4(t)$. The era of radiation ended and the era of matter began when these densities were equal, $\rho_m(t) = \rho_r(t)$. We can estimate this time by using the density ratios $\Omega_{m0}$ and $\Omega_{r0}$ (11.476 & 11.479). We find

$$ \frac{a}{a_0} = \frac{\rho_{r0}}{\rho_{m0}} = \frac{\Omega_{r0}}{\Omega_{m0}} = \frac{9.0606 \times 10^{-5}}{0.3089} = 2.93318 \times 10^{-4} \quad (11.498) $$

which is a redshift of $z = a_0/a - 1 = 3408.3$. Our integral (11.475) gives the time as $t(3408.3) = 50,953$ years after the time of infinite redshift. The temperature then was $T = 9,400$ K or $kT = 0.81$ eV.

### 11.59 The era of matter ($w = 0$)

A universe composed only of dust or non-relativistic collisionless matter has no pressure. Thus $p = w\rho = 0$ with $\rho \neq 0$, and so $w = 0$. Conservation of energy (11.461), or equivalently (11.466), implies that the matter density falls with the volume as

$$ \rho_m = \rho_{m0} \left( \frac{a_0}{a} \right)^3. \quad (11.499) $$

The era of matter began about 50,953 after the time of infinite redshift when the matter density $\rho_m$ first exceeded the radiation density $\rho_r$. Most of the matter is of an unknown kind that interacts very weakly with photons and
11.59 The era of matter \((w = 0)\)

The era of matter \((w = 0)\)

Figure 11.3 The scale factor ratio \(a/a_0\) (solid), the radiation density \(\rho_r\) (dotdash), and the matter density \(\rho_m\) (dashed) are plotted as functions of the time (11.475) in kyr after the time of infinite redshift. The era of radiation ends at \(t = 50,953\) years when the two densities are equal to \(1.055 \times 10^{-16} \text{ kg/m}^3\), and \(a/a_0 = 2.933 \times 10^{-4}\).

is called dark matter. Baryons amount to only about 15.7\% of \(\rho_m\), so if they were the principal kind of matter, the era of radiation would have lasted much longer.

Near the middle of the era of matter, at \(t \sim 5\) Gyr, the radiation density and the dark-energy density are unimportant, and we need take only the matter density into account in order to approximate the evolution of the scale factor. The density then varies as \(\rho \sim \rho_m \propto 1/a^3\), and so the quantity

\[ f = \frac{4\pi G \rho_m a^3}{3} \]  

is constant. The resulting Friedmann equations (11.445 & 11.446) are

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho_m + 3p) = -\frac{4\pi G \rho_m}{3} \quad \text{or} \quad \ddot{a} = -\frac{f}{a^2} \]
For a flat universe, $k = 0$, we get the **Einstein-de Sitter model**

\[
a(t) = \left[ \frac{3\sqrt{f}}{\sqrt{2}} t \right]^{2/3}.
\]

(11.503)

For a closed universe, $k = 1$, we use example 6.53 to integrate

\[
\dot{a} = \sqrt{2f/a - 1}
\]

(11.504)

to

\[
t = 2f \arctan \sqrt{a/(2f - a)} - \sqrt{a(2f - a)}.
\]

(11.505)

For an open universe, $k = -1$, we use example 6.54 to integrate

\[
\dot{a} = \sqrt{2f/a + 1}
\]

(11.506)

to

\[
t = \sqrt{a(2f + a)} - 2f \ln \left[ \sqrt{a} + \sqrt{a + 2f} \right].
\]

(11.507)

\[
t - t_0 = [a(2f + a)]^{1/2} - f \ln \left\{ 2[a(2f + a)]^{1/2} + 2a + 2f \right\}.
\]

(11.508)

**Transparency:** Some 380,000 years after inflation at a redshift of $z = 1090$, the universe had cooled to about $T = 3000$ K or $kT = 0.26$ eV—a temperature at which less than 1% of the hydrogen is ionized. Ordinary matter became a gas of neutral atoms rather than a plasma of ions and electrons, and the universe suddenly became transparent to light. This moment of last scattering and first transparency often and inexplicably is called **recombination**.

11.60 **The era of dark energy** ($w = -1$)

About 3.606 billion years ago or 10.193 Gyr after inflation at a redshift of $z = 0.3079$, the matter density, falling as $1/a^3$, dropped below the **dark-energy** density $\rho_\Lambda = 6.0084 \times 10^{-27}$ kg/m$^3$ or (2.256 meV)$^4$. The age of the universe is 13.799 billion years. For the past 3.606 billion years, this
constant energy density, called **dark energy**, has accelerated the expansion of the universe approximately as in the de Sitter model (11.486)

\[
a(t) = a(t_m) \exp \left( \frac{(t - t_m)}{\sqrt{8\pi G \rho_v / 3}} \right) \tag{11.509}
\]

in which \( t_m = (10.427 \pm 0.07) \times 10^9 \) years.

