(*e*) Show that

$$H = -\lambda \begin{vmatrix} f_{LL} & f_{LK} & f_L \\ f_{KL} & f_{KK} & f_K \\ f_L & f_K & 0 \end{vmatrix}$$

where H is defined as in Eq. (8-14) and λ is marginal cost.

(*f*) Show that $H = \lambda(y^2/K^2) f_{LL} = \lambda(y^2/L^2) f_{KK}$ if $f(L, K)$ is homogeneous of degree 1. Show, therefore, that there can be no "stage I" or "stage III" of the production process if the isoquants are convex to the origin.

8. Derive an expression analogous to Eq. (8-44) for the cross-effects $\partial x_i^*/\partial w_j$ and $\partial x_i^s/\partial w_j$, $i \neq j$. Show that if x_i and x_j are either both substitutes or both complements to x_n, then $\partial x_i^*/\partial w_j \leq \partial x_i^s/\partial w_j$.

9. Derive an expression analogous to Eq. (8-49) showing the relationship between the profit-maximizing and cost-minimizing cross-effects $\partial x_i^p/\partial w_j$ and $\partial x_i^y/\partial w_j, i \neq j$. If x_i and x_j are both normal factors, which cross-effect is larger? Can these short- and long-run cross-effects have different signs?

SELECTED REFERENCES

Alchian, A. A.: "Costs and Outputs," in M. Abramovitz (ed.), *The Allocation of Economic Resources*, Stanford University Press, Stanford, CA, 1959.

Allen, R. G. D.: *Mathematical Economics*, Macmillan & Co., Ltd., London, 1956.

Coase, R. H.: "The Nature of the Firm," *Economica*, (N.S.), **4**:331–351, 1937. Reprinted in American Economic Association, *Readings in Price Theory*, Richard D. Irwin, Chicago, 1952.

Fuss, M., and D. McFadden.: *Production Economics: A Dual Approach to Theory and Applications*, North Holland, Amsterdam, 1978.

Hicks, J. R.: *Value and Capital*, 2d ed., Clarendon Press, Oxford, 1946.

Samuelson, P. A.: *Foundations of Economic Analysis*, Harvard University Press, Cambridge, MA, 1947.

Shephard, Ronald W.: *Cost and Production Functions*, Princeton University Press, Princeton NJ, 1953; also the revised version of this book which has become a classic, *The Theory of Cost and Production Functions*, Princeton University Press, Princeton, NJ, 1970.

CHAPTER 9

COST AND PRODUCTION FUNCTIONS: SPECIAL TOPICS

9.1 HOMOGENEOUS AND HOMOTHETIC PRODUCTION FUNCTIONS†

An interesting and important class of production functions is the homothetic production functions, of which the homogeneous functions are a subset. A production function is homogeneous of degree r if when all inputs are increased (decreased) by the same proportion, output increases (decreases) by the rth power of that increase. Formally, if $f(x_1, \ldots, x_n)$ is homogeneous of degree r,

$$f(tx_1, \ldots, tx_n) \equiv t^r f(x_1, \ldots, x_n)$$

Several properties of homogeneous functions in general were noted in an earlier chapter, especially Euler's theorem, already used extensively in other contexts. In addition, the geometric property that

$$\frac{f_i(tx_1, \ldots, tx_n)}{f_j(tx_1, \ldots, tx_n)} \equiv \frac{f_i(x_1, \ldots, x_n)}{f_j(x_1, \ldots, x_n)}$$

i.e., that the slopes of the level curves are the same along every point of a given ray out of the origin, was proved using the homogeneity of degree $r - 1$ of the first partials f_i and f_j.

†The student may wish to review the sections in Chap. 3 on homogeneity.

However, homogeneous functions are not the only functions with this geometric property. Consider any monotonic transformation $F(z)$ of a homogeneous production function $z = f(x_1, \ldots, x_n)$. That is, consider $y = H(x_1, \ldots, x_n) = F(f(x_1, \ldots, x_n))$, where $F'(z) > 0$. The requirement $F'(z) > 0$ ensures that z and y move in the same direction; e.g., when z increases, y must increase. The slope of a level curve $H(x_1, \ldots, x_n) = y$ in the $x_i x_j$ plane is

$$\frac{H_i}{H_j} = \frac{F'(z) f_i}{F'(z) f_j} = \frac{f_i}{f_j}$$

But we already know that f_i/f_j is invariant under a radial expansion. Hence the function $H(x_1, \ldots, x_n) = F(f(x_1, \ldots, x_n))$ also exhibits this property.

The class of functions $y = H(x_1, \ldots, x_n) = F(f(x_1, \ldots, x_n))$, where $F' \neq 0$ and $f(x_1, \ldots, x_n)$ is a homogeneous function, is called the *homothetic functions*. In fact, no generality is lost if $f(x_1, \ldots, x_n)$ is restricted to *linear* homogeneous functions, i.e., functions homogeneous of degree 1. The reason is that if $f(x_1, \ldots, x_n)$ is homogeneous of degree r, then $[f(x_1, \ldots, x_n)]^{1/r}$ is homogeneous of degree 1:

$$[f(tx_1, \ldots, tx_n)]^{1/r} \equiv [t^r f(x_1, \ldots, x_n)]^{1/r} \equiv t[f(x_1, \ldots, x_n)]^{1/r}$$

Taking the rth root of f can be incorporated into the monotonic transformation $y = F(z)$. That is, $F(z)$ itself can be thought of as a composite function, the first part of which is taking the rth root of $f(x_1, \ldots, x_n)$ and the second part whatever transformation yields $H(x_1, \ldots, x_n)$. Hence we can define as the class of homothetic functions all functions $H(x_1, \ldots, x_n) \equiv F(f(x_1, \ldots, x_n))$, where $f(x_1, \ldots, x_n)$ is homogeneous of degree 1 and $F' \neq 0$.

The statement that the slopes of the level curves are invariant under radial expansion or contraction of the original point, i.e., when $x_1, \ldots, x_n$ is replaced by $tx_1, \ldots, tx_n$, can be expressed another way. The slope of the level curve (surface) at any point is H_i/H_j. This is just another function of the x_i's; that is, define

$$\frac{H_i}{H_j} \equiv h_{ij}(x_1, \ldots, x_n)$$

The function $h_{ij}(x_1, \ldots, x_n)$ designates the (negative) slope of the level surface of H in the $x_i x_j$ plane. This slope is unchanged under $x_1, \ldots, x_n \to tx_1, \ldots, tx_n$. But this is simply a statement that $h_{ij}(x_1, \ldots, x_n)$ is homogeneous of degree 0, that is, that $h_{ij}(tx_1, \ldots, tx_n) \equiv h_{ij}(x_1, \ldots, x_n)$. It can in fact be shown by more advanced methods that homotheticity can be defined in this manner also; i.e., if $h_{ij}(x_1, \ldots, x_n)$ is homogeneous of degree 0 for all $x_i x_j$ planes, then $H(x_1, \ldots, x_n)$ must have the form $H(x_1, \ldots, x_n) \equiv F(f(x_1, \ldots, x_n))$, where $f(x_1, \ldots, x_n)$ is homogeneous of degree 1 and $F' \neq 0$.

Example. Consider the production function $y = H(x_1, x_2) = x_1 x_2 + x_1^2 x_2^2$. This function is not homogeneous, as can readily be verified. It is homothetic, however, since $H(x_1, x_2) \equiv z + z^2$, where $z = x_1 x_2$. That is, $H(x_1, x_2) \equiv F(f(x_1, x_2))$, where $F(z) = z + z^2$. Note that $F'(z) = 1 + 2z \neq 0$, since production is presumed to be nonnegative. The slope of a level curve of $H(x_1, x_2)$ is

$$-\frac{H_1}{H_2} = -\frac{x_2 + 2x_1 x_2^2}{x_1 + 2x_1^2 x_2}$$
$$= -\frac{x_2(1 + 2x_1 x_2)}{x_1(1 + 2x_1 x_2)} = -\frac{x_2}{x_1}$$

Note that $F'(z) = 1 + 2x_1 x_2$ appears in the numerator and denominator. Hence, $H_1/H_2 = h_{12}(x_1, x_2) = x_2/x_1$. The function h_{12} is clearly homogeneous of degree 0: $h_{12}(tx_1, tx_2) \equiv tx_2/tx_1 \equiv x_2/x_1 \equiv h_{12}(x_1, x_2)$. Thus, the level curves of $x_1 x_2 + x_1^2 x_2^2$ have the same slope at all points along any given ray out of the origin.

Still another way to express homotheticity is to state that the output elasticities for all factors are equal at any given point. That is, $\epsilon_{1y} = \epsilon_{2y} = \cdots = \epsilon_{ny}$. This is clear from the geometry of straight-line expansion paths. Consider Fig. 9-1. Any increase, say, in output from y to y' will result in a new tangency point B along a straight line through the origin and the former tangency point A. The triangles OAx_1^0 and $OB(tx_1^0)$ are similar; hence, x_1 increases by $OB/OA \equiv t$. But, clearly, x_2 increases by $OB/OA \equiv t$ also, for the same reason. Hence for homothetic production functions, output elasticities are equal in all factors.

