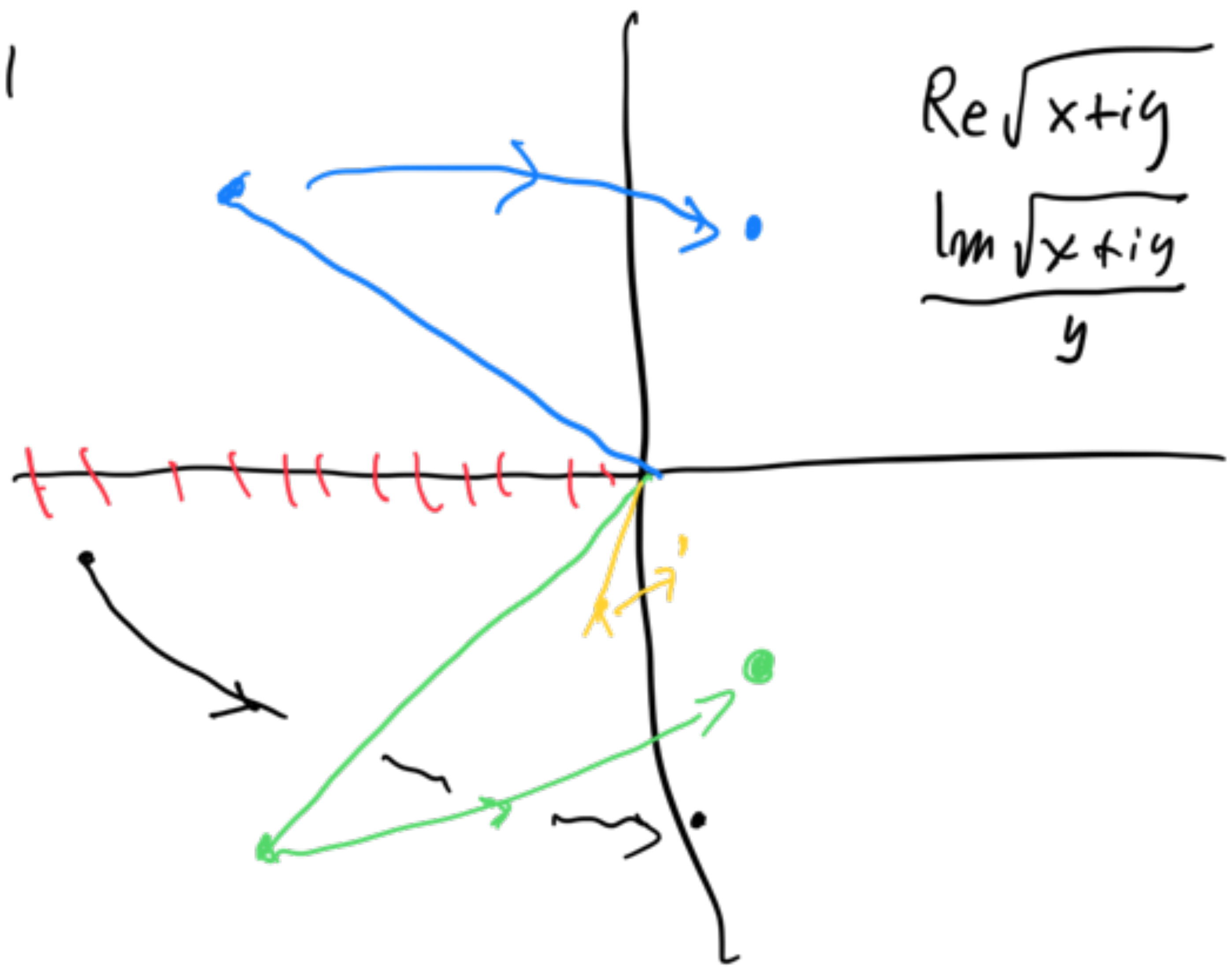
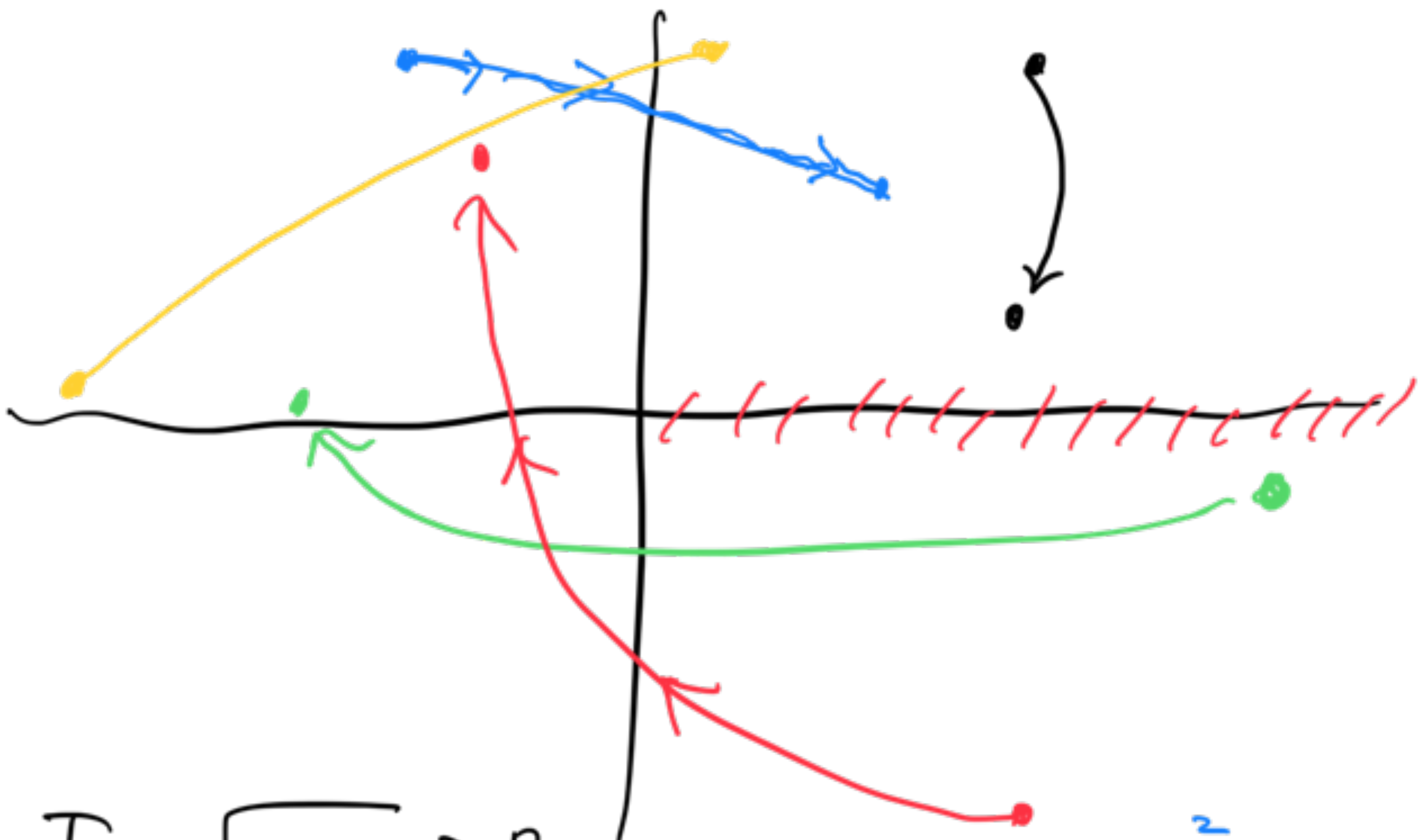
6.31