### 11.61 History

Observations and measurements on the largest scales show no spatial curvature and are consistent with a flat universe, \( k = 0 \). So the evolution of the scale factor \( a(t) \) is given by the \( k = 0 \) equations (11.484, 11.493, 11.503, & 11.509) for a flat universe. If we also simplify the history of the universe
by imagining that its energy density was due to a single component during each era, then we can sketch its evolution as follows.

During the brief era of inflation, the scale factor $a(t)$ grew as in the de Sitter model (11.484)

$$a(t) = a(0) \exp \left( t \sqrt{8\pi G \rho_i / 3} \right)$$  \hspace{1cm} (11.510)

in which $\rho_i$ is the positive energy density that drove inflation.

During the 50,952 years of the era of radiation, $a(t)$ grew as $\sqrt{t}$ as in (11.493)

$$a(t) = \left( 2 (t - t_i) \sqrt{8\pi G \rho(t_r) a^4(t_r) / 3} \right)^{1/2} + a(t_i)$$  \hspace{1cm} (11.511)

where $t_i$ is the time at the end of inflation, and $t_r$ is any time during the era of radiation.

During the 10.193 billion years of the matter era, $a(t)$ grew as (11.503)

$$a(t) = \left[ (t - t_r) \sqrt{3\pi G \rho(t'_m) a(t'_m) + a^{3/2}(t_r)} \right]^{2/3} + a(t_r)$$ \hspace{1cm} (11.512)

where $t_r$ is the time at the end of the radiation era, and $t'_m$ is any time in the matter era. By 380,000 years, the temperature had dropped to 3000 K, the universe had become transparent, and the CMBR had begun to travel freely.

Over the past 3.606 billion years of the era of vacuum dominance, $a(t)$ has been growing exponentially (11.509)

$$a(t) = a(t_m) \exp \left( (t - t_m) \sqrt{8\pi G \rho_v / 3} \right)$$ \hspace{1cm} (11.513)

in which $t_m$ is the time at the end of the matter era, and $\rho_v$ is the density of dark energy, which while vastly less than the energy density $\rho_i$ that drove inflation, currently amounts to 69.11% of the total energy density. Curiously, the standard model of cosmology begins and ends with the de Sitter model, although with vastly different rates of expansion.

**11.62 Yang-Mills theory**

The gauge transformation of an abelian gauge theory like electrodynamics multiplies a single charged field by a spacetime-dependent phase factor $\phi'(x) = \exp(i q \theta(x)) \phi(x)$. Yang and Mills generalized this gauge transformation to one that multiplies a vector $\phi$ of matter fields by a spacetime
dependent unitary matrix $U(x)$

$$\phi'_a(x) = \sum_{b=1}^n U_{ab}(x) \phi_b(x) \quad \text{or} \quad \phi'(x) = U(x) \phi(x) \quad (11.514)$$

and showed how to make the action of the theory invariant under such non-abelian gauge transformations. (The fields $\phi$ are scalars for simplicity.)

Since the matrix $U$ is unitary, inner products like $\phi^\dagger(x) \phi(x)$ are automatically invariant

$$\phi^\dagger(x) \phi(x) = \phi^\dagger(x) U^\dagger(x) U(x) \phi(x) \quad (11.515)$$

But inner products of derivatives $\partial^i \phi^\dagger \partial_i \phi$ are not invariant because the derivative acts on the matrix $U(x)$ as well as on the field $\phi(x)$.

Yang and Mills made derivatives $D_i \phi$ that transform like the fields $\phi$ by

$$(D_i \phi)' = U D_i \phi. \quad (11.516)$$

To do so, they introduced gauge-field matrices $A_i$ that play the role of the connections $\Gamma_i$ in general relativity and set

$$D_i = \partial_i + A_i \quad (11.517)$$

in which $A_i$ like $\partial_i$ is antihermitian. They required that under the gauge transformation (11.514), the gauge-field matrix $A_i$ transform to $A'_i$ in such a way as to make the derivatives transform as in (11.516)

$$(D_i \phi)' = (\partial_i + A'_i) \phi' = (\partial_i + A'_i) U \phi = U D_i \phi = U (\partial_i + A_i) \phi. \quad (11.518)$$

So they set

$$(\partial_i + A'_i) U \phi = U (\partial_i + A_i) \phi \quad \text{or} \quad (\partial_i U) \phi + A'_i U \phi = U A_i \phi. \quad (11.519)$$

and made the gauge-field matrix $A_i$ transform as

$$A'_i = U A_i U^{-1} - (\partial_i U) U^{-1}. \quad (11.520)$$

Thus under the gauge transformation (11.514), the derivative $D_i \phi$ transforms as in (11.516), like the vector $\phi$ in (11.514), and the inner product of covariant derivatives

$$\left[ (D^i \phi)^\dagger D_i \phi \right]' = (D^i \phi)^\dagger U^\dagger U D_i \phi = (D^i \phi)^\dagger D_i \phi \quad (11.521)$$

remains invariant.