This result can be shown algebraically by noting that a straight-line expansion path implies that the ratio x_j/x_i, the slope of the ray out to that point in the $x_i x_j$ plane, is the same for any output level as long as factor prices are held constant. That is,

$$\frac{\partial(x_j^*/x_i^*)}{\partial y} \equiv 0$$

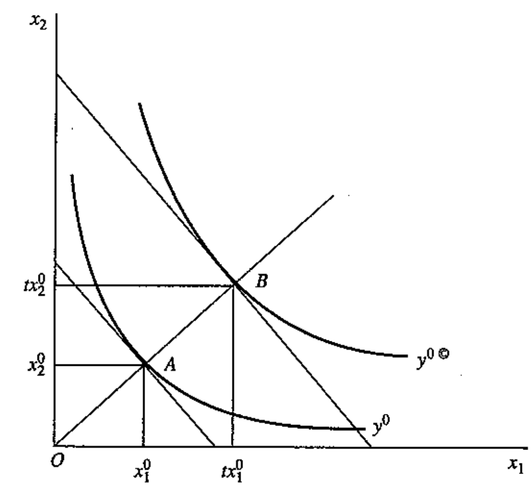

FIGURE 9-1
Homothetic Production Functions. The level curves of homothetic production functions are all radial expansions of one another; i.e., at the intersections of any ray out of the origin and the level curves, the slopes are all the same. Put another way, if output is increased autonomously, i.e., holding factor prices constant, the new tangency point will lie along the ray projecting the old tangency point from the origin. By similar triangles, it is clear that x_1 increases in the same proportion t as x_2 does when y is increased.

Using the quotient rule and multiplying through by x_i^2 yields

$$x_i^* \frac{\partial x_j^*}{\partial y} - x_j^* \frac{\partial x_i^*}{\partial y} \equiv 0$$

After multiplication by y and division by $x_i^* x_j^*$ this leads to

$$\frac{y}{x_j^*}\frac{\partial x_j^*}{\partial y} \equiv \frac{y}{x_i^*}\frac{\partial x_i^*}{\partial y}$$

or

$$\epsilon_{jy} \equiv \epsilon_{iy} \quad i, j = 1, \ldots, n \tag{9-1}$$

The value of this common output elasticity can be found by applying Eq. (8-63). Since $\epsilon_{iy} = \epsilon_{jy} = \epsilon_y$, say, this constant can be removed from the summation, yielding

$$\epsilon_y \sum \kappa_i' = 1$$

However,

$$\sum \kappa_i' = \frac{\sum w_i x_i}{(\text{MC})y} = \frac{\text{AC}}{\text{MC}}$$

Thus

$$\epsilon_y = \frac{\text{MC}}{\text{AC}} \tag{9-2}$$

The common value of output elasticity, for homothetic functions, is the ratio of marginal to average cost. Therefore, for firms with increasing average costs, the factors are all output-elastic; that is, $\epsilon_y = \epsilon_{iy} > 1$ for all factors; for firms with declining (average) cost, factors are all output-inelastic. Also, if the firm is at the minimum point of its AC curve, the output elasticities of its factors are all unity if the production function is homothetic.

9.2 THE COST FUNCTION: FURTHER PROPERTIES

We have already shown that $C^*(w_1, w_2, y)$ is homogeneous of degree 1 in w_1 and w_2, or, more generally, for the n-factor firm, $C^*(w_1, \ldots, w_n, y)$ is homogeneous of degree 1 in $w_1, \ldots, w_n$. Again, since $C^* = \sum w_i x_i^*(w_1, \ldots, w_n, y)$, and since the $x_i^*(w_1, \ldots, w_n, y)$'s are homogeneous of degree 0 in $w_1, \ldots, w_n$,

$$\begin{aligned} C^*(tw_1, \ldots, tw_n, y) &\equiv \sum t w_i x_i^*(tw_1, \ldots tw_n, y) \\ &\equiv t \sum w_i x_i^*(w_1, \ldots, w_n, y) \\ &\equiv tC^*(w_1, \ldots, w_n, y) \end{aligned}$$

Suppose in addition that the production function $y = f(x_1, \ldots, x_n)$ is homogeneous of some degree $r > 0$ in $x_1, \ldots, x_n$. In this case, we shall demonstrate that the cost

function can be partitioned into

$$C^*(w_1, \ldots, w_n, y) \equiv y^{1/r} A(w_1, \ldots, w_n) \tag{9-3}$$

where it is to be noted that the function $A(w_1, \ldots, w_n)$ is a function of factor prices only. In the case where $r = 1$, that is, $f(x_1, \ldots, x_n)$ exhibits constant returns to scale,

$$C^*(w_1, \ldots, w_n, y) \equiv y\text{AC}(w_1, \ldots, w_n) \tag{9-4}$$

where $A(w_1, \ldots, w_n)$ becomes the average cost function AC. But average cost AC $(w_1, \ldots, w_n)$ is a function of factor prices only, i.e., independent of output level. This is of course as it must be; if a firm exhibits constant returns to scale, AC = MC = constant, i.e., a function of factor prices only at every level of output.

We shall prove some of these results for the case of differentiable functions. For simplicity, we shall deal with functions of only two variables, i.e., the two-factor case. The generalizations to n factors are straightforward and are left as exercises for the student. Remember, as always, that y is a parameter in the cost minimization model.

Equation (9-3) is intuitively plausible. Consider Fig. 9-2. Suppose the firm is initially at point $\mathbf{x}^0$ utilizing inputs $\mathbf{x}^0 = (x_1^0, x_2^0)$. Some level of cost $C(\mathbf{x}^0)$ would exist. Suppose now both inputs were doubled, to $(2x_1^0, 2x_2^0) = \mathbf{x}^1$. Then since the production function is homothetic (indeed, homogeneous), the new cost-minimizing tangency will lie on a ray from the origin extending past the original point $\mathbf{x}^0$ to point

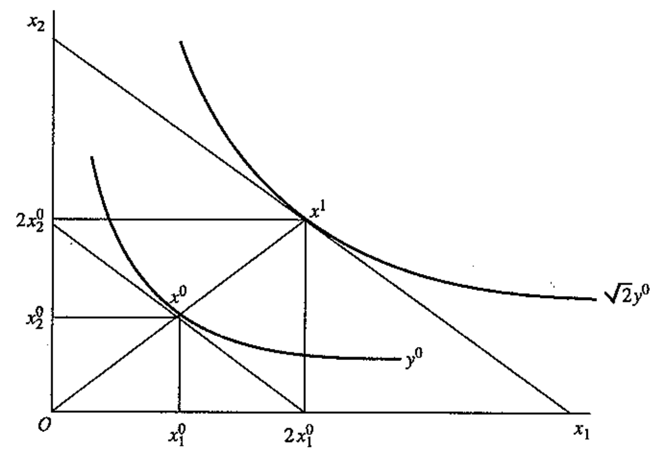

FIGURE 9-2
A Production Function Homogeneous of Degree 1/2. When input levels x_1^0, x_2^0 are doubled, say, output increases by the factor $2^{1/2} = \sqrt{2}$. However, since $C = w_1x_1 + w_2x_2$, cost doubles; that is, $C(x^0) = \frac{1}{2}C(x^1)$. This means that a doubling of cost is accompanied by a $\sqrt{2}$-fold increase in y; that is, cost and output are related as $C = Ay^2$. The constant of proportionality is constant only in that it does not involve y output. It *is* a function of factor prices; that is, $A = A(w_1, w_2)$.

$\mathbf{x}^1$ at twice the input levels. At $\mathbf{x}^1$, the cost $C(\mathbf{x}^1)$ is clearly twice $C(\mathbf{x}^0)$, since both inputs have exactly doubled while factor prices remain the same. Hence, $C(\mathbf{x}^1) = 2\,C(\mathbf{x}^0)$. However, y^0, output at $\mathbf{x}^0$, has grown only to $2^{1/2}y^0 = \sqrt{2}y^0$, since the production function is homogeneous of degree 1/2. This means that, holding factor prices constant, cost and output are related in the proportion $C = Ay^2$, since a doubling, say, of cost is accompanied by an increase of output of the factors of $\sqrt{2}$. The proportionality constant A, in fact, must be dependent on factor prices; that is, $A = A(w_1, w_2)$. For a different slope of the isocost line, the proportionality constant will be different; however, cost and output will still have the general relation (9-3).

The preceding reasoning cannot be applied to general nonhomogeneous functions. (It can be applied in a more complicated fashion, and we shall do so, to general homothetic functions.) If the production function is nonhomothetic, a given increase in output is not related to a simple proportionate expansion of all inputs. Instead, the ratios of one factor to another will change. Hence, the cost function will necessarily be a more complicated function than (9-3), wherein factor prices and output are all mixed together and not separable into two parts, one related to output and the other to factor prices.