$$\frac{\operatorname{Re}\sqrt{x+iy}}{} > 0$$

$$\frac{\operatorname{Im}\sqrt{x+iy}}{y} > 0$$



$$\operatorname{Im}\sqrt{x+iy} > 0$$

$$x+iy = \left(\alpha\frac{y}{|y|} + i\beta\right)^2 = \alpha^2 - \beta^2$$

$$\frac{\operatorname{Re}\sqrt{x+iy}}{y} > 0$$

$$\sqrt{x+iy} = \frac{y}{|y|}\alpha + i\beta$$

$$\sqrt{x+iy} = \alpha + i\beta\frac{y}{|y|}$$

$$(x+iy) = \alpha^2 - \beta^2 + 2i\alpha\beta\frac{y}{|y|}$$

$$\left.\begin{aligned} x &= \alpha^2 - \beta^2 \\ |y| &= 2\alpha\beta \end{aligned}\right\} \quad \alpha^2 = x + \beta^2 = x + \frac{y^2}{4\alpha^2}$$

$$\alpha^4 - x\alpha^2 - \frac{y^2}{4} = 0 \qquad \alpha^2 = \frac{x \pm \sqrt{x^2+y^2}}{2}$$

$$0 < \alpha^2 \Rightarrow \alpha = \sqrt{\frac{x+\sqrt{x^2+y^2}}{2}}$$

$$\beta^2 = \alpha^2 - x = \frac{x+\sqrt{x^2+y^2}}{2} - x = \frac{\sqrt{x^2+y^2}-x}{2}$$

$$\beta = \sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}$$

$$i^i = ? \qquad i = e^{\frac{i\pi}{2}} e^{2\pi n i}$$

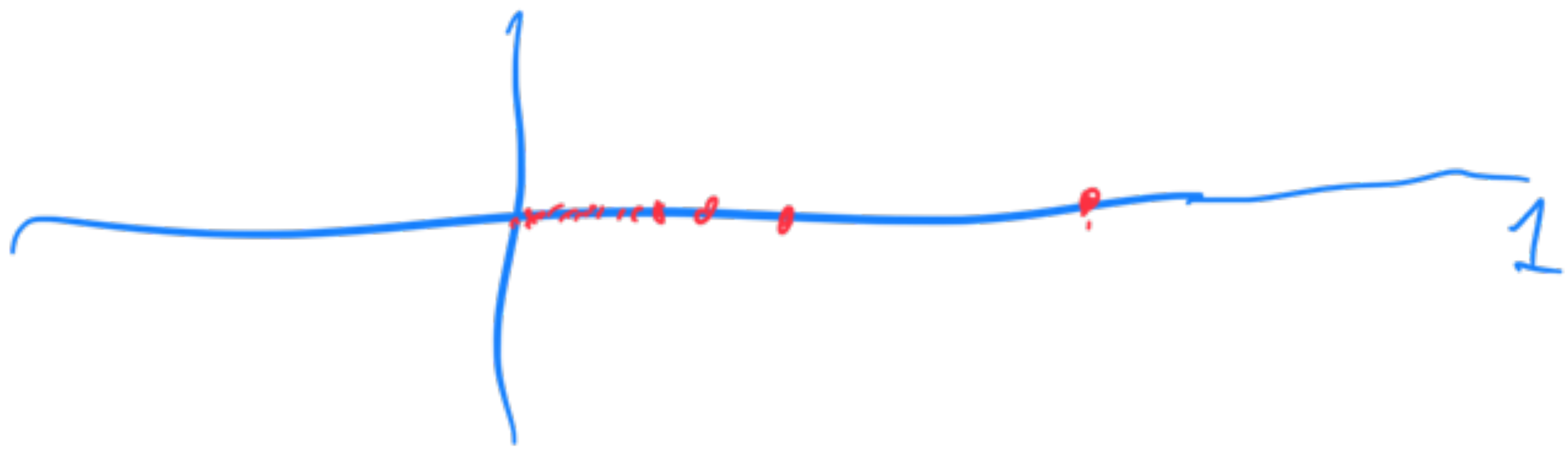
n an integer

$$i = e^{i(\frac{\pi}{2}+2\pi n)}$$

$$i^i \neq e^{i\cdot i(\frac{\pi}{2}+2\pi n)} = e^{-\frac{\pi}{2}-2\pi n}$$

$$i^i = e^{i(\frac{\pi}{2}+2\pi n)e^{i(\frac{\pi}{2}+2\pi n)}}$$

$$= e^{i^2(\frac{\pi}{2}+2\pi n)} = e^{-\frac{\pi}{2}-2\pi n}$$



7.8

$$\underbrace{\qquad}_{V}$$

$$V = IR + \frac{Q}{C}$$

$$\dot{V} = \dot{I}R + \frac{I}{C}$$

$$\dot{I} + \frac{I}{RC} = \frac{\dot{V}}{R}$$

$$r = \frac{1}{RC} \qquad s = \frac{\dot{V}(t)}{R}$$

$$\alpha(t) = \exp\left(\int^t \frac{dt'}{RC}\right) = e^{\frac{t}{RC}}$$

$$y(t) = e^{-\frac{t}{RC}}\left[I(0) + \int_0^t e^{t'/RC}\frac{\dot{V}(t')}{R}dt'\right]$$