To make an invariant action density for the gauge-field matrices $A_i$, they
used the transformation law (11.518) which implies that $D'_i U \phi = UD_i \phi$ or $D'_i = UD_i U^{-1}$. So they defined their generalized Faraday tensor as

$$F_{ik} = [D_i, D_k] = \partial_i A_k - \partial_k A_i + [A_i, A_k]$$

(11.522)

so that it transforms covariantly

$$F_{ik}' = UF_{ik}U^{-1}.$$  

(11.523)

They then generalized the action density $F_{ik}F^{ik}$ of electrodynamics to the trace $\text{Tr} (F_{ik}F^{ik})$ of the square of the Faraday matrices which is invariant under gauge transformations since

$$\text{Tr} \left( UF_{ik}U^{-1}UF^{ik}U^{-1} \right) = \text{Tr} \left( UF_{ik}F^{ik}U^{-1} \right) = \text{Tr} \left( F_{ik}F^{ik} \right).$$

(11.524)

As an action density for fermionic matter fields, they replaced the ordinary derivative in Dirac’s formula $\overline{\psi}(\gamma^i \partial_i + m)\psi$ by the covariant derivative (11.517) to get $\overline{\psi}(\gamma^i D_i + m)\psi$ (Chen-Ning Yang 1922–, Robert L. Mills 1927–1999).

In an abelian gauge theory, the square of the 1-form $A = A_i dx^i$ vanishes $A^2 = A_i A_k dx^i \wedge dx^k = 0$, but in a nonabelian gauge theory the gauge fields are matrices, and $A^2 \neq 0$. The sum $dA + A^2$ is the Faraday 2-form

$$F = dA + A^2 = (\partial_i A_k + A_i A_k) \ dx^i \wedge dx^k = \frac{1}{2} \ (\partial_i A_k - \partial_k A_i + [A_i, A_k]) \ dx^i \wedge dx^k = \frac{1}{2} F_{ik} \ dx^i \wedge dx^k.$$  

(11.525)

The scalar matter fields $\phi$ may have self-interactions described by a potential $V(\phi)$ such as $V(\phi) = \lambda (\phi^\dagger \phi - m^2/\lambda)^2$ which is positive unless $\phi^\dagger \phi = m^2/\lambda$. The kinetic action of these fields is $(D_i \phi)^\dagger D_i \phi$. At low temperatures, these scalar fields assume mean values $\langle 0 \mid \phi \mid 0 \rangle = \phi_0$ in the vacuum with $\phi_0^\dagger \phi_0 = m^2/\lambda$ so as to minimize their potential energy density $V(\phi)$ and their kinetic action $(D_i \phi)^\dagger D_i \phi = (\partial^i \phi + A^i \phi)^\dagger (\partial_i \phi + A_i \phi)$ is approximately $\phi_0^\dagger A^i A_i \phi_0$. The gauge-field matrix $A_{ab} = i t_{ab}^\alpha A^\alpha$ is a linear combination of the generators $t^\alpha$ of the gauge group. So the action of the scalar fields contains the term $\phi_0^\dagger A^i A_i \phi_0 = -M^2_{\alpha\beta} A^\alpha_{\alpha} A_{\beta}$ in which the mass-squared matrix for the gauge fields is $M^2_{\alpha\beta} = \phi_0^\dagger t_{ab}^\alpha t_{bc}^\beta \phi_0$. This Higgs mechanism gives masses to those linear combinations $b_{\beta i} A_{\beta}$ of the gauge fields for which $M^2_{\alpha\beta} b_{\beta i} = m^2_\alpha b_{\alpha i} \neq 0$.

The Higgs mechanism also gives masses to the fermions. The mass term $m$ in the Yang-Mills-Dirac action is replaced by something like $c \phi$ in which $c$ is a constant, different for each fermion. In the vacuum and at low temperatures, each fermion acquires as its mass $c \phi_0$. On 4 July 2012, physicists at CERN’s
Large Hadron Collider announced the discovery of a Higgs-like particle with a mass near 125 GeV/c² (Peter Higgs 1929–).