In proving (9-3), we shall use the following relationship, already discussed in the first discussion of interpreting λ, the Lagrange multiplier of the constrained cost minimization problem, as marginal cost. Since $C^* \equiv w_1x_1^* + w_2x_2^*$, then since $w_1 = \lambda^* f_1$ and $w_2 = \lambda^* f_2$,

$$C^* \equiv \lambda^*(f_1x_1^* + f_2x_2^*)$$

However, for homogeneous functions, $f_1x_1 + f_2x_2 \equiv ry$, where r is the degree of homogeneity. Hence for homogeneous functions,

$$C^* \equiv \lambda^* ry \tag{9-5a}$$

or

$$\frac{C^*}{y} \equiv r\frac{\partial C^*}{\partial y} \tag{9-5b}$$

The question now is: What general functional form $C^*(w_1, w_2, y)$ has the property of obeying Eqs. (9-5), which say that average cost C^*/y is proportional to marginal cost, the factor of proportionality being the constant r? This question is answered by integrating the partial differential Eq. (9-5*b*).

Rearranging the terms in (9-5*b*) yields

$$\frac{\partial C^*}{C^*} \equiv \frac{1}{r}\frac{\partial y}{y} \tag{9-6}$$

The differential notation ∂C^* is used rather than dC^* to remind us that in that differentiation, w_1 and w_2 were being held constant. Integrating both sides of (9-6) gives

$$\int \frac{\partial C^*}{C^*} \equiv \frac{1}{r}\left(\int \frac{\partial y}{y}\right) + K(w_1, w_2) \tag{9-7}$$

As in all integrations, an arbitrary constant appears. However, since this was a *partial* differential equation with respect to y, the constant term can include any arbitrary function of the variables held constant in the original differentiation, i.e., the factor prices here. In fact, the theory of partial differential equations assures us that the inclusion of an arbitrary function in the integration constant of the variable held fixed in the partial differentiation yields the general solution to the partial differential equation.

Performing the indicated integration in Eq. (9-7) yields

$$\log C^* = \frac{1}{r}\log y + \log A(w_1, w_2) \tag{9-8}$$

Here, we have written the constant term $K(w_1, w_2)$ as $\log A(w_1, w_2)$. There is no loss of generality involved, since any real number is the logarithm of some positive number. This manipulation, however, permits us to rewrite (9-8) as

$$\log C^* \equiv \log [y^{(1/r)}A(w_1, w_2)] \tag{9-9}$$

since the logarithm of a product is the sum of the individual logarithms, and $\log a^b \equiv b \log a$. Since the logarithms (9-8) and (9-9) are equal (identical, in fact), their antilogarithms are equal, i.e.,

$$C^* \equiv y^{(1/r)}A(w_1, w_2) \tag{9-10}$$

which was to be proved.

That (9-10) is a solution of the partial differential Eq. (9-5*b*) can be seen by substitution:

$$\lambda^* = \frac{\partial C^*}{\partial y} = \frac{1}{r}y^{[(1/r)-1]}A(w_1, w_2)$$

Substituting this into the right-hand side of Eq. (9-5*a*) yields

$$\frac{1}{r}y^{[(1/r)-1]}Ary \qquad \text{or} \qquad y^{(1/r)}A$$

But this is identically the left-hand side, C^*. By definition, since the substitution of the form $C^* = y^{(1/r)}A(w_1, w_2)$ into the equation $C^* = \lambda^* ry$ makes that equation an identity, $C^* = y^{(1/r)}A(w_1, w_2)$ is a solution of (9-5). And, it is the most general solution of (9-5) because of the inclusion of the arbitrary *function* $A(w_1, w_2)$ as the constant of integration. It is also clear that the integration constant must be positive; otherwise positive outputs would be associated with imaginary (involving $\sqrt{-1}$) costs.

To recapitulate, what has been shown is that if the production function is homogeneous of any degree r ($r > 0$), then costs, output, and factor prices are related in the multiplicatively separable fashion $C^* = y^{(1/r)}A(w_1, w_2)$. Equivalently, for homogeneous production functions, average costs are always proportional to marginal costs, the factor of proportionality being the degree of homogeneity r; that is, $C^*/y \equiv r\partial C^*/\partial y$.

Either Eq. (9-5) or (9-10) can be used to show the relationship of the degree of homogeneity to the slope of the marginal and average cost functions. From (9-10),

$$\mathrm{MC} = \frac{\partial C^*}{\partial y} = \frac{1}{r} y^{[(1/r)-1]} A(w_1, w_2)$$

and thus

$$\frac{\partial \mathrm{MC}}{\partial y} = \frac{1}{r}\left(\frac{1}{r} - 1\right) y^{[(1/r)-2]} A(w_1, w_2)$$

By inspection, if $r < 1$, $\partial\mathrm{MC}/\partial y > 0$; that is, for a homogeneous production function exhibiting decreasing returns to scale, marginal costs (not surprisingly) are always increasing. Similarly, if $r > 1$, $\partial\mathrm{MC}/\partial y < 0$; that is, falling marginal costs are associated with homogeneous production functions exhibiting increasing returns to scale. Lastly, if $r = 1$, the constant-returns-to-scale case, marginal cost is constant and equal to $A(w_1, w_2)$ for all levels of output.

Alternatively, from (9-5*b*), if $r > 1$, say, AC > MC. Since marginal cost is always below average cost, AC must always be falling, with similar reasoning holding for $r < 1$ and $r = 1$. Also, differentiating (9-5*a*) partially with respect to y yields

$$\frac{\partial C^*}{\partial y} \equiv \lambda^* \equiv r\left(\lambda^* + y\frac{\partial \lambda^*}{\partial y}\right)$$

Solving for $\partial\lambda^*/\partial y$, that is, $\partial\mathrm{MC}/\partial y$, gives

$$\frac{\partial \mathrm{MC}}{\partial y} \equiv \frac{1}{ry}\mathrm{MC}(1 - r) \tag{9-11}$$

from which the preceding results can be read directly.

Homothetic Functions

Let us now consider the functional form of the cost function associated with the general class of homothetic production functions, $y = F(f(x_1, x_2))$, where $f(x_1, x_2)$ is homogeneous of degree 1, and $F'(z) > 0$, where $z = f(x_1, x_2)$. Proceeding as before, we have

$$\begin{aligned} C^* &\equiv w_1x_1^* + w_2x_2^* \\ &\equiv \lambda^*(F'(z)f_1)x_1^* + \lambda^*(F'(z)f_2)x_2^* \\ &\equiv \lambda^*F'(z)(f_1x_1^* + f_2x_2^*) \end{aligned}$$

or

$$C^* \equiv \lambda^* F'(z) z \tag{9-12}$$

using Euler's theorem. Now y is a monotonic transformation of z; that is, $F'(z) > 0$. This means that if z were plotted against y, the resulting curve would always be upward-sloping. Under these conditions, a unique value of z will be associated with

any value of y; that is, the function $y = F(z)$ is "invertible" to $z = F^{-1}(y)$. The situation is the same as expressing demand curves as $p = p(x)$ (price as a function of quantity) instead of the more common $x = x(p)$ (quantity as a function of price). Thus we can write

$$C^* \equiv \lambda^* F'(F^{-1}(y))[F^{-1}(y)]$$

or, combining all the separate functions of y,

$$C^* \equiv \lambda^* G(y) \tag{9-13}$$

That is, for homothetic functions, the cost function can be written as marginal cost times some function of y only, $G(y)$. If the homothetic function were in fact homogeneous of some degree r, then $G(y) = ry$, a particularly simple form, as indicated in Eq. (9-5*a*). As before, the question is: What general functional form of $C^*(w_1, w_2, y)$ satisfies the partial differential Eq. (9-13)? That is, what restrictions on the form of $C^*(w_1, w_2, y)$ are imposed by the structure (9-13)?

This question is answered as before by integrating the differential Eq. (9-13). Separating the y terms and remembering that $\lambda^* = \partial C^*/\partial y$, we have

$$\frac{\partial C^*}{C^*} \equiv \frac{\partial y}{G(y)} \tag{9-14}$$

The critical thing to notice about (9-14) is that the right-hand side is a function of y only. We shall assume that some integral function of $1/G(y)$ exists, and we shall designate that integral function as $\log J(y)$. Also, an arbitrary constant of integration must appear, and, as in the homogeneous case, this constant is not really a constant but an arbitrary *function* of the remaining variables, w_1 and w_2, which are treated as constants when the cost function is differentiated partially with respect to y. This constant function will be designated $\log A(w_1, w_2)$. Thus, integrating (9-14) gives

$$\int \frac{\partial C^*}{C^*} \equiv \int \frac{\partial y}{G(y)} + \log A(w_1, w_2)$$

which yields

$$\log C^* \equiv \log J(y) + \log A(w_1, w_2)$$

Using the rules of logarithms and taking antilogarithms, we have

$$C^* \equiv J(y)A(w_1, w_2) \tag{9-15}$$

What Eq. (9-15) says is that for homothetic productions, the cost function can be written as the product of two functions: a function of output y and another function of factors prices only. $C^*(w_1, w_2, y)$ is said to be multiplicately separable in y and the factor prices.

That C^* should have this form is entirely reasonable. Recall that a homothetic function is simply a monotonic function of a linear homogeneous function. It is as if the isoquants of a linear homogeneous (constant-returns-to-scale) production function were relabeled through some technological transformation, represented by $F(z)$.

But it is only a transformation of output values, not a change in the shapes of the isoquants themselves. Since the cost function for a linear homogeneous production function can be written $C^* = yA(w_1, w_2)$, and one gets a homothetic function by operating on output y alone, not surprisingly the only change induced in the cost function is the replacement of y by some more complicated function of y, designated $J(y)$ in Eq. (9-15).