$$= e^{-\frac{t}{RC}}\frac{1}{R}\int_0^t e^{+t'/RC}\dot{V}(t')\,dt'$$

$$\dot{V}(t') = V(\omega)\cos(\omega t')$$

$$V(t') = V\omega \cos(\omega$$

$$= e^{-\frac{t}{RC}} \frac{V\omega}{R} \int_0^t e^{+t'/RC} \cos(\omega t')\,dt'$$

$$= \frac{\omega V}{2R} e^{-\frac{t}{RC}} \int_0^t \left(e^{t'\left(\frac{1}{RC}+i\omega\right)} + e^{t'\left(\frac{1}{RC}-i\omega\right)} \right) dt'$$

$$= \frac{\omega V}{2R} e^{-\frac{t}{RC}} \left[\frac{e^{t\left(\frac{1}{RC}+i\omega\right)}}{\frac{1}{RC}+i\omega} + \frac{e^{t\left(\frac{1}{RC}-i\omega\right)}}{\frac{1}{RC}-i\omega} \right]$$

$$= \frac{\omega VC}{2(1+\omega^2R^2C^2)} \left[-2e^{-\frac{t}{RC}} + 2\cos\omega t + 2\omega RC\sin\omega t \right]$$

$$= \frac{\omega VC}{1+\omega^2R^2C^2} \left(\cos\omega t + \omega R \sin\omega t - e^{-t/RC} \right)$$

$\dot{y} = -e^{-y}$ $\qquad$ $y(0)=1$

$y(t) = \log(e - t)$

$\dot{y} = e^{-y}$ $\qquad$ $y = \log(t+e)$

Ex. 7.45 Hermite polynomials

$$y'' - x^2 y + \lambda y = 0 \qquad -\infty < x < \infty$$

↑ diverges as $x \to \infty$ is essential singularity

For huge x $\qquad y'' \approx x^2 y$

So $\quad y(x) \sim e^{-x^2/2} \qquad y' = -x e^{-x^2/2}$

$y'' = x^2 e^{-x^2/2} - e^{-x^2/2}$ so $y'' \approx x^2 y$

for huge x. So we set

$$y(x) = e^{-x^2/2} h(x) \text{ and get}$$

$$y' = -xy + h' e^{-x^2/2} = e^{-x^2/2}(h' - xh)$$

$$y'' = e^{-x^2/2}\left(-xh' + x^2 h + h'' - h - xh'\right)$$

$$= e^{-x^2/2}\left(h'' - 2xh' + (x^2-1)h\right)$$

So $y'' + (\lambda - x^2)y = 0 \Rightarrow$

$$e^{-\frac{x^2}{2}}\left[h'' - 2xh' + (x^2-1)h + (\lambda - x^2)h\right] = 0$$

$$h'' - 2xh' + (\lambda - 1)h = 0$$ So

we let $h(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$.

So $h' = \sum a_n (n+r) x^{n+r-1}$

$$h'' = \sum (n+r)(n+r-1) a_n x^{n+r-2}$$

$$\sum (n+r)(n+r-1) a_n x^{n+r-2} - \sum 2(n+r) x^{n+r} a_n + \sum (\lambda - 1) a_n x^{n+r} = 0$$

$r(r-1) = 0$ so $r = 0$ & $r = 1$

For $r = 0$

$$0 = \sum n(n-1) a_n x^{n-2} - \sum 2n x^n a_n + \sum (\lambda - 1) a_n x^n$$

$$0 = \sum \left\{ (n+2)(n+1) a_{n+2} + [-2n + \lambda - 1] a_n \right\} x^n = 0$$

$$(n+2)(n+1) a_{n+2} = (2n + 1 - \lambda) a_n$$

$$a_{n+2} = \frac{2n+1-\lambda}{(n+2)(n+1)} a_n$$

Unless $\lambda = 2n+1$, the series is like $a_{n+2} \sim \frac{2}{n+2} a_n$

or $a_n \sim \frac{2}{n} a_{n-2}$

or $a_{2n} \sim \frac{2^n a_0}{(2n)!!}$

So $h(x) \sim \sum a_{2n} x^{2n} \sim \sum \frac{2^n x^{2n}}{(2n)!!} a_0$

$$\sim \sum \frac{(x^2)^n}{n!} a_0 \sim e^{x^2}.$$

But then $g(x) \sim e^{-x^2/2} e^{x^2}$

$$\sim e^{x^2/2}$$

is <u>not</u> square integrable.

So $\lambda = 2n+1$ and then $h(x)$ is a polynomial, a Hermite polynomial.

For $r=1$

$$a_{n+2} = \frac{2n+1-\lambda}{(n+2)(n+1)} a_n$$

and again

$$\lambda = 2n+1$$

is an eigenvalue.

Here $y'' + (\lambda - x^2)y = 0$ or

$$y'' + Q(x)y = 0$$

$P(x) = 0$ $\qquad Q(x) = \lambda - x^2$

$z = \frac{1}{x}$ $\qquad \frac{Q(1/z)}{z^4} = \frac{\lambda - 1/z^2}{z^4} = \frac{\lambda}{z^4} - \frac{1}{z^6}$

diverges as $z \to 0$.

And $\dfrac{Q(1/z)}{z^2} = \dfrac{\lambda}{z^2} - \dfrac{1}{z^4}$ also diverges

So $x = \pm\infty$ are essential singularities of the Hermite equation.

H atom

$$(r^2 R')' + (\alpha r^2 + \beta r + \gamma) R = 0$$

$$r^2 R'' + 2rR' + (\alpha r^2 + \beta r + \gamma) R = 0$$

$$R'' + \frac{2}{r} R' + \left(\alpha + \frac{\beta}{r} + \frac{\gamma}{r^2}\right) R = 0$$

$$P(r) = \frac{2}{r} \qquad Q(r) = \alpha + \frac{\beta}{r} + \frac{\gamma}{r^2}$$

$r = 0$ is a singular point

But $r P(r) = 2$ and $r^2 Q = \alpha r^2 + \beta r + \gamma$ are both finite at $r = 0$.

So $r = 0$ is a regular singular point of the non-rel. H-atom eq.

still singular

But $r=\infty$ is an essentially singular point.