11.63 Spin-one-half fields in general relativity

The flat-space action density (10.307) for a spin-one-half field \( \psi \) is

\[
L = -\overline{\psi} [\gamma^a (\partial_a + A_a) + m] \psi
\]  

(11.526)

in which \( a \) is a flat-space index, \( A_a \) is a matrix of gauge fields, \( \psi \) is a 4-component Dirac or Majorana field, \( \gamma^a \beta = i \gamma^a \gamma^0 \), and \( m \) is a constant or a mean value of a scalar field. One may use the tetrad fields \( e^a_\mu(x) \) of section 11.21 to turn the flat-space indices \( a \) into curved-space indices. Since derivatives and gauge fields intrinsically are generally covariant vectors, one replaces \( \gamma^a (\partial_a + A_a) \) by \( \gamma^a e^a_\mu (\partial_\mu + A_\mu) \). The next step is to correct for the effect of the derivative \( \partial_\mu \) on the field \( \psi \) by making the derivative generally covariant as well as gauge covariant. The required Einstein connection is (Weinberg, 1972, sec. 12.5)

\[
E_\mu = \frac{i}{2} J^{ab} e^a_\mu e^b_\nu e_{bc;\mu}
\]  

(11.527)

in which the generators (10.302) of the Lorentz group \( J^{ab} \) are the commutators of Dirac’s 4 × 4 gamma matrices (10.305)

\[
J^{ab} = \frac{-i}{4} \left[ \gamma^a , \gamma^b \right],
\]  

(11.528)

the covariant derivative of the tetrad \( e_{bc} \) is

\[
e_{bc;\mu} = e_{bc;\mu} - e_{bc} \Gamma^a_{\nu\mu},
\]  

(11.529)

and the Levi-Civita affine connection (11.231) is \( \Gamma^a_{\nu\mu} = e^a_\sigma e^{\sigma}_{\mu\nu} \). Thus the generally covariant, gauge-covariant action density of a spin-one-half field is

\[
L' = -\overline{\psi} [\gamma^a e^a_\mu (\partial_\mu + A_\mu + E_\mu)] \psi.
\]  

(11.530)

11.64 Gauge Theory and Vectors

This section is optional on a first reading.

We can formulate Yang-Mills theory in terms of vectors as we did relativity. To accommodate noncompact groups, we will generalize the unitary
matrices \( U(x) \) of the Yang-Mills gauge group to nonsingular matrices \( V(x) \) that act on \( n \) matter fields \( \psi^a(x) \) as

\[
\psi'^a(x) = \sum_{a=1}^{n} V^b_a(x) \psi^b(x).
\] (11.531)

The field

\[
\Psi(x) = \sum_{a=1}^{n} e_a(x) \psi^a(x)
\] (11.532)

will be gauge invariant \( \Psi'(x) = \Psi(x) \) if the vectors \( e_a(x) \) transform as

\[
e'^a(x) = \sum_{b=1}^{n} e_b(x) V^{-1b}_a(x).
\] (11.533)

In what follows, we will sum over repeated indices from 1 to \( n \) and often will suppress explicit mention of the spacetime coordinates. In this compressed notation, the field \( \Psi \) is gauge invariant because

\[
\Psi' = e'_a \psi'^a = e_b V^{-1b}_a V^c_a \psi^c = e_b \delta^b_c \psi^c = e_b \psi^b = \Psi
\] (11.534)

which is \( e'^T \psi' = e^T V^{-1} e \psi = e^T \psi \) in matrix notation.

The inner product of two basis vectors is an internal “metric tensor”

\[
e^a \cdot e_b = \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} e^a_\alpha \eta_{\alpha\beta} e_b^\beta = \sum_{a=1}^{N} e^a_\alpha e_b^\alpha = g^{ab}
\] (11.535)

in which for simplicity I used the the \( N \)-dimensional identity matrix for the metric \( \eta \). As in relativity, we’ll assume the matrix \( g^{ab} \) to be nonsingular. We then can use its inverse to construct dual vectors \( e^a = g^{ab} e_b \) that satisfy \( e^{a\dagger} \cdot e_b = \delta^a_b \).

The free Dirac action density of the invariant field \( \Psi \)

\[
\bar{\psi}(\gamma^i \partial_i + m) \psi = \bar{\psi}_a e^{a\dagger}(\gamma^i \partial_i + m) e_b \psi^b = \bar{\psi}_a \left[ \gamma^i (\delta^a_{\beta} \partial_i + e^{a\dagger} \cdot e_{b,i}) + m \delta^a_b \right] \psi^b
\] (11.536)

is the full action of the component fields \( \psi^b \)

\[
\bar{\psi}(\gamma^i \partial_i + m) \psi = \bar{\psi}_a (\gamma^i D^a_{ib} + m \delta^a_b) \psi^b = \bar{\psi}_a \left[ \gamma^i (\delta^a_{\beta} \partial_i + A^a_{ib}) + m \delta^a_b \right] \psi^b
\] (11.537)

if we identify the gauge-field matrix as \( A^a_{ib} = e^{a\dagger} \cdot e_{b,i} \) in harmony with the definition (11.231) of the affine connection \( \Gamma^k_{i\ell} = e^k \cdot e_{i,\ell} \).