The correctness of (9-15) as a solution to (9-13) can be checked heuristically as follows. When this form, $C^* = J(y)A(w_1, w_2)$, is substituted into (9-13), the right-hand side must be identically C^*. Performing the indicated operations gives $\lambda^* = J'(y)A(w_1, w_2)$, and thus

$$C^* = J'(y)A(w_1, w_2) \times \text{some function of } y$$

and (9-15) is therefore of the requisite form.

9.3 THE DUALITY OF COST AND PRODUCTION FUNCTIONS

At this juncture let us recapitulate the analysis of production and cost functions. The starting point of the analysis was the assumption of a well-defined quasi-concave production function, i.e., one whose isoquants are convex to the origin. We asserted that the firm would always minimize the total factor cost of producing any given output level, as this was the only postulate consistent with wealth or profit maximization. The first-order conditions of the implied constrained minimization problem were then solved, in principle, for the factor demand relations $x_i = x_i^*(w_1, w_2, y)$, along with the Lagrange multiplier (identified as marginal cost) $\lambda = \lambda^*(w_1, w_2, y)$. The comparative statics relations were developed yielding certain sign restrictions on some of the partial derivatives of the previous demand relations, namely, $\partial x_i^*/\partial w_i < 0$.

These demand relations were then substituted into the expression for total cost, $C = w_1x_1 + w_2x_2$, yielding the total cost function

$$C^*(w_1, w_2, y) = w_1x_1^* + w_2x_2^*$$

It was shown via the envelope theorem that $\partial C^*/\partial w_i = x_i^*$, $\partial C^*/\partial y = \lambda^*$. Also, certain properties of the cost function regarding homogeneity and functional form were derivable from assumptions about the production function.

We now pose a new question. We have seen how it is possible to derive cost functions from production functions. Is it possible, and if so, how, to derive production functions from cost functions? That is, suppose one were given a cost function that satisfied the properties implied by the usual analysis of production functions. Is it possible to identify with that cost function some unique production function that would generate that cost function? The answer in general is yes; there is, in fact, a duality between production and cost functions: the existence of one implies, for well-behaved functions, the unique existence of the other. We shall now investigate these matters.

A critical step in the construction of the cost function was inverting the solution of the first-order relations $w_i - \lambda f_i = 0$, $y - f(x_1, x_2) = 0$ to obtain the demand relations $x_i = x_i^*(w_1, w_2, y)$. The uniqueness of these solutions is guaranteed by the sufficient second-order conditions for constrained minimum, which in turn guarantees that the Jacobian matrix of the first-order equations, i.e., the cross-partials of the Lagrangian $\mathscr{L}$, has a nonzero determinant. These sufficient second-order conditions also imply that $\partial x_i^*/\partial w_i < 0$, $i = 1, 2$. However, $x_i^* = \partial C^*/\partial w_i$. Hence,

$$\frac{\partial x_i^*}{\partial w_i} = \frac{\partial^2 C^*}{\partial w_i^2} < 0 \tag{9-16}$$

That is, the cost function has the property that the second partials with respect to the factor prices are negative. As was shown in the previous chapter, the cost functions for any well-behaved production function are weakly concave in the factor prices. Again, for the two-factor case, $C^*(w_1, w_2, y)$ is linear homogeneous in w_1, w_2. Thus, as shown earlier in a different manner, since $x_i^*(w_1, w_2, y)$ is a first partial of C^* with respect to a factor price, x_i^* is homogeneous of degree 0 in w_1, w_2. Hence by Euler's theorem,

$$\frac{\partial x_1^*}{\partial w_1}w_1 + \frac{\partial x_1^*}{\partial w_2}w_2 \equiv 0$$

Similarly

$$\frac{\partial x_2^*}{\partial w_1}w_1 + \frac{\partial x_2^*}{\partial w_2}w_2 \equiv 0$$

Eliminating w_1 and w_2 (noting that $\partial x_1^*/\partial w_2 = \partial x_2/\partial w_1^* = C_{12}^*$, $C_{ii}^* = \partial x_i^*/\partial w_i$) reveals that

$$C_{11}^*C_{22}^* - C_{12}^{*2} = 0 \tag{9-17}$$

The determinant of the cross-partials of C^* with respect to the factor prices equals 0. In fact, C^* cannot be strictly concave in w_1 and w_2 because it is linearly homogeneous in w_1 and w_2; that is, radial expansions of w_1 and w_2 produce *linear* expansions of C^*. This result easily generalizes to the case of n factors using the methodology of Chap. 7.

Consider now the problem of constructing a production function from a cost function. Before proceeding, we would check to see whether in fact the given $C^*(w_1, w_2, y)$ exhibited "weak" concavity in w_1 and w_2 and linear homogeneity in w_1 and w_2. Assume that these conditions are met. Then the implied factor demands are

$$x_1^*(w_1, w_2, y) = \frac{\partial C^*}{\partial w_1}$$

$$x_2^*(w_1, w_2, y) = \frac{\partial C^*}{\partial w_2}$$

However, x_1^* and x_2^* are homogeneous of degree 0 in w_1 and w_2; hence they can be written

$$x_1^*(w_1, w_2, y) \equiv x_1^*\left(1, \frac{w_2}{w_1}, y\right) \equiv g_1(w, y)$$
$$\tag{9-18}$$
$$x_2^*(w_1, w_2, y) \equiv x_2^*\left(1, \frac{w_2}{w_1}, y\right) \equiv g_2(w, y)$$

where $w = w_2/w_1$. But (9-18) represents two equations in the four variables x_1, x_2, w, and y. Under the mathematical conditions that the Jacobian of these equations is nonzero, i.e., that

$$J = \begin{vmatrix} g_{1w} & g_{1y} \\ g_{2w} & g_{2y} \end{vmatrix} \neq 0$$

these equations can be used to eliminate the variable w. This will leave one equation in x_1, x_2, and y, say

$$h(x_1, x_2, y) = 0$$

Solving this equation for $y = f(x_1, x_2)$ yields the production function.

How stringent is the assumption that the above Jacobian determinant be nonzero? The partials g_{1w} and g_{2w} are essentially the slopes of the factor demand relations (a reciprocal slope in the case of g_{1w}) with respect to changes in *relative* prices. In particular, using the chain rule leads to

$$\frac{\partial x_1^*}{\partial w_1} = \frac{\partial x_1^*}{\partial w}\frac{\partial w}{\partial w_1} = \frac{\partial g_1}{\partial w}\left(-\frac{w_2}{w_1^2}\right) < 0$$

and hence $g_{1w} = -(w_1^2/w_2)(\partial x_1^*/\partial w_1) > 0$. Similarly,

$$\frac{\partial x_2^*}{\partial w_2} = \frac{\partial x_2^*}{\partial w}\frac{\partial w}{\partial w_2} = g_{2w}\frac{1}{w_1}$$

and hence

$$g_{2w} = w_1 \frac{\partial x_2^*}{\partial w_2^*} < 0$$

If both factors are normal, as would be the case for homothetic production functions, then $g_{1y}, g_{2y} > 0$ and J has the sign pattern

$$\begin{vmatrix} + & + \\ - & + \end{vmatrix} > 0$$

implying that $J > 0$, and thus $J \neq 0$. In the nonhomothetic case, it would be pure coincidence if $J = 0$; hence it is not implausible to assert $J \neq 0$. Hence in general we shall expect to find a unique production function associated with any well-specified cost function. This is not to say that it will be *easy* to find either the production function or the cost function from the other. In general, the equations

to be solved, i.e., the first-order relations in the case of deriving the cost functions or Eq. (9-18) in the case of deriving the production function, will be complicated nonlinear functions. But we can be assured that the functions exist, in principle, and that they are unique.

Example. We previously have found the cost function associated with a Cobb-Douglas production function. It had the same multiplicatively separable form. Let us see how Eq. (9-18) can be used to reverse the process. Suppose C^* is given to us or estimated econometrically as

$$C^* = y^k w_1^\alpha w_2^{1-\alpha} \tag{9-19}$$

where $0 < \alpha < 1$ (to ensure that $C_1^* = x_1^* > 0, C_2^* = x_2^* > 0, C_{11}^*, C_{22}^* < 0$) and the exponents of w_1 and w_2 sum to unity (to ensure C^* homogeneous of degree 1 in w_1 and w_2). The parameter k can take on unrestricted positive values. What production function will generate this cost function?