We first set $R = r^l S$

$$R' = l r^{l-1} S + r^l S'$$

$$r^2 R' = l r^{l+1} S + r^{l+2} S'$$

$$(r^2R')' = l(l+1) r^l S + (l+2) r^{l+1} S' + r^{l+2} S''$$

Need to cancel $l(l+1) r^l$ with $(\alpha r^2 + \beta r + \gamma) r^l S$. So $\gamma = -l(l+1)$.

Next $S = e^{-\delta r} s$ and find

$r^2 R'' \sim r^2 \delta^2 + \alpha r^2 = 0$ and $\delta = \sqrt{-\alpha}$

So we need $\alpha \leq 0$

$$\frac{d}{dx} \to \frac{d}{d(-x)} = -\frac{d}{dx}$$

$$-\hbar^2 \quad d^2$$

$$2m \quad dx^2 \qquad\qquad h_2 - - 1$$

$$-\frac{\hbar^2}{2m} \vec{\nabla}^2$$

$$V(x) = V(-x)$$

$$V(\vec{r}) = V(-\vec{r}).$$

$$L(x)y(x) = 0$$

$$L(-x) \qquad y(-x)$$

$$L(-x)\,y(-x) = 0$$

$$L(-x) = \pm L(x)$$

$$\pm L(x)\,y(-x) = 0$$

$$y(x), \quad y(-x)$$

$y(x) + y(-x)$ even

$y(x) - y(-x)$ odd

Wronski

$$Y_{ij} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

$$W = |Y| = y_1 y_2' - y_2 y_1' \neq 0$$

$$W' = y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1''$$

$$= y_1 y_2'' - y_2 y_1''$$

$$y_1'' + P y_1' + Q y_1 = 0$$

$$y_2'' + P y_2' + Q y_2 = 0$$

$$W' = y_1 (-P y_2' - Q y_2) - y_2 (-P y_1' - Q y_1)$$

$$W' = y_1 \ldots$$

$$= -P y_1 y_2' + P y_2 y_1'$$

$$= -P W$$

$$\frac{W'}{W} = -P = (\log W)'$$

$$\log W(x) = -\int^{x} P(x')\,dx'$$

$$W(x) = W(x_0)\, e^{-\int_{x_0}^{x} P(x')\,dx'}$$

$$= y_1 y_2' - y_2 y_1'$$

$$= y_1^2 \frac{d}{dx} \frac{y_2}{y_1} = y_2' y_1 - y_1^2 y_2 \frac{y_1'}{y_1^2}$$

$$= y_1 y_2' - y_2 y_1'$$

$$W = y_1^2 \left(\frac{y_2}{y_1}\right)' = e^{-\int^{x} P(x')\,dx'}$$

$$\left(\frac{y_2}{y_1}\right)' = \frac{1}{y_1^2} e^{-\int P}$$

x'

$$\frac{y_2}{y_1} = \int^{x} \frac{e^{-\int^{x'} p\, dx'}}{y_1^2(x')}\, dx'$$

$$Y = \begin{pmatrix} y_1^{(\ell_1)}(x_1) & y_2^{(\ell_1)}(x_1) \\ y_2^{(\ell_2)}(x_1) & y_2^{(\ell_2)}(x_2) \end{pmatrix}$$

$$YC = B$$

$$Y \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$c_1 y_1(x_1) + c_2 y_2(x_1) = b_1$$

$$c_1 y_1(x_2) + c_2 y_2(x_2) = b_2$$

$$C = Y^{-1} B$$

$$Y = \begin{pmatrix} y_1(x_1) & y_2(x_1) \\ y_1(x_2) & y_2(x_2) \end{pmatrix}$$

$$Y_{jk} = y_k(x_j)$$

$$\begin{cases} \alpha \sinh a + \beta \cosh a = c \\ \alpha \sinh(-a) + \beta \cosh(-a) = d \end{cases}$$

