

The maximization of *society's* total value of output depends upon further assumptions about property rights and one individual's effects on others.¹ We shall return to this issue in Chap. 19 on welfare economics.

PROBLEMS

1. What is the difference, in a many-commodity model, between diminishing marginal rate of substitution between any pair of commodities, and quasi-concavity of the utility function? Which is the more restrictive concept?
2. Why does the proposition "More is preferred to less" imply downward-sloping indifference curves?
3. What dependence, if any, does the homogeneity of degree 0 of the money-income-held-constant demand curves have on the homogeneity of the consumer's utility function?
4. Show that the marginal utility of money income, λ^M , is homogeneous of degree -1.
5. Consider the utility functions of the form $U = x_1^{\alpha_1} x_2^{\alpha_2}$. Show that the implied demand curves are

$$x_1^M = \frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{M}{p_1}$$

$$x_2^M = \frac{\alpha_2}{\alpha_1 + \alpha_2} \frac{M}{p_2}$$

- Find λ^M and $U^*(x_1^M, x_2^M)$, and verify that $\lambda^M = \partial U^* / \partial M$.
6. Prove the elasticity formulas (10-53), (10-54), (10-59), (10-60), and (10-61) for the n -commodity case.
 7. Is it possible to define complements in consumer theory by saying that the marginal utility of x_i increases when more x_j is consumed? (*Hint*: What mathematical term is being defined, and is it invariant to a monotonic transformation?)
 8. Substitutes can be defined by the sign of the *gross* (including income effects) cross-effects of prices on quantities, or the *net* effect (i.e., *not* including income effects). That is, one may define " x_i is a substitute for x_j " if:
 - (i) $\frac{\partial x_i^M}{\partial p_j} > 0$
 - or
 - (ii) $\frac{\partial x_i^U}{\partial p_j} > 0$
 (with the reverse sign on the inequality for "complements").
 - (a) Which term is likely to be the more observable (empirically)?
 - (b) Are these terms invariant to a monotonic transformation of the utility function?
 - (c) According to the preceding definitions, if x_i is a substitute for x_j , is x_j necessarily a substitute for x_i ?
 9. Considering Hicks' "third law" and the preceding definition (ii) of substitutes and complements, show that there is a tendency toward substitution of commodities in the sense

¹Smith's famous passages, quoted above, in fact appear in a section of the book entitled *Economic Liberalism. The Ideal* (emphasis added); Smith went on to consider *The Reality*, dwelling on such problems as monopoly, tariffs, and the like.

that

$$\sum_{i \neq j} p_i \delta_{ij} = \sum_{i \neq j} p_j \delta_{ij} > 0$$

10. Describe the effects of a monotonic transformation of the utility function on:
 - (a) The rate of change of the marginal utility of one good with respect to a change in another good.
 - (b) The law of diminishing marginal utility.
 - (c) The slopes of demand curves.
 - (d) The values of income elasticities.
 - (e) The homogeneity of the demand functions.
 - (f) The size and sign of the marginal utility of income.
11. For the utility maximization model, show that

$$\frac{\partial x_i^M}{\partial M} = \frac{-1}{\lambda^M} \left(\frac{\partial \lambda^M}{\partial p_i} + x_i^M \frac{\partial \lambda^M}{\partial M} \right)$$

where λ^M is the marginal utility of money income.

12. Suppose a consumer will have income x_1^0 this year and x_2^0 next year. He or she consumes x_1 this year and x_2 next year, being able to borrow and lend at interest rate r . Assume the consumer maximizes the utility of consumption over these two years.
 - (a) Derive the comparative statics for this problem. Will an increase in this year's income necessarily lead to an increase in consumption this year?
 - (b) Prove that the consumer will be better off (worse off) if the interest rate rises if he or she was a net saver (dissaver) this year.
13. Consider the utility maximization problem, max $U(x_1, x_2)$ subject to $p_1 x_1 + p_2 x_2 = 1$, where prices have been "normalized" by setting $M = 1$. Let $U^*(p_1, p_2)$ be the indirect utility function, and λ be the Lagrange multiplier.
 - (a) Show that $\lambda^M = (\partial U / \partial x_1) x_1^* + (\partial U / \partial x_2) x_2^*$
 - (b) Show that $\partial U^* / \partial p_1 = -\lambda^M x_1^*$, $\partial U^* / \partial p_2 = -\lambda^M x_2^*$.
 - (c) Show that $\lambda^M = -[(\partial U^* / \partial p_1) p_1 + (\partial U^* / \partial p_2) p_2]$.
 - (d) Prove that if $U(x_1, x_2)$ is homogeneous of degree r in (x_1, x_2) , then $U^*(p_1, p_2)$ is homogeneous of degree $-r$ in (p_1, p_2) .
14. Consider the class of utility functions that are "additively separable," i.e.,