By the envelope theorem (Shephard's lemma) $\partial C^*/\partial w_i = x_i^*$. Hence

$$x_1^* = y^k \alpha w_1^{\alpha-1} w_2^{1-\alpha} = \alpha y^k \left(\frac{w_2}{w_1}\right)^{1-\alpha}$$

Similarly,

$$x_2^* = y^k(1-\alpha)w_1^\alpha w_2^{-\alpha} = (1-\alpha)y^k\left(\frac{w_2}{w_1}\right)^{-\alpha}$$

Letting $w = w_2/w_1$, let us eliminate this variable. The asterisks are redundant here and will be dropped to save notational clutter. It will be easiest if we take logarithms of both sides of the equation. Then

$$\log x_1 = \log \alpha + k \log y + (1-\alpha)\log w$$
$$\log x_2 = \log(1-\alpha) + k \log y - \alpha \log w$$

Multiply the first equation by α and the second by $1-\alpha$ and add:

$$\alpha \log x_1 + (1-\alpha)\log x_2 = \alpha \log \alpha + (1-\alpha)\log(1-\alpha) + k \log y$$

or

$$\log x_1^\alpha x_2^{1-\alpha} = \log \alpha^\alpha (1-\alpha)^{1-\alpha} y^k$$

Taking antilogarithms and rearranging slightly, we get

$$y = K x_1^{\alpha/k} x_2^{(1-\alpha)/k} \tag{9-20}$$

where $K = [1/\alpha^\alpha(1-\alpha)^{1-\alpha}]^{1/k}$. Equation (9-20) is the production function associated with the cost function (9-19). As expected, it is of the Cobb-Douglas, or multiplicatively separable, type, and is homogeneous of degree $1/k$, since C^* was homogeneous of degree k in y.

The Importance of Duality

The duality of cost and production functions is important for reasons other than mathematical elegance. Economists will have occasion to estimate factor demand and cost functions. There are basically two ways to approach this problem. One way is to estimate, by some procedure, the underlying production function for some activity and to then calculate, by inverting the implied first-order relations, the factor demand curves (holding output constant). The cost function can then be calculated also. This, however, is a very arduous procedure. Production functions are largely unobservable. The data points will represent a sampling of input and output levels that will have taken place at different times, as factor or output prices changed. And of what use is knowledge of the production function itself? Largely, it is to derive implications regarding factor usage and cost considerations when various parameters, e.g., factor and output prices, change.

It would seem to make more sense to start with estimating the cost functions or the factor demand curves directly; i.e., some functional form of the cost function could be asserted, say a logarithmic linear function, and costs could be estimated directly. However, this procedure would always be subject to the criticism that the estimated cost or demand functions were beasts without parents, i.e., they were derived from fictitious, or nonexistent, production processes. And that would be a serious criticism indeed.

However, the duality results of the previous sections rescue this simpler approach. We can be assured that if a cost function satisfies some elementary properties, i.e., linear homogeneity and concavity in the factor prices, then there in fact is some real, unique underlying production function. Thus, the cost function will be more plausible.

Moreover, the cost function may be easier to estimate, econometrically, than the production function. The cost function is a function of factor prices and output levels, all of which are potentially observable, possibly easily so. What is more, once estimated, the cost function can be used to derive directly the constant output factor demand curves using the relation $x_i^* = \partial C^*/\partial w_i$. Thus, the simpler approach of estimating cost functions is apt to be more useful than the more complicated procedure of estimating production functions. The duality results assure us that procedure is in fact theoretically sound.

9.4 ELASTICITY OF SUBSTITUTION; THE CONSTANT-ELASTICITY-OF-SUBSTITUTION (CES) PRODUCTION FUNCTION

Neoclassical production theory recognizes the possibility of substituting one factor of production for another. The existence of more than one point on an isoquant is equivalent to such an assertion. However, we have not yet considered any quantitative measurement of the degree to which one factor can in fact be so substituted for another.

Consider a production function with L-shaped isoquants, represented in Fig. 9-3. This function can be written algebraically as $y = \min[(x_1/a_1), (x_2/a_2)]$, where a_1 and a_2 are constants. This function describes an activity for which no effective substitution is possible. For any wage ratio, the cost-minimizing firm will always operate at the elbow of the isoquants. The marginal product of each factor

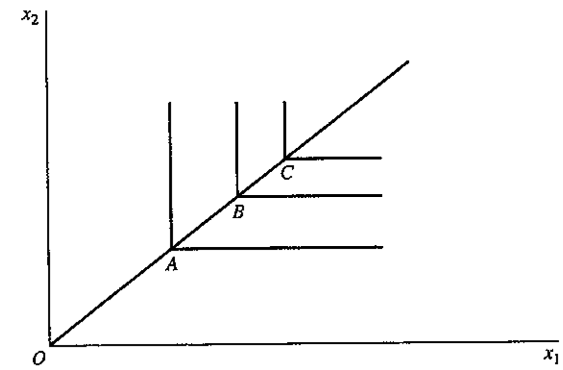

FIGURE 9-3
The Fixed-Coefficient Production Function. This production function is given by $y = \min(x_1/a_1, x_2/a_2)$, where a_1 and a_2 are parameters. No substitution among the factors is worthwhile; the marginal products of x_1 or x_2 are 0 at all points except along the corners of the production function. Although extensively used in input-output analysis and short-term forecasting models, it is doubtful that this is a useful way to look at the real world.

is 0 unless it is combined in a fixed proportion with the other input. (For this reason, this production function is described as one of *fixed coefficients*.)

How shall the degree of substitutability of one factor for another be described? Consider the Cobb-Douglas production function $y = x_1^\alpha x_2^{1-\alpha}$, where, say, x_1 is labor and x_2 is capital. A cost-minimizing firm satisfies the first-order conditions of the Lagrangian

$$\mathcal{L} = w_1x_1 + w_2x_2 + \lambda\left(y - \left(x_1^\alpha x_2^{1-\alpha}\right)\right)$$

or, in this case,

$$w_1 - \lambda\alpha x_1^{\alpha-1}x_2^{1-\alpha} = 0 \qquad (9\text{-}21a)$$

$$w_2 - \lambda(1-\alpha)x_1^\alpha x_2^{-\alpha} = 0 \qquad (9\text{-}21b)$$

$$y - x_1^\alpha x_2^{1-\alpha} = 0 \qquad (9\text{-}21c)$$

Upon division, Eqs. (9-21a) and (9-21b) yield

$$\frac{x_2}{x_1} = \frac{(1-\alpha)w_1}{\alpha w_2} \qquad (9\text{-}22)$$

This expression can also be derived from the constant-output factor demand curves derived earlier:

$$x_1^* = \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\left(\frac{w_2}{w_1}\right)^{1-\alpha} y \qquad (9\text{-}23a)$$

$$x_2^* = \left(\frac{\alpha}{1-\alpha}\right)^{-\alpha}\left(\frac{w_2}{w_1}\right)^{-\alpha} y \qquad (9\text{-}23b)$$

Equation (9-22) says that for this production function, the capital-labor ratio is (1) independent of the level of output, and (2) a function only of the *ratio* of the wage rates (rental rate on capital to the labor wage rate). We shall shortly consider the generality of this situation.

We can therefore conceive of the capital-labor ratio as a simple function of the wage ratio. If we let $u = x_2/x_1$, $w = w_2/w_1$ for notational ease, Eq. (9-22) becomes

$$u = \frac{k}{w} \tag{9-24}$$

where $k = (1-\alpha)/\alpha$. How does x_2/x_1 vary when w_2/w_1 varies? From Eq. (9-24),

$$\frac{du}{dw} = -\frac{k}{w^2} \tag{9-25}$$

where the expression is negative, as expected. Although this actual rate of change in the capital-labor ratio is a measure of substitutability, a more frequent measure is the dimensionless elasticity analog,

$$\sigma = -\frac{du/u}{dw/w} = -\frac{w}{u}\frac{du}{dw} \tag{9-26}$$

the (approximate) percentage change in the input ratio per percentage change in factor prices. A minus sign is added to make the measure positive. This measure σ is called the *elasticity of substitution*. Applying Eq. (9-26) to the Cobb-Douglas case gives

$$\sigma = \frac{w}{u}\frac{k}{w^2} = \frac{w^2}{k}\frac{k}{w^2} = 1$$

Thus, the elasticity of substitution for a Cobb-Douglas production function is constant along the whole range of any isoquant and equal to 1.

The Cobb-Douglas production function $y = x_1^{\alpha}x_2^{1-\alpha}$ is a special case of production functions that exhibit constant elasticity of substitution (CES) along any isoquant. We shall investigate these important functions, deriving their functional form and other properties. These functions have wide application in empirical work on production processes.