$x = \pm a$

$$-\alpha \sinh a + \beta \cosh a = d$$

$$\beta \cosh a = d + \alpha \sinh a$$

$$2\alpha \sinh a + d = c$$

$$\alpha = \frac{c-d}{2\sinh a}$$

$(c-d)/2$

$$\beta = \frac{d + \cdots}{\cos ka} = \frac{c+d}{2\cos ka}$$

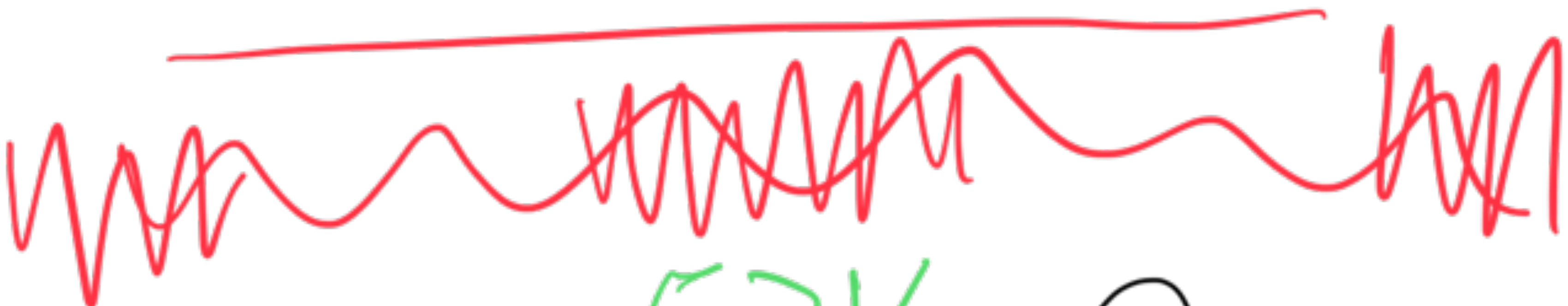
$$-y'' = k^2 y$$

$\sin ka \qquad \cos ka$

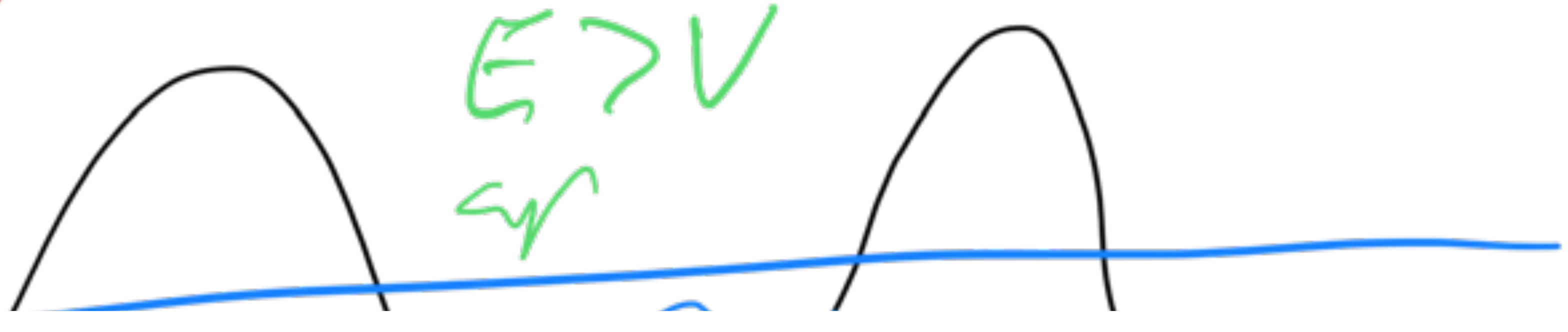
$$y(x) = \frac{c-d}{2\sin ka}\sin kx + \frac{c+d}{2\cos ka}\cos kx$$

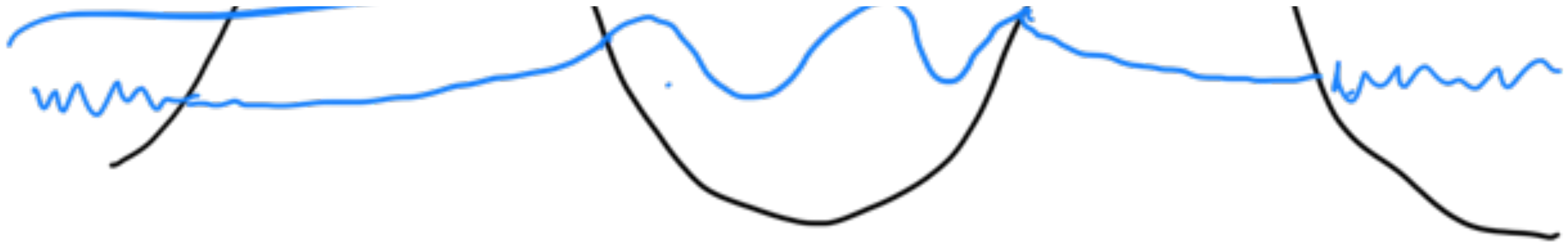
$$W_0'' = \frac{1}{2}\frac{\frac{2m}{\hbar^2}V'}{\sqrt{-2m(E-V)/\hbar^2}}$$

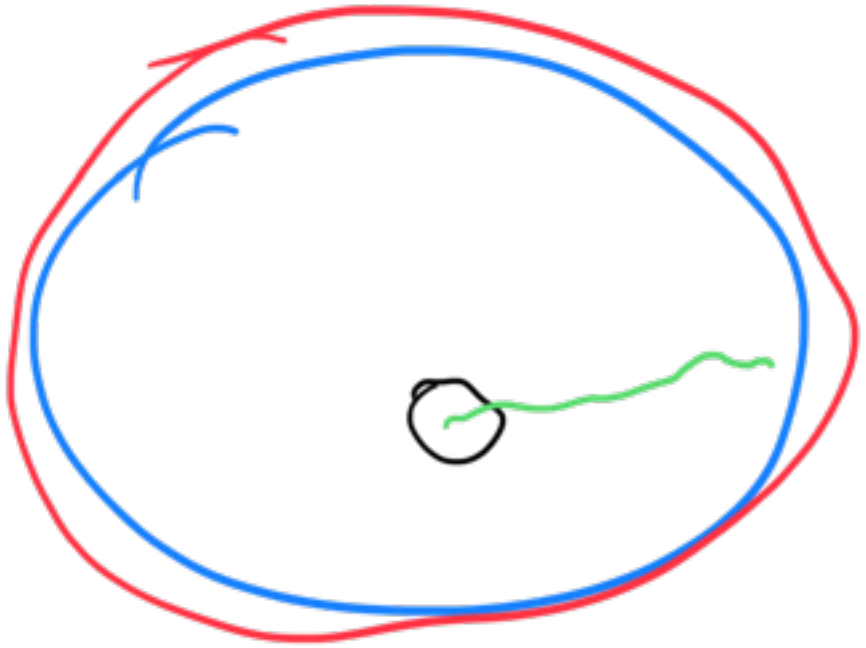
$$\frac{mV'}{\sqrt{-2m(E-V)/\hbar^2}} + W_1'^2 = -2m\frac{(E-V)}{\hbar^2}$$