Under the gauge transformation \( e'_a = e_b V^{-1b}_a \), the metric matrix transforms as

\[
g'_{ab} = V^{-1c}_a g_{cd} V^{-1d}_b \quad \text{or as} \quad g' = V^{-1} g V^{-1}
\] (11.538)
in matrix notation. Its inverse goes as $g^{-1} = V g^{-1} V^\dagger$.

The gauge-field matrix $A_{i b}^a = e^{a\dagger} e_{b,i} = g^{a c} e^c_{b,i}$ transforms as

$$A_{i b}^a = g^{a c} e^c_{b,i} = V_a^c A_{i b}^c V_{b i}^{-1}d + V_a^c V_{b i}^{-1}c \tag{11.539}$$

or as $A'_{i} = V A_i V^{-1} + V \partial_i V^{-1} = V A_i V^{-1} - (\partial_i V) V^{-1}$.

By using the identity $e^{a\dagger} e_{c,i} = - e^{a\dagger} e_{c,i}$, we may write (exercise 11.45) the Faraday tensor as

$$F_{i j b} = [D_i, D_j]_{b} = e^{a\dagger} e_{b,j} - e^{a\dagger} e_{b,j} e^{c\dagger} e_{b,j} - e^{a\dagger} e_{b,i} + e^{a\dagger} e_{b,i} e^{c\dagger} e_{b,i}. \tag{11.540}$$

If $n = N$, then

$$\sum_{c=1}^{n} e^{c\dagger} e^{\beta c} = \delta^{\alpha\beta} \quad \text{and} \quad F_{i j b} = 0. \tag{11.541}$$

The Faraday tensor vanishes when $n = N$ because the dimension of the embedding space is too small to allow the tangent space to have different orientations at different points $x$ of spacetime. The Faraday tensor, which represents internal curvature, therefore must vanish. One needs at least three dimensions in which to bend a sheet of paper. The embedding space must have $N > 2$ dimensions for $SU(2)$, $N > 3$ for $SU(3)$, and $N > 5$ for $SU(5)$.

The covariant derivative of the internal metric matrix

$$g_{i i} = g_{i i} - g A_i - A_i^\dagger g \tag{11.542}$$

does not vanish and transforms as $(g_{i i})' = V^{-1i} g_{i i} V^{-1}$. A suitable action density for it is the trace $\text{Tr}(g_{i i} g^{-1} g_{i i} g^{-1})$. If the metric matrix assumes a (constant, hermitian) mean value $g_0$ in the vacuum at low temperatures, then its action is

$$m^2 \text{Tr} \left[(g_0 A_i + A_i^\dagger g_0^{-1}) (g_0 A_i + A_i^\dagger g_0^{-1}) g_0^{-1} \right] \tag{11.543}$$

which is a mass term for the matrix of gauge bosons

$$W_i = g_0^{1/2} A_i g_0^{-1/2} + g_0^{-1/2} A_i^\dagger g_0^{1/2}. \tag{11.544}$$

This mass mechanism also gives masses to the fermions. To see how, we write the Dirac action density (11.537) as

$$\bar{\psi}_a \left[ \gamma^i (\delta^a b \partial_i + A_i^a_{b i}) + m \delta^a_b \right] \psi^b = \bar{\psi}^a \left[ \gamma^i (g_{a b} \partial_i + g_{a c} A_{i b}^c) + m g_{a b} \right] \psi^b. \tag{11.545}$$

Each fermion now gets a mass $m c_i$ proportional to an eigenvalue $c_i$ of the hermitian matrix $g_0$.

This mass mechanism does not leave behind scalar bosons. Whether Nature ever uses it is unclear.
This section is optional on a first reading.