The concept of the elasticity of substitution is not dependent on the behavioral assertion of cost minimization. The concept can as easily be described as the percentage change in the input ratio per percentage change in the marginal rate of substitution (MRS) since the cost-minimizing firm always sets $w_1/w_2 = f_1/f_2 =$ MRS. Thus we can write, as an alternative definition,

$$\sigma = -\frac{f_1/f_2}{x_1/x_2}\frac{d(x_1/x_2)}{d(f_1/f_2)} \tag{9-27}$$

(Note that we are considering the inverse ratios x_1/x_2 instead of x_2/x_1, etc. As we shall shortly see, this is of no consequence.) Let us evaluate this expression. Along any isoquant, $x_2 = x_2(x_1)$. Then $dx_2/dx_1 = -f_1/f_2$, and therefore we can write (9-27) (using the chain rule) as

$$\sigma = -\frac{f_1x_2}{f_2x_1}\frac{d(x_1/x_2)/dx_1}{d(f_1/f_2)/dx_1}$$

Evaluating the terms in the second fraction yields

$$\begin{aligned}\frac{d(x_1/x_2)}{dx_1} &= \left(x_2 - x_1\frac{dx_2}{dx_1}\right)\frac{1}{x_2^2}\\ &= \left(x_2 + x_1\frac{f_1}{f_2}\right)\frac{1}{x_2^2}\\ &= \frac{1}{f_2x_2^2}(f_1x_1 + f_2x_2)\end{aligned}$$

Similarly, $d(f_1/f_2)/dx_1$ is simply $-d^2x_2/dx_1^2$, since $dx_2/dx_1 = -f_1/f_2$. From Chap. 3,

$$-\frac{d^2x_2}{dx_1^2} = \frac{1}{f_2^3}(f_2^2f_{11} - 2f_1f_2f_{12} + f_1^2f_{22})$$

Combining these expressions leads to

$$\sigma = -\frac{f_1x_2}{f_2x_1}\frac{f_2^3}{f_2x_2^2}\frac{f_1x_1 + f_2x_2}{(f_2^2f_{11} - 2f_1f_2f_{12} + f_1^2f_{22})}$$

or

$$\sigma = -\frac{f_1f_2(f_1x_1 + f_2x_2)}{x_1x_2(f_2^2f_{11} - 2f_1f_2f_{12} + f_1^2f_{22})} \tag{9-28}$$

This rather cumbersome expression for σ can be drastically simplified in the important special case of linear homogeneous production functions. First, the numerator immediately becomes f_1f_2y, upon application of Euler's theorem. For the denominator, since $f(x_1, x_2)$ is homogeneous of degree 1, f_1 and f_2 are homogeneous of degree 0. Hence, applying Euler's theorem to f_1 and f_2, we have

$$f_{11}x_1 + f_{12}x_2 \equiv 0 \qquad \text{or} \qquad f_{11} = -f_{12}\frac{x_2}{x_1}$$

Similarly,

$$f_{22} = -f_{12}\frac{x_1}{x_2}$$

Making these substitutions leads to

$$\begin{aligned}x_1x_2(f_2^2f_{11} - 2f_1f_2f_{12} + f_1^2f_{22}) &= -x_1x_2f_{12}\left(f_2^2\frac{x_2}{x_1} + 2f_1f_2 + f_1^2\frac{x_1}{x_2}\right)\\ &= -f_{12}(f_2^2x_2^2 + 2f_1f_2x_1x_2 + f_1^2x_1^2)\\ &= -f_{12}(f_1x_1 + f_2x_2)^2 = -f_{12}y^2\end{aligned}$$

Therefore, for linear homogeneous production functions

$$\sigma = -\frac{f_1 f_2 y}{-f_{12} y^2} = \frac{f_1 f_2}{y f_{12}} \tag{9-29}$$

a drastic simplification of Eq. (9-28) indeed.

A curiosity concerning Eqs. (9-28) and (9-29) is that they are symmetric between x_1 and x_2. That is, the identical expression results when the subscripts are interchanged. Thus we can speak of the elasticity of substitution between x_1 and x_2 rather than the elasticity of substitution of x_2 for x_1, or of x_1 for x_2. It does not matter whether x_2/x_1 is related to f_2/f_1 or x_1/x_2 is related to f_1/f_2 by derivatives in σ. The formula is the same either way.

Formula (9-29) can be related to the expression for the rate of change of one factor with respect to another. Recalling Eq. (8-28*b*) on the comparative statics of cost minimization, we have

$$\frac{\partial x_2^*}{\partial w_1} = -\frac{H_{12}}{H}$$

where

$$H = \begin{vmatrix} -\lambda^* f_{11} & -\lambda^* f_{12} & -f_1 \\ -\lambda^* f_{21} & -\lambda^* f_{22} & -f_2 \\ -f_1 & -f_2 & 0 \end{vmatrix}$$

Thus

$$\frac{\partial x_1^*}{\partial w_2} = \frac{-f_1 f_2}{\lambda^*\left(f_{11} f_2^2 - 2 f_{12} f_1 f_2 + f_{22} f_1^2\right)}$$

which, for linear homogeneous production functions, becomes (as before, dropping the asterisks to remove clutter)

$$\frac{\partial x_1}{\partial w_2} = \frac{f_1 f_2 x_1 x_2}{\lambda y^2 f_{12}}$$

or

$$\frac{y\lambda}{x_1 x_2}\frac{\partial x_1}{\partial w_2} = \frac{f_1 f_2}{y f_{12}} = \sigma$$

Noting that $\lambda = w_2/f_2$, we can write this as

$$\frac{y}{f_2 x_2}\frac{w_2}{x_1}\frac{\partial x_1}{\partial w_2} = \sigma$$

Letting $\kappa_i = f_i x_i / y$ ($\kappa_1 + \kappa_2 = 1$, by Euler's theorem) and denoting the cross-elasticity of demand by ϵ_{12},

$$\epsilon_{12} = \kappa_2 \sigma \tag{9-30}$$

Thus, the elasticity of substitution is related in this simple fashion to the cross-elasticity of (constant-output) factor demand. And, of course,

$$\sigma = \frac{1}{\kappa_2}\epsilon_{12} = \frac{1}{\kappa_1}\epsilon_{21}$$

Knowledge of σ at any point would undoubtedly be a useful technological datum for empirical work. Beyond the strictly qualitative results of comparative statics, measurement of the degree of responsiveness to changes in parameters is an essential part of any science. Hence, it would be useful to be able to estimate a quantity like σ. A useful first approximation in so doing is to assume that the production process is linear homogeneous and exhibits *constant* elasticity of substitution everywhere. That is, σ is the same at all factor combinations. What would such production functions look like? We have already shown that the Cobb-Douglas function has the property $\sigma = 1$ everywhere. What about other values of σ?

Return to Eq. (9-26), $\sigma = -(w/u)(du/dw)$, where $u = x_2/x_1$, $w = w_2/w_1$. Strictly speaking, we should in general write

$$\sigma = -\frac{w}{u}\frac{\partial u}{\partial w}$$

since in general $u = x_2/x_1$ will not be a function of the wage ratio w_2/w_1 only but will also depend on the output level y. However, consider the case first of homothetic production functions. The cost function for all homothetic production functions can be written

$$C^* = J(y)A(w_1, w_2) \tag{9-31}$$

where $A(w_1, w_2)$ is linear homogeneous. (Any cost function is linear homogeneous in the factor prices.) Using the envelope theorem (Shephard's lemma), we have

$$x_1 = J(y)A_1(w_1, w_2) \tag{9-32a}$$

$$x_2 = J(y)A_2(w_1, w_2) \tag{9-32b}$$

where $A_1 = \partial A/\partial w_1$, etc. Since A_1 and A_2 are first partials of a linear homogeneous function, they are homogeneous of degree 0 in w_1 and w_2. But then

$$A_1(w_1, w_2) = A_1\left(1, \frac{w_2}{w_1}\right) = B_1(w)$$

and so forth, and therefore we can write

$$x_1 = J(y)B_1(w) \tag{9-33a}$$

$$x_2 = J(y)B_2(w) \tag{9-33b}$$

(In fact, only the factor demands of homothetic production functions have this functional form.) Dividing Eq. (9-33*b*) by (9-33*a*) gives

$$u = \frac{x_2}{x_1} = \frac{B_2(w)}{B_1(w)} = B(w)$$

That is, for all homothetic production functions, the ratio of factor inputs is a function of the ratio of wage rates only, not at all a function of output y. This of course is geometrically obvious, since the isoquants of homothetic production functions are merely radial blowups of each other. Hence, in formula (9-26) it is valid, for homothetic production functions, to write $\sigma = -(w/u)(du/dw)$, since u is indeed some well-defined function of w only.

Suppose now, maintaining the assumption of homotheticity, that σ is constant everywhere. The class of homothetic functions having constant elasticity of substitution consists of those which satisfy the differential equation

$$-\frac{w}{u}\frac{du}{dw} = \sigma = \text{constant}$$

Let us solve this differential equation. Rearranging variables gives

$$\frac{du}{u} = -\sigma\frac{dw}{w}$$

Integrating both sides and denoting the arbitrary constant of integration as $\log c$, we have

$$\log u = -\sigma \log w + \log c = \log cw^{-\sigma}$$

or

$$u = cw^{-\sigma} = c\left(\frac{1}{w}\right)^{\sigma} \tag{9-34}$$

where, of necessity, $c > 0$. Thus, all such production functions must have the property that the capital-labor ratio is proportional to the wage ratio raised to some power, that power being the negative of the elasticity of substitution. What production functions satisfy (9-34)? For cost-minimizing firms, $1/w = w_1/w_2 = f_1/f_2 = -\partial x_2/\partial x_1$, the slope of an isoquant at some arbitrary output level y. Rewriting (9-34) in terms of the original variables yields

$$\frac{x_2}{x_1} = c\left(\frac{f_1}{f_2}\right)^{\sigma}$$

or, taking roots ($k = c^{1/\sigma}$),

$$\left(\frac{x_2}{x_1}\right)^{1/\sigma} = -k\frac{\partial x_2}{\partial x_1}$$

Now k is any positive number. We can, for convenience, write $k = (1-\alpha)/\alpha$, where $0 < \alpha < 1$. As α varies between 0 and 1, k varies from 0 to ∞, so no generality is lost. Separating variables gives

$$\alpha\frac{\partial x_1}{x_1^{1/\sigma}} = -\frac{(1-\alpha)\partial x_2}{x_2^{1/\sigma}} \tag{9-35}$$

We have to distinguish two cases now when integrating this expression. When $\sigma = 1$, logarithms will be involved, whereas when $\sigma \neq 1$, the integrals will be simple polynomials.