$$U(x_1, x_2) \equiv U^1(x_1) + U^2(x_2).$$
 - (a) Find the first- and second-order conditions for utility maximization for these utility functions. Show that diminishing marginal utility in at least one good is implied.
 - (b) Show that if there is diminishing marginal utility in *each* good, then both goods are "normal," i.e., not inferior.
 - (c) Show that this specification does not imply $\partial x_i^M / \partial p_j = 0, i \neq j$.
 - (d) Show, however, that if $\partial x_i^M / \partial p_j \equiv \partial x_j^M / \partial p_i \equiv 0$, then $U(x_1, x_2) \equiv \alpha_1 \log x_1 + \alpha_2 \log x_2$.
 - (e) Assume now that x_1 is a Giffen good, i.e., $\partial x_1^M / \partial p_1 > 0$. Prove that $\partial \lambda^M / \partial M > 0$.
15. Consider the two-good utility maximization model and assume x_1 is a Giffen good, i.e., $\partial x_1^M / \partial p_1 > 0$. Prove that $\partial x_2^M / \partial p_1$ and $\partial x_2^U / \partial p_1$ must be of opposite signs.
16. Derive an expression analogous to Eq. (10-42) for the difference between $\partial x_i^U / \partial p_j$ and $\partial x_i^M / \partial p_j, i \neq j$. Show that if x_i and x_j are either both net substitutes or both net complements of x_n , the Hicksian cross-elasticities of demand are numerically smaller in the long run than in the short run.

17. Let $U = f(x_1, x_2) + g(x_3, x_4) = (x_1^2/2 + x_1x_2) + (x_3^2/2 + x_3x_4)$. Show that if $p_1 > p_2$, this utility function achieves an interior constrained maximum subject to a linear budget constraint and that $\partial x_i^M / \partial p_j \neq 0$, $i = 1, 2, j = 3, 4$. Show also that if f^* is the utility-maximizing value of f ; $\partial f^* / \partial p_i \neq 0$, $j = 3, 4$. That is, strongly separable utility functions do not imply the possibility of "two-stage" budgeting.

18. The Hicksian "real income," or utility-held-constant demand curves are written

$$x_1 = x_1^U(p_1, p_2, U^0)$$

Suppose now, when p_1 changes, U^0 is also adjusted to that maximum amount achievable so as to keep money income M constant, i.e.,

$$U^0 = U^*(p_1, p_2, M)$$

is that functional relationship which keeps M constant by adjusting utility, when p_1 or p_2 changes. Thus, the money-income-held-constant demand curves can be written

$$x_1^M(p_1, p_2, M) \equiv x_1^U(p_1, p_2, U^*(p_1, p_2, M))$$

(a) Show that the income effect on x_1 is proportional to the "utility effect" on x_1 , i.e., the change in x_1^U when U is changed, the factor of proportionality being the marginal utility of money income.

(b) Show that

$$\frac{\partial x_1^M}{\partial p_2} \equiv \frac{\partial x_1^U}{\partial p_2} - x_2 \frac{\partial x_1^M}{\partial M}$$

(This is an alternative derivation of the Slutsky equation to that given in the text.)
19. In a leading economics text, the following form of the "law of diminishing marginal rate of substitution" is given: The more of one good a consumer has, holding the quantities of all other goods constant, the smaller the marginal evaluation of that good becomes in terms of all other goods, i.e., the indifference curves become less steep. (Sketch this condition graphically.)

(a) This is a postulate about the slopes of indifference curves, i.e., about the term $(-U_1/U_2)$. What is the sign, according to this postulate, of $\partial(-U_1/U_2)/\partial x_1$, $\partial(-U_1/U_2)/\partial x_2$?

(b) Show that this postulate implies that the indifference curves are convex to the origin.

(c) Suppose this postulate is violated for good 2. Show that x_1 is an inferior good. Show that if the postulate is violated for good 1 also, then the indifference curves are concave to the origin.

(d) Show that the preceding postulate rules out inferior goods (for the two-good case).

(e) Show that in part (c), in which the indifference curves are still assumed to be convex to the origin, the marginal evaluation of x_2 increases the more it is consumed relative to x_1 . Explain intuitively.

(f) Show that in a three-good world, the preceding postulate is insufficiently strong to imply indifference curves which are convex to the origin.

20. An historically important class of utility functions includes those functions which exhibit vertically parallel indifference curves; i.e., with x_1 on the horizontal axis and x_2 on the vertical axis, the slopes of all indifference curves are the same at any given level of x_1 . For these utility functions:

(a) Prove graphically and algebraically that the income effect on x_1 equals 0.

(b) Show that the "ordinary" demand curve for x_1 , $x_1^M(p_1, p_2, M)$ and the compensated

demand curve for x_1 , $x_1^U(p_1, p_2, U)$ are identical by showing that at any point, the slopes of x_1^M and x_1^U are the same, and that the shifts in x_1^M and x_1^U are the same with respect to a change in p_2 , the price of the second good.

(c) Consider the utility function $U = x_2 + \log x_1$. Show that this function has vertically parallel indifference curves.

(d) For $U = x_2 + \log x_1$, show also that the price consumption paths with respect to changes in p_1 are horizontal, i.e., that the amount of x_2 consumed is independent of the price of good 1.

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