In gauge theory, what plays the role of spacetime? Could it be the group manifold? Let us consider the gauge group $SU(2)$ whose group manifold is the 3-sphere in flat euclidian 4-space. A point on the 3-sphere is explained in example 10.31. The coordinates $r^a = r_a$ are not vectors. The three basis vectors are

$$e_a = \frac{\partial p}{\partial r^a} = \left( \mp \frac{r_a}{\sqrt{1 - r^2}}, \delta^2_a, \delta^3_a \right) \quad (11.547)$$

and so the metric $g_{ab} = e_a \cdot e_b$ is

$$g_{ab} = \frac{r_a r_b}{1 - r^2} + \delta_{ab} \quad (11.548)$$

or

$$\|g\| = \frac{1}{1 - r^2} \begin{pmatrix} 1 - r_2^2 - r_3^2 & r_1 r_2 & r_1 r_3 \\ r_2 r_1 & 1 - r_1^2 - r_3^2 & r_2 r_3 \\ r_3 r_1 & r_3 r_2 & 1 - r_1^2 - r_2^2 \end{pmatrix}. \quad (11.549)$$

The inverse matrix is

$$g^{bc} = \delta_{bc} - r_b r_c. \quad (11.550)$$

The dual vectors

$$e^b = g^{bc} e_c = \left( \mp r_b \sqrt{1 - r^2}, \delta^1_b - r_b r_1, \delta^2_b - r_b r_2, \delta^3_b - r_b r_3 \right) \quad (11.551)$$

satisfy $e^b \cdot e_a = \delta^b_a$.

There are two kinds of affine connections $e^b \cdot e_{a,c}$ and $e^b \cdot e_{a,i}$. If we differentiate $e_a$ with respect to an $SU(2)$ coordinate $r_c$, then

$$E^b_{c a} = e^b \cdot e_{a,c} = r_b \left( \delta_{ac} + \frac{r_a r_c}{1 - r^2} \right) \quad (11.552)$$

in which we used $E$ (for Einstein) instead of $\Gamma$ for the affine connection. If we differentiate $e_a$ with respect to a spacetime coordinate $x^i$, then

$$E^b_{i a} = e^b \cdot e_{a,i} = e^b \cdot e_{a,c} r^c_{i} = r_b r^c_{i} \left( \delta_{ac} + \frac{r_a r_c}{1 - r^2} \right). \quad (11.553)$$

But if the group coordinates $r_a$ are functions of the spacetime coordinates $x^i$, then there are 4 new basis 4-vectors $e_i = e_a r_{a,i}$. The metric then is a
Exercises

7 × 7 matrix ∥ g ∥ with entries $g_{a,b} = e_a \cdot e_b$, $g_{a,k} = e_a \cdot e_k$, $g_{i,b} = e_i \cdot e_b$, and
$g_{i,k} = e_i \cdot e_k$ or

$$
∥ g ∥ = \begin{pmatrix} g_{a,b} & g_{a,b} r_{b,k} \\ g_{a,b} r_{a,i} & g_{a,b} r_{a,i} r_{b,k} \end{pmatrix}
$$

(11.554)

Further Reading


Exercises

11.1 Compute the derivatives (11.22 & 11.23).
11.2 Show that the transformation $x \to x'$ defined by (11.16) is a rotation
and a reflection.
11.3 Show that the matrix (11.40) satisfies the Lorentz condition (11.39).
11.4 If $\eta = L \eta L^T$, show that $\Lambda = L^{-1}$ satisfies the definition (11.39) of a
Lorentz transformation $\eta = \Lambda^T \eta \Lambda$.
11.5 The LHC is designed to collide 7 TeV protons against 7 TeV protons for
a total collision energy of 14 TeV. Suppose one used a linear accelerator
to fire a beam of protons at a target of protons at rest at one end of
the accelerator. What energy would you need to see the same physics
as at the LHC?
11.6 Use Gauss’s law and the Maxwell-Ampère law (11.87) to show that the
microscopic (total) current 4-vector $j = (c \rho, \mathbf{j})$ obeys the continuity
equation $\dot{\rho} + \nabla \cdot \mathbf{j} = 0$.
11.7 Show that if $M_{ik}$ is a covariant second-rank tensor with no particular
symmetry, then only its antisymmetric part contributes to the 2-form
$M_{ik} dx^i \wedge dx^k$ and only its symmetric part contributes to the quantity
$M_{ik} dx^i dx^k$.
11.8 In rectangular coordinates, use the Levi-Civita identity (1.453) to de-
rive the curl-curl equations (11.90).
11.9 Derive the Bianchi identity (11.92) from the definition (11.79) of the
Faraday field-strength tensor, and show that it implies the two homo-
genous Maxwell equations (11.82).
11.10 Show that if \( A \) is a \( p \)-form, then 
\[
d(AB) = dA \wedge B + (-1)^p A \wedge dB.
\]

11.11 Show that if \( \omega = a_{ij} dx^i \wedge dx^j / 2 \) with \( a_{ij} = -a_{ji} \), then
\[
d\omega = \frac{1}{3!} (\partial_k a_{ij} + \partial_i a_{jk} + \partial_j a_{ki}) \ dx^i \wedge dx^j \wedge dx^k. \tag{11.555}
\]

11.12 (a) Derive the metric (11.153) for the hyperboloid \( H^2 \) from the inner products of \( e_r \) and \( e_\phi \) by using the metric \( \eta = \text{diag}(1,1,-1) \). (b) Derive the line element (11.154) and metric (11.153) of the hyperboloid \( H^2 \) from its line element (11.150) in the embedding space \( \mathbb{R}^3 \).