Case 1. Let $\sigma = 1$. Integrating both sides of (9-35) yields

$$\alpha\int\frac{\partial x_1}{x_1} = -(1-\alpha)\left(\int\frac{\partial x_2}{x_2}\right) + \log g(y)$$

The arbitrary constant of integration can in general be any function of y, since y was held constant in determining the slope $\partial x_2/\partial x_1$. Again, since Eq. (9-35) is really a *partial* differential equation, the arbitrary constant of integration can involve any function of the variable or variables held constant, in this case output y. For convenience, we have denoted this constant of integration $\log g(y)$. Performing the indicated operations, we have

$$\alpha \log x_1 = -(1-\alpha)\log x_2 + \log g(y)$$

or

$$g(y) = x_1^{\alpha}x_2^{1-\alpha}$$

Up to this point, the only assumption about the form of the production function we have made is that it is homothetic. Indeed, assuming $g(y)$ is monotonic, we can write

$$y = F\left(x_1^{\alpha}x_2^{1-\alpha}\right) \tag{9-36}$$

where F is the inverse function of g; that is, if $z = g(y)$, $y = g^{-1}(z) = F(z) = F(x_1^{\alpha}x_2^{1-\alpha})$. Equation (9-36) has the required form for homotheticity, being a function of a linear homogeneous function. If now we insist that $y = f(x_1, x_2) = F(x_1^{\alpha}x_2^{1-\alpha})$ be homogeneous of some degree s, then

$$f(x_1, x_2) = kx_1^{\alpha_1}x_2^{\alpha_2} \tag{9-37}$$

where $\alpha_1 = \alpha s$, $\alpha_2 = (1-\alpha)s$, and thus $\alpha_1 + \alpha_2 = s$. If $f(x_1, x_2)$ is to be linear homogeneous, with $\sigma = 1$, then

$$f(x_1, x_2) = kx_1^{\alpha}x_2^{1-\alpha} \tag{9-38}$$

Equations (9-36) to (9-38) represent the general functional forms of production functions that exhibit constant elasticity of substitution equal to unity everywhere ($\sigma = 1$) and, in addition, are, respectively, homothetic, homogeneous of degree s, and linear homogeneous. Consider now the second case, $\sigma \neq 1$.

Case 2. If $\sigma \neq 1$, integrating both sides of Eq. (9-35) yields

$$\alpha\int\frac{\partial x_1}{x_1^{1/\sigma}} = -(1-\alpha)\int\frac{\partial x_2}{x_2^{1/\sigma}} + g(y)$$

where again, the arbitrary constant of integration is some function of output y, designated $g(y)$, since y is held constant in finding the slope $\partial x_2/\partial x_1$ of an isoquant. Performing the indicated operations and rearranging yields, incorporating the factor $(-1/\sigma) + 1$ into $g(y)$,

$$g(y) = \alpha x_1^{(-1/\sigma)+1} + (1-\alpha)x_2^{(-1/\sigma)+1} \tag{9-39}$$

It will simplify matters if we let $\rho = 1 - (1/\sigma)$; that is, $\sigma = 1/(1-\rho)$; then

$$g(y) = \alpha x_1^\rho + (1-\alpha)x_2^\rho \tag{9-40}$$

Assuming again that $g(y)$ is monotonic, (9-40) can be written

$$y = F\left(\alpha x_1^\rho + (1-\alpha)x_2^\rho\right) \tag{9-41}$$

Equation (9-41) is the most general form of homothetic production functions exhibiting constant elasticity of substitution. If we wish $y = f(x_1, x_2)$ to be homogeneous of degree 1, then

$$y = k\left(\alpha x_1^\rho + (1-\alpha)x_2^\rho\right)^{1/\rho} \tag{9-42}$$

Equation (9-42) is what is commonly referred to as the CES production function. It assumes linear homogeneity. The elasticity of substitution, of course, varies between 0 and ∞. When $\sigma \to 0, \rho \to -\infty$; when $\sigma = 1, \rho = 0$, and when $\sigma \to +\infty, \rho \to +1$. Hence the range of values for ρ is $-\infty < \rho < 1$. When $\sigma \to 0 (\rho \to -\infty)$, the isoquants become L-shaped; i.e., the function becomes a fixed-proportions production function. When $\sigma \to \infty (\rho \to +1)$, the isoquants become straight lines, as inspection of (9-42) reveals.

Although we have proved that when $\sigma = 1$ $(\rho = 0)$, the CES production function becomes Cobb-Douglas, that fact is not obvious from Eq. (9-42). In order to show this result directly, we need a mathematical theorem known as L'Hôpital's rule.

L'Hôpital's rule. Suppose that $f(x)$ and $g(x)$ both tend to 0 (have a limit of 0) as $x \to 0$. Then if the ratio $f'(x)/g'(x)$ exists,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} \tag{9-43}$$

The limit of the ratio of the functions, if it exists, equals the ratio of the derivatives of $f(x)$ and $g(x)$, respectively.

The formal proof of this theorem can be found in any advanced calculus text; we shall not present it here.

Consider the CES function (9-42) again, and take the logarithms of both sides:

$$\log y = \log k + \frac{\log\left(\alpha x_1^\rho + (1-\alpha)x_2^\rho\right)}{\rho} \tag{9-44}$$

The right-hand side of (9-44) consists, aside from the constant, of a ratio of two functions, each of which tends to 0 as $\rho \to 0$. We find the limit as $\rho \to 0$, letting $f(\rho)$ = numerator, remembering that if $y = a^t$, $dy/dt = a^t \log a$:

$$f'(\rho) = \frac{1}{\alpha x_1^\rho + (1-\alpha)x_2^\rho}\left[x_1^\rho \alpha \log x_1 + x_2^\rho (1-\alpha)\log x_2\right]$$

$$\lim_{\rho\to 0} f'(\rho) = \frac{1}{1}[\alpha \log x_1 + (1-\alpha)\log x_2]$$
$$= \log x_1^\alpha x_2^{1-\alpha}$$

The denominator of (9-44) is simply ρ, and thus $g'(\rho) = 1$; hence, $\lim_{\rho\to 0} g'(\rho) = 1$. Therefore, as $\rho \to 0$,

$$\log y = \log k + \log x_1^\alpha x_2^{1-\alpha}$$

or

$$y = k x_1^\alpha x_2^{1-\alpha}$$

the Cobb-Douglas function, as expected.

The factor demands and the cost functions associated with the CES production function can be derived using the cost minimization hypothesis. Formally, the problem is

minimize

$$w_1 x_1 + w_2 x_2 = C$$

subject to

$$\alpha_1 x_1^\rho + \alpha_2 x_2^\rho = y^\rho$$

where $\alpha_1 + \alpha_2 = 1$.

The Lagrangian is $\mathcal{L} = w_1 x_1 + w_2 x_2 + \lambda(y^\rho - (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho))$; differentiating with respect to x_1, x_2, and eliminating λ yields (eliminating the *'s to save notational clutter)

$$\frac{w_1}{w_2} = \frac{\alpha_1 x_1^{\rho-1}}{\alpha_2 x_2^{\rho-1}}$$

Multiplying through by (x_1/x_2),

$$\frac{w_1 x_1}{w_2 x_2} = \frac{\alpha_1 x_1^\rho}{\alpha_2 x_2^\rho}$$

Now add 1 to both sides of this equation (which adds the denominator of each side to the respective numerator):

$$\frac{C}{w_2 x_2} = \frac{y^\rho}{\alpha_2 x_2^\rho}$$

Solving for x_2,

$$x_2 = C^{1/(1-\rho)} y^{-\rho/(1-\rho)} w_2^{-1/(1-\rho)} \alpha_2^{1/(1-\rho)}$$

and by symmetry,

$$x_1 = C^{1/(1-\rho)} y^{-\rho/(1-\rho)} w_1^{-1/(1-\rho)} \alpha_1^{1/(1-\rho)}$$

Therefore

$$w_2 x_2 = C^{1/(1-\rho)} y^{-\rho/(1-\rho)} w_2^{-\rho/(1-\rho)} \alpha_2^{1/(1-\rho)}$$

and

$$w_1 x_1 = C^{1/(1-\rho)} y^{-\rho/(1-\rho)} w_1^{-\rho/(1-\rho)} \alpha_1^{1/(1-\rho)}$$

Adding produces total cost; therefore

$$C = C^{1/(1-\rho)} y^{-\rho/(1-\rho)} \left(\alpha_1^{1/(1-\rho)} w_1^{-\rho/(1-\rho)} + \alpha_2^{1/(1-\rho)} w_2^{-\rho/(1-\rho)}\right)$$

and thus

$$C = y\left(\alpha_1^{1/(1-\rho)} w_1^{-\rho/(1-\rho)} + \alpha_2^{1/(1-\rho)} w_2^{-\rho/(1-\rho)}\right)^{(1-\rho)/-\rho} \quad (9\text{-}45)$$

We can derive the constant-output factor demands using the envelope theorem result $\partial C^*/\partial w_i = x_i^*$:

$$\frac{\partial C^*}{\partial w_1} = y\left(\frac{1-\rho}{-\rho}\right)\left(\alpha_1^{1/(1-\rho)} w_1^{-\rho/(1-\rho)} + \alpha_2^{1/(1-\rho)} w_2^{-\rho/(1-\rho)}\right)^{1/-\rho} \times \left(\alpha_1^{1/(1-\rho)} w_1^{-1/(1-\rho)}\right)\left(\frac{-\rho}{1-\rho}\right)$$

or

$$\frac{\partial C^*}{\partial w_1} = x_1^* = y\left(\alpha_1^{1/(1-\rho)} w_1^{-\rho/(1-\rho)} + \alpha_2^{1/(1-\rho)} w_2^{-\rho/(1-\rho)}\right)^{1/-\rho} \alpha_1^{1/(1-\rho)} w_1^{-1/(1-\rho)} \quad (9\text{-}46)$$

with a similar expression for x_2^*.