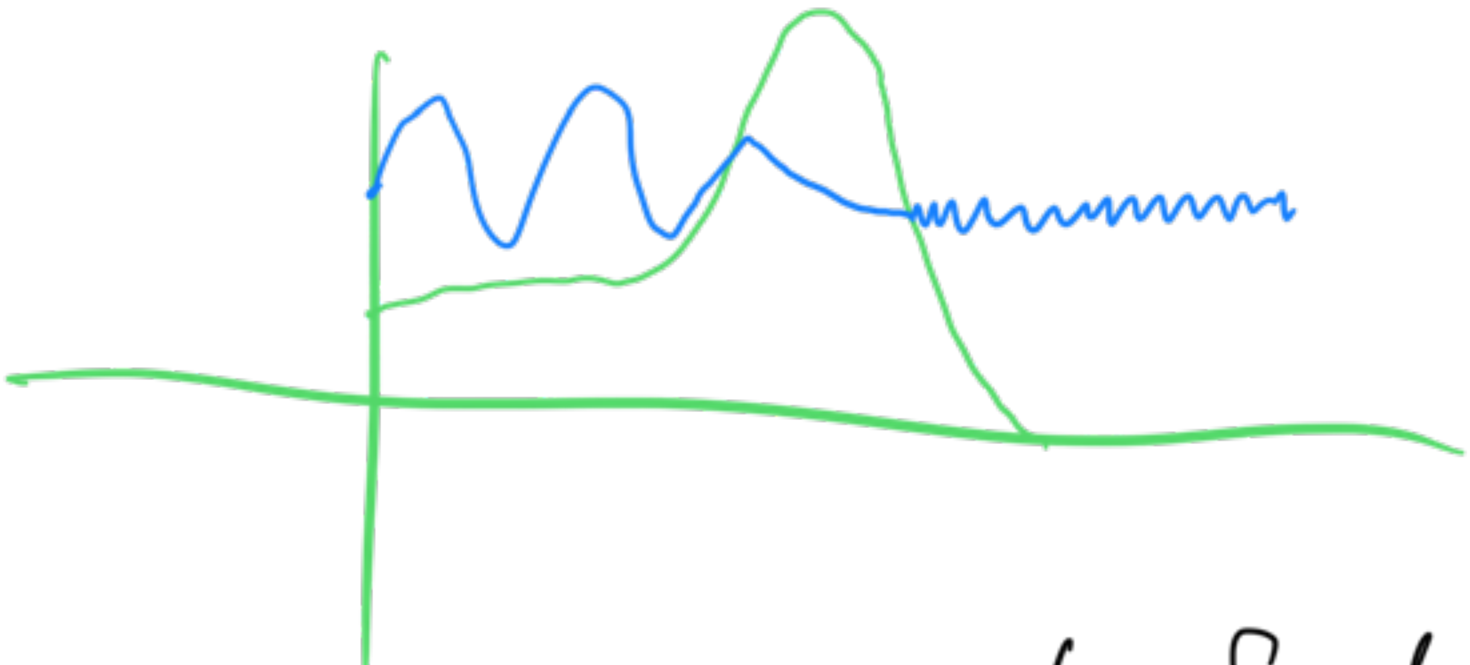
$E > V$







tunneling



scale factor

$$H = \frac{\dot a}{a}$$

$$\cancel{\ddot z} + \dot z^2 + \left(\frac{c^2 m}{\hbar}\right)^2 = 0$$

$$z_0 = \frac{i c^2 m}{}$$

h

$$\phi = e^{z_1} = \int^{t} \frac{i c^2 m}{\hbar} - \frac{3\dot{a}}{2a}\, dt$$

$$= \left(\cos \int^{t} \frac{c^2 m}{\hbar}\, dt'\right) a^{-3/2}$$

$$L_0 = h_2 \frac{d^2}{dx^2} + h_1 \frac{d}{dx} + h_0$$

$$L_1 = -\frac{1}{h_2} L_0$$

$$= -\frac{d^2}{dx^2} - \frac{h_1}{h_2}\frac{d}{dx} - \frac{h_0}{h_2}$$

$$L = \rho L_1 = -\rho \left[\frac{d^2}{dx^2} - \frac{h_1}{h_2}\frac{d}{dx} - \frac{h_0}{h_2}\right]$$

$$= \frac{e^{\int h_1/h_2}}{-h_2} L_0 = \rho L_0$$

$$L_0 u = \lambda u$$

$$L u = p L_0 u = p \lambda u$$

$$\exp\left[-\int^x \frac{p'}{p} dx'\right] = \exp\left[-\int (\log p)' dx'\right]$$