11.14 Use the flat-space formula (11.183) to compute the change \( dp \) due to \( d\rho, d\phi, \) and \( dz \), and so derive the expressions (11.184) for the orthonormal basis vectors \( \hat{\rho}, \hat{\phi}, \) and \( \hat{z} \).

11.15 Similarly, derive (11.192) from (11.190).

11.16 Use the definition (11.207) to show that in flat 3-space, the dual of the Hodge dual is the identity: \( **(dx^i) = dx^i \) and \( **(dx^i \wedge dx^k) = dx^i \wedge dx^k. \)

11.17 Use the definition of the Hodge star (11.218) to derive (a) two of the four identities (11.219) and (b) the other two.

11.18 Show that Levi-Civita’s 4-symbol obeys the identity (11.223).

11.19 Show that \( \epsilon_{\ell m n} e^{p m n} = 2 \delta_\ell^p \).

11.20 Show that \( \epsilon_{k l m n} e^{p l m n} = 3! \delta_k^p \).

11.21 (a) Using the formulas (11.192) for the basis vectors of spherical coordinates in terms of those of rectangular coordinates, compute the derivatives of the unit vectors \( \hat{r}, \hat{\theta}, \) and \( \hat{\phi} \) with respect to the variables \( r, \theta, \) and \( \phi \) and express them in terms of the basis vectors \( \hat{r}, \hat{\theta}, \) and \( \hat{\phi} \). (b) Using the formulas of (a) and our expression (6.28) for the gradient in spherical coordinates, derive the formula (11.329) for the laplacian \( \nabla \cdot \nabla \).

11.22 Consider the torus with coordinates \( \theta, \phi \) labeling the arbitrary point
\[
p = (\cos \phi (R + r \sin \theta), \sin \phi (R + r \sin \theta), r \cos \theta) \tag{11.556}
\]
in which \( R > r \). Both \( \theta \) and \( \phi \) run from 0 to \( 2\pi \). (a) Find the basis vectors \( e_\theta \) and \( e_\phi \). (b) Find the metric tensor and its inverse.

11.23 For the same torus, (a) find the dual vectors \( \hat{e}^\theta \) and \( \hat{e}^\phi \) and (b) find the nonzero connections \( \Gamma^i_{jk} \) where \( i, j, \) & \( k \) take the values \( \theta, \theta, \) & \( \phi \).

11.24 For the same torus, (a) find the two Christoffel matrices \( \Gamma_\theta \) and \( \Gamma_\phi \), (b) find their commutator \( [\Gamma_\theta, \Gamma_\phi] \), and (c) find the elements \( R^\theta_{\theta \theta \theta}, R^\phi_{\phi \phi \phi}, R^\theta_{\phi \phi \phi} \), and \( R^\phi_{\theta \theta \phi} \) of the curvature tensor.
11.25 Find the curvature scalar $R$ of the torus with points (11.556). **Hint:**
In these four problems, you may imitate the corresponding calculation for the sphere in Sec. 11.48.

11.26 By differentiating the identity $g_{ik} g_{kl} = \delta^i_l$, show that $\delta g^{ik} = - g^{is} g^{kt} \delta g_{st}$ or equivalently that $d g^{ik} = - g^{is} g^{kt} d g_{st}$.

11.27 Just to get an idea of the sizes involved in black holes, imagine an isolated sphere of matter of uniform density $\rho$ that as an initial condition is all at rest within a radius $r_b$. Its radius will be less than its Schwarzschild radius if

$$r_b < \frac{2MG}{c^2} = 2 \left( \frac{4}{3} \pi r_b^3 \rho \right) \frac{G}{c^2}.$$  \hspace{1cm} \text{(11.557)}

If the density $\rho$ is that of water under standard conditions (1 gram per cc), for what range of radii $r_b$ might the sphere be or become a black hole? Same question if $\rho$ is the density of dark energy.

11.28 For the points (11.427), derive the metric (11.430) with $k = 1$. Don’t forget to relate $d\chi$ to $dr$.

11.29 For the points (11.428), derive the metric (11.430) with $k = 0$.

11.30 For the points (11.429), derive the metric (11.430) with $k = -1$. Don’t forget to relate $d\chi$ to $dr$.

11.31 Suppose the constant $k$ in the Roberson-Walker metric (11.426 or 11.430) is some number other than 0 or $\pm 1$. Find a coordinate transformation such that in the new coordinates, the Roberson-Walker metric has $k = k/|k| = \pm 1$. **Hint:** You also can change the scale factor $a$.