Generalizations to *n* Factors

Consider again the definition of elasticity of substitution given in Eq. (9-27) but now assume that the two factors in question are two of n factors that enter the production function:

$$\sigma_{ij} = -\frac{f_i/f_j}{x_i/x_j}\frac{d(x_i/x_j)}{d(f_i/f_j)} \quad (9\text{-}47)$$

This number is a measure of how fast the ratio of two inputs changes when the marginal rate of substitution between them changes. In order for this definition to make sense, the other factors must be held constant at some parametric levels $x_k = x_k^0, k \neq i, j$. When more than two factors are involved, a marginal rate of substitution of one variable for another can only be defined in some two-dimensional subspace of the original space, i.e., along a plane (hyperplane) parallel to the x_i, x_j axes, in which the other variables are held constant. Thus definitions of elasticity of substitution analogous to Eq. (9-27), for the n-factor case, are "partial" elasticities of substitution. By holding the other factors constant, they do not represent the full degree of substitution possibilities present in the production function. These partial measures would be especially deceptive if one or more of the factors held constant were either close substitutes or highly complementary to the variable factors.

As an alternative, one could develop elasticities of substitution based on Eq. (9-26):

$$\sigma_{ij}^* = -\frac{w_{ij}}{u_{ij}}\frac{\partial u_{ij}}{\partial w_{ij}} \quad (9\text{-}48)$$

where $w_{ij} = w_i/w_j$, $u_{ij} = x_i/x_j$. In this definition, all other *wages* are to be held constant with the other factors allowed to vary. This definition overcomes most of the objections stated above for the fixed-input definition (9-47). Clearly, σ_{ij}^* will relate to the cross-elasticities of factor demand. As such, they are less of a technological datum of the production function but most likely a more useful concept, since in reality it will be unlikely that the other factors will remain constant.

The obvious generalization of the CES functional form to many factors

$$y = A\left(\alpha_1 x_1^\rho + \cdots + \alpha_n x_n^\rho\right)^{1/\rho} \quad (9\text{-}49)$$

has been shown to yield constant elasticities of the type given in (9-48); that is, the other factor *prices* are held fixed.† They are also called the *Allen elasticities*.‡ However, all the partial elasticities are equal to each other and to $1/(1-\rho)$. Also, when $\rho = 0$ ($\sigma_{ij} = 1$), the form reduces, as in the two-factor case, to a Cobb-Douglas or multiplicatively separable function

$$y = A x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

where $\sum \alpha_i = 1$ to preserve linear homogeneity.

The Generalized Leontief Cost Function

A cost function developed by Erwin Diewert§ has been found to be useful in empirical analysis. This functional form is

$$C^* = y \sum\sum \beta_{ij} w_i^{1/2} w_j^{1/2} \qquad i, j = 1, \ldots, n \quad (9\text{-}50)$$

In order for this function to satisfy the requirements of a cost function, it must display symmetry, i.e., $\beta_{ij} = \beta_{ji}$. The constant-output factor demands can be obtained by differentiation with respect to the wages:

$$x_i^* = \frac{\partial C^*}{\partial w_i} = y \sum_k \beta_{ik}\left(\frac{w_k}{w_i}\right)^{1/2} \qquad i = 1, \ldots, n \quad (9\text{-}51)$$

Differentiating further,

$$\frac{\partial x_i^*}{\partial w_j} = \frac{1}{2} y \beta_{ij}\left(\frac{1}{w_i w_j}\right)^{1/2}$$

Note that $\beta_{ij} = \beta_{ji}$ is required in order that $\partial x_i^*/\partial w_j = \partial x_j^*/\partial w_i$.

†See H. Uzawa, "Production Functions with Constant Elasticities of Substitution," *The Review of Economic Studies*, **29**:291–299, October 1962.

‡See R. G. D. Allen, *Mathematical Analysis for Economists*, MacMillan & Co., Ltd., London, 1938; reprinted by St. Martin's Press, New York, 1967.

§W. Erwin Diewert, "An Application of the Shephard Duality Theorem: A Generalized Leontief Production Function," *Journal of Political Economy*, **79**:481–507, June 1971.

The reason this function is called a generalized Leontief function is that input-output analysis, as developed by Wassily Leontief, utilizes "fixed coefficient" technology, i.e., L-shaped isoquants, indicating an absence of substitution possibilities among factors of production. In the special case where $\beta_{ij} = 0, i \neq j, x_i = y\beta_{ii}$. In that case, therefore,

$$\frac{x_i}{x_j} = \frac{\beta_{ii}}{\beta_{jj}}$$

That is, the ratio of inputs is independent of output level and factor prices. This describes the Leontief-style technology. We shall further explore models of this nature in Chap. 17 on linear programming (General Equilibrium I).

Additional functional forms will be analyzed in the context of utility theory, though some functions have been useful in both production and consumer theory.

PROBLEMS

1. If a production function is homogeneous of degree $r > 1$ $(r < 1)$, it exhibits increasing (decreasing) returns to scale. The converse, however, is false. Explain.
2. Suppose all firms in a competitive industry have the same production function, $y = f(x_1, x_2)$, where $f(x_1, x_2)$ is homogeneous of degree $r < 1$. Show that all firms in this industry will be receiving "rents," i.e., positive accounting profits. To which factor of production do these rents accrue? In the long run, if entry is free in this industry, what will be the industry price, output, and number of firms?
3. Find the production function associated with each of the following cost functions:
   (a) $C = \sqrt{w_1 w_2} e^{y/2}$
   (b) $C = w_2[1 + y + \log(w_1/w_2)]$
   (c) $C = y(w_1^2 + w_2^2)^{1/2}$
4. It is often said that the reason for U-shaped average cost curves is indivisibility of some factors. However, indivisibility does not necessarily lead to such properties. Suppose a firm's production function is homogeneous of some degree. Suppose the production function is also homogeneous in any $n - 1$ factors when the nth factor is held fixed at some level. Show that the only function with these properties is the multiplicatively separable form $y = kx_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.
5. What class of *homothetic* functions $y = f(x_1, \ldots, x_n)$ is also homothetic in any $n - 1$ factors, with the nth factor held fixed at some level?
6. Show that for homothetic production functions, the output at which average cost is a minimum is independent of factor prices.
7. Suppose a production function $y = f(x_1, x_2)$ is homothetic, that is, $f(x_1, x_2) = F(h(x_1, x_2))$, where $h(x_1, x_2)$ is linear homogeneous. Show that the elasticity of substitution is given by $\sigma = (h_1 h_2)/h_{12}h$.

BIBLIOGRAPHY

Allen, R. G. D.: *Mathematical Analysis for Economists,* Macmillan & Co., Ltd., London, 1938; reprinted by St. Martin's Press, New York, 1967.

Arrow, K. J., H. Chenery, B. Minhas, and R. M. Solow: "Capital-Labor Substitution and Economic Efficiency," *The Review of Economics and Statistics,* **43**:225–250, 1961. The seminal paper on CES production functions.

Blackorby, Charles, and R. Robert Russell: "Will the Real Elasticity of Substitution Please Stand Up? A Comparison of the Allen/Uzawa and Morishima Elasticities," *American Economic Review,* **79**:882–888, September 1989.

Carlson, Sune: *A Study on the Theory of Production,* Kelley & Millman, New York, 1956.

Diewert, W. E.: "Applications of Duality Theory," Department of Manpower and Immigration, Canada, 1973.

Frisch, Ragnar: *Theory of Production,* Rand McNally & Company, Chicago, 1965.

Hicks, J. R.: *Value and Capital,* 2d ed., Clarendon Press, Oxford, 1946.

Jorgenson, D. W., L. R. Christensen, and L. J. Lau: "Transcendental Logarithmic Production Frontiers," *Review of Economics and Statistics,* **55**:28–45, February 1973.

McFadden, Daniel: "Constant Elasticity of Substitution Production Functions," *Review of Economic Studies,* **30**:73–83, June 1963.

Samuelson, P. A.: *Foundations of Economic Analysis,* Harvard University Press, Cambridge, MA, 1947.

Shephard, Ronald W.: *Cost and Production Functions,* Princeton University Press, Princeton, NJ, 1953; also the revised version of this book, 1970, which has become a classic.

Uzawa, H.: "Production Functions with Constant Elasticities of Substitution," *The Review of Economic Studies,* **29**:291–299, October 1962.