$$= \exp\left[-\log p(x) - c\right]$$

$$= \frac{d}{p(x)}$$

$$p = \frac{\hbar}{i} \frac{d}{dx}$$ hermitian

$$\vec{L} = \vec{x} \wedge \vec{p} = \vec{x} \times \vec{p}$$

$\vee$

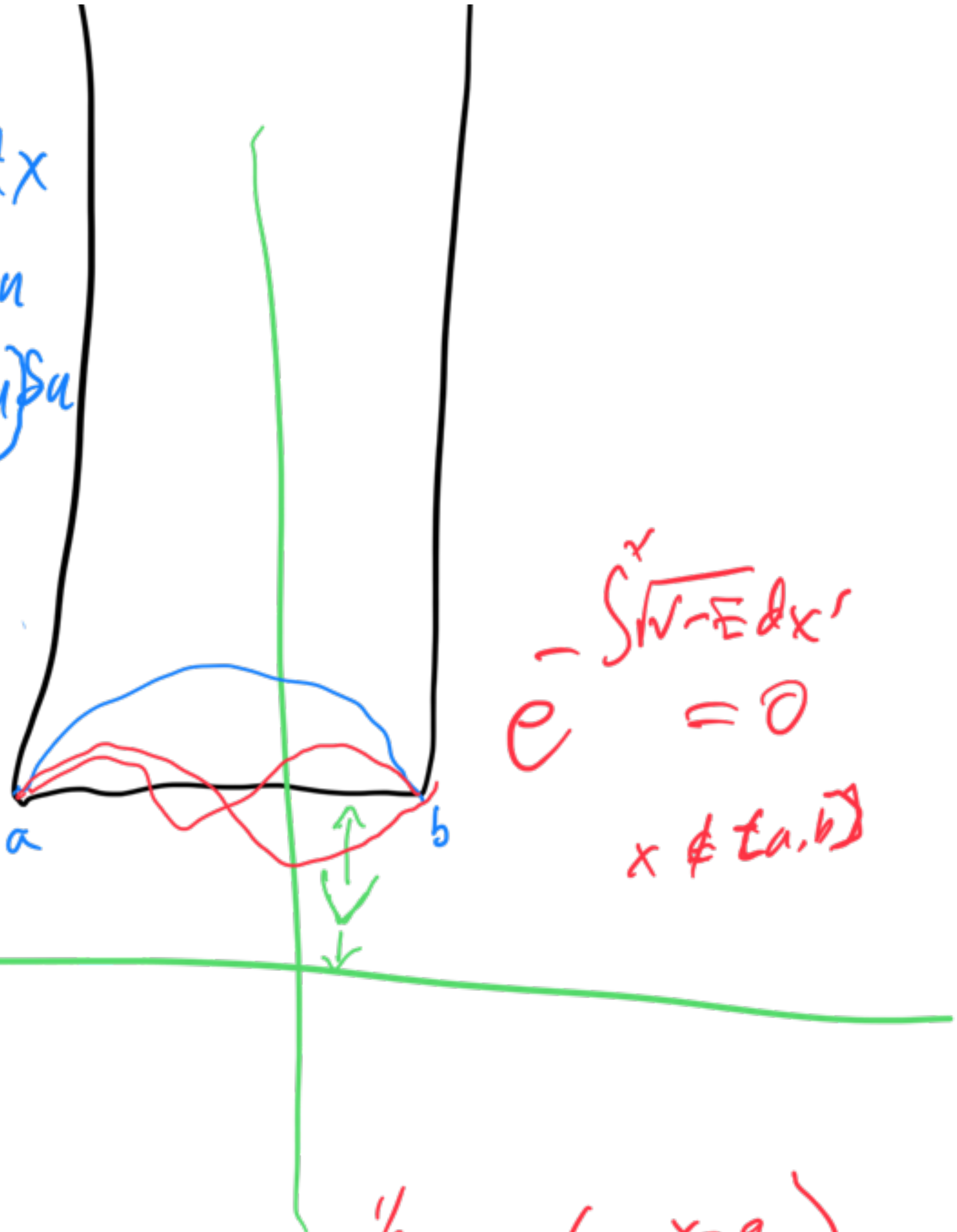
$$\int u'^2 + p\,u^2\,dx$$

$$\int u'\delta u' + p\,u\,\delta u$$

$$= \int -u''\delta u + p\,u\,\delta u$$

$$\Rightarrow$$

$$-u'' + p\,u = \lambda u$$

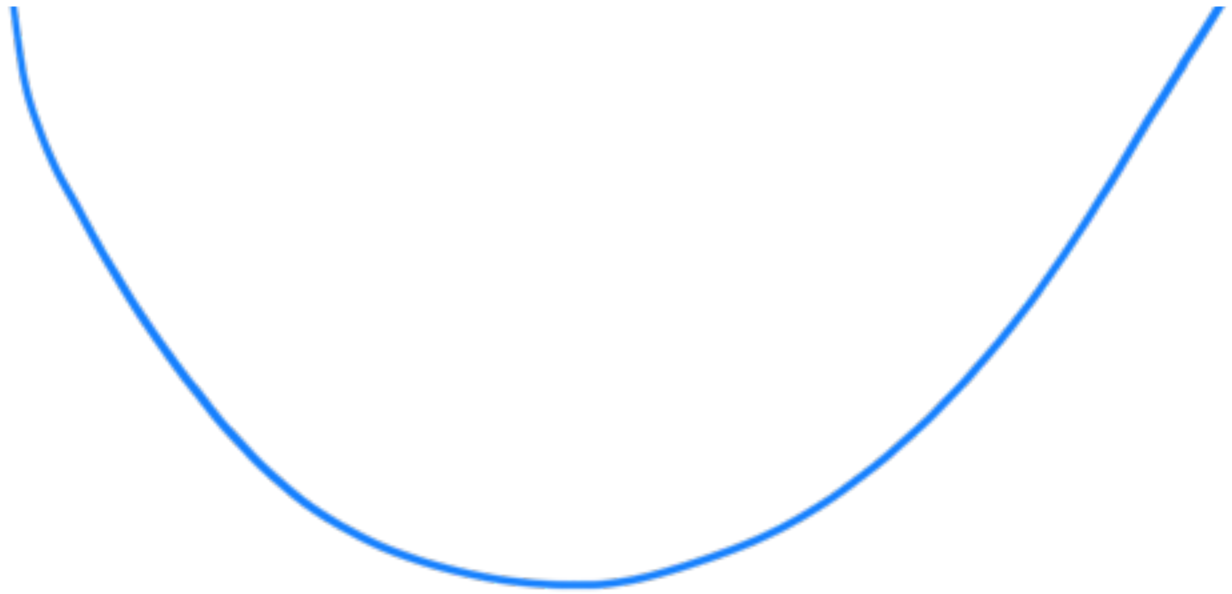


$$e^{-\int^{x}\sqrt{V-E}\,dx'} = 0$$

$$x \notin [a,b]$$

$$u_n(x) = \left(\frac{2}{b-a}\right)^{1/2} \sin\left(n\pi\,\frac{x-a}{b-a}\right)$$

$$\lambda_n = E[u_n] = \frac{1}{2m}\left(\frac{n\pi\hbar}{b-a}\right)^2$$



$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}u + \frac{m\omega^3}{2}x^2 u = \lambda u$$

$$= \hbar\omega\left[\frac{m\omega}{2\hbar}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right) + \frac{1}{2}\right]u$$

ground state

$$\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\langle x|0\rangle = 0$$

$$u_0(x) = \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$$

$$u_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{2\hbar}\right)^{\frac{n}{2}}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)^n u_0(x).$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i\,p}{m\omega}\right)$$

$$[a, a^\dagger] = 2\frac{m\omega}{2\hbar}\frac{\hbar}{m\omega} = 1 \qquad [q,p] = i\hbar$$

$$|n\rangle = \frac{1}{\sqrt{n!}} a^{\dagger n} |0\rangle \qquad [x,p] = i\hbar$$

$$a|z\rangle = z|z\rangle$$

$$\psi_n(x) = \langle x | n \rangle. \qquad z = x + iy$$

$$0 = x^2 y'' + x\,p(x)\,y' + q(x)\,y$$

where $x\,p(x) = x^2 P(x)$

$$q(x) = x^2 Q(x)$$

and

$$0 = y'' + P(x)\,y' + Q(x)\,y.$$

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum a_n x^{n+r}$$

$$y'(x) = \sum (n+r)\,a_n\,x^{n+r-1}$$

$$x^{n+r-2}$$

$$y''(x) = \sum (n+r)(n+r-1) a_n x$$

So $0 = x^2 y'' + x p y' + q y$ is

$$0 = \sum (n+r)(n+r-1) a_n x^{n+r}$$
$$+ p(x) \sum (n+r) a_n x^{n+r}$$
$$+ q(x) \sum a_n x^{n+r}$$

So

$$0 = \sum \left[(n+r)(n+r-1) + p(x)(n+r) + q(x) \right] a_n x^{n+r}$$

The lowest power of x here is for $n=0$ given by the indicial equation:

$$0 = r(r-1) + p(0) r + q(0)$$

$$0 = r^2 + (p(0)-1) r + q(0)$$

$$r = \frac{1 - p(0) \pm \sqrt{(1-p(0))^2 - 4 q(0)}}{2}.$$

Fuchs said use bigger root

$$r = \frac{1 - p(0) + \sqrt{(1-p(0))^2 - 4 q(0)}}{2}.$$

But now if

$$p(x) = \sum_{j=0}^{\infty} p_j x^j$$

$$q(x) = \sum_{j=0}^{\infty} q_j x^j$$

So set $m + n + j = k + r$ $\quad n = k - j$

$m + j = k$

So j can't exceed k.