11.32 Derive the affine connections in Eq.(11.434).

11.33 Derive the affine connections in Eq.(11.435).

11.34 Derive the affine connections in Eq.(11.436).

11.35 Derive the spatial Einstein equation (11.446) from (11.410, 11.430, 11.441, 11.443, & 11.444).

11.36 Assume there had been no inflation, no era of radiation, and no dark energy. In this case, the magnitude of the difference $|\Omega - 1|$ would have increased as $t^{2/3}$ over the past 13.8 billion years. Show explicitly how close to unity $\Omega$ would have had to have been at $t = 1$ s so as to satisfy the observational constraint $|\Omega_0 - 1| < 0.036$ on the present value of $\Omega$.

11.37 Derive the relation (11.464) between the energy density $\rho$ and the Robertson-Walker scale factor $a(t)$ from the conservation law (11.461) and the equation of state $p_i = w_i \rho_i$.

11.38 Use the Friedmann equations (11.445 & 11.469) for constant $\rho = -p$ and $k = 1$ to derive (11.485) subject to the boundary condition that $a(t)$ has its minimum at $t = 0$. 
11.39 Use the Friedmann equations (11.445 & 11.469) with $w = -1$, $\rho$ constant, and $k = -1$ to derive (11.486) subject to the boundary condition that $a(0) = 0$.

11.40 Use the Friedmann equations (11.445 & 11.469) with $w = -1$, $\rho$ constant, and $k = 0$ to derive (11.484). Show why a linear combination of the two solutions (11.484) does not work.

11.41 Use the conservation equation (11.490) and the Friedmann equations (11.445 & 11.469) with $w = 1/3$, $k = 0$, and $a(0) = 0$ to derive (11.493).

11.42 Show that if the matrix $U(x)$ is nonsingular, then

$$\left( \partial_i U \right) U^{-1} = -U \partial_i U^{-1}. \quad (11.558)$$

11.43 The gauge-field matrix is a linear combination $A_k = -ig t^b A_k^b$ of the generators $t^b$ of a representation of the gauge group. The generators obey the commutation relations

$$[t^a, t^b] = if_{abc} t^c \quad (11.559)$$

in which the $f_{abc}$ are the structure constants of the gauge group. Show that under a gauge transformation (11.520)

$$A_i^t = U A_i U^{-1} - \left( \partial_i U \right) U^{-1} \quad (11.560)$$

by the unitary matrix $U = \exp(-ig \lambda^a t^a)$ in which $\lambda^a$ is infinitesimal, the gauge-field matrix $A_i$ transforms as

$$-ig A_i^t a^a = -ig A_i^a - ig^2 f_{abc} \lambda^a A_i^b t^c + ig \partial_i \lambda^a t^a. \quad (11.561)$$

Show further that the gauge field transforms as

$$A_i^t a^a = A_i^a - \partial_i \lambda^a - g f_{abc} A_i^b \lambda^c. \quad (11.562)$$

11.44 Show that if the vectors $e_a(x)$ are orthonormal, then $e^a\ddagger e_{c,i} = -e_{a,i}^\ddagger e_c$.

11.45 Use the identity of exercise 11.44 to derive the formula (11.540) for the nonabelian Faraday tensor.

11.46 Using the tricks of section 11.41, show that $\delta \sqrt{g} = -\frac{1}{2} \sqrt{g} g_{ik} \delta g^{ik}$. This relation and the definition (11.386) $R = g^{ik} R^n_{ink}$ imply that the first-order change in the Einstein-Hilbert action is (11.413) apart from an irrelevant surface term (Carroll, 2003, chap 4.3) due to $g^{ik} \sqrt{g} \delta R^n_{ink}$.

11.47 Write Dirac’s action density in the explicitly hermitian form $L_D = -\frac{1}{2} \bar{\psi} \gamma^i \partial_i \psi - \frac{1}{2} \left[ \bar{\psi} \gamma^i \partial_i \psi \right]^\dagger$ in which the field $\psi$ has the invariant form $\bar{\psi} = e_a \psi_a$ and $\bar{\psi} = i \psi^0 \gamma^0$. Use the identity $\left[ \bar{\psi} a^i \gamma^i \psi_b \right]^\dagger = -\bar{\psi} b^i \gamma^i \psi_a$ to
show that the gauge-field matrix $A_i$ defined as the coefficient of $\bar{\psi}_a \gamma^i \psi_b$ as in $\bar{\psi}_a \gamma^i (\partial_i + i A_{iab}) \psi_b$ is hermitian $A_{iab}^* = A_{iba}$. 