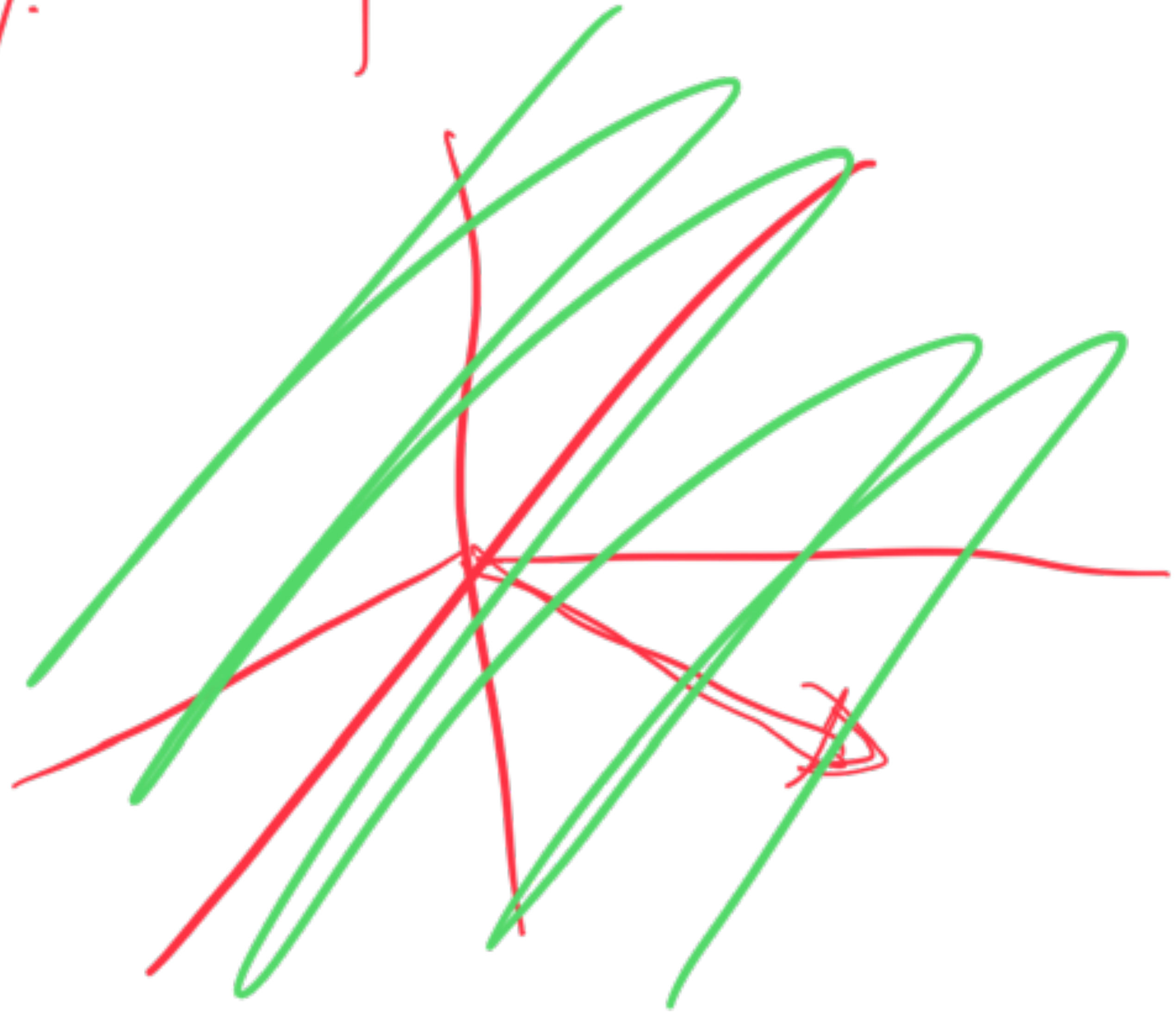
$$f(x) = \sum_{i=1}^{\infty} a_i u_i(x)$$

$$\int_a^b dx\, \rho\, u_j(x) f(x) = \int_a^b \sum_{i=1}^{\infty} a_i u_i(x) u_j(x) \rho\, dx$$

$$= \sum_{i=1}^{\infty} a_i \delta_{ij} = a_j.$$

$$f(x) = \int_a^b f(y)\, \delta(x-y)\, dy$$

e.g. $\rho(r) = 4\pi r -$



$\phi \quad \lambda$

$$\lambda = \frac{h}{p} \text{ or } \frac{\hbar}{p}$$

$$p = \frac{h}{\lambda} \text{ or } \frac{\hbar}{\lambda}$$

$$p = \frac{1}{\lambda}$$

if we want to understand Nature

at $\lambda \to 0$

$$p = \frac{1}{\lambda} \to \infty$$

$$N = \int_a^b dx\, u_k(x) \left(f(x) - \sum_{j=1}^{n} c_j u_j(x) \right)$$

$$N(r_n, u_k) = 0$$

$$\underbrace{E(r_n, r_n)}_{\text{bounded}} = E(f,f) - 2\sum c_k \lambda_k c_k$$

$$+\sum c_k c_\ell \lambda_k \delta_{k\ell}$$

$$= E(f,f) - \sum \lambda_k c_k^2$$

neg

$$E(r_n, r_n) \le E(f,f)$$

$$\|r_n\|^2 \le \frac{E(f,f)}{\lambda_{n+1}}$$

bdd.

$\lambda_{n+1} \to \infty$

$\to$ zero

$$f \in D$$

$$f_n(x) = \sum_{i=1}^{n} u_i(x)\, c_i$$

$$0 \leftarrow \int_a |f_n(x) - f(x)| \rho \, dx$$

as $n \to \infty$.

$$f(x) = \sum c_i u_i(x)$$

$$c_i = \int u_i(y) f(y) \rho(y) \, dy$$

$$f(x) = \sum u_i(x) \int u_i(y) f(y) \rho(y) \, dy$$

$$= \int f(y) \left[\sum u_i(x) u_i(y) \rho(y) \right] dy$$

$$= \int f(y) \, \delta(x-y) \, dy$$

$$\sum^{\infty} \ldots \rho(y) \sum^{\infty} u_i(x) u_i(y)$$

$$\boxed{\delta(x-y) = \rho(y)\sum_{i=0} u_i(x)\,u_i(y)}$$

$$= \rho(x)\sum_{i=0}^{\infty} u_i(x)\,u_i(y)$$

$$\delta(x-y) = \rho^{\alpha}(x)\,\rho^{1-\alpha}(y)\sum_{i=1}^{\infty} u_i(x)\,u_i(y)$$

$$L u = -(xu')' + \frac{n^2 u}{x} = \lambda x u$$

$$\lambda = z^2_{n,k}$$

$$\Box = \Delta - \frac{\partial^2}{\partial t^2} = \Delta - \partial_0^2$$

$$x^0 = ct$$

$\theta(x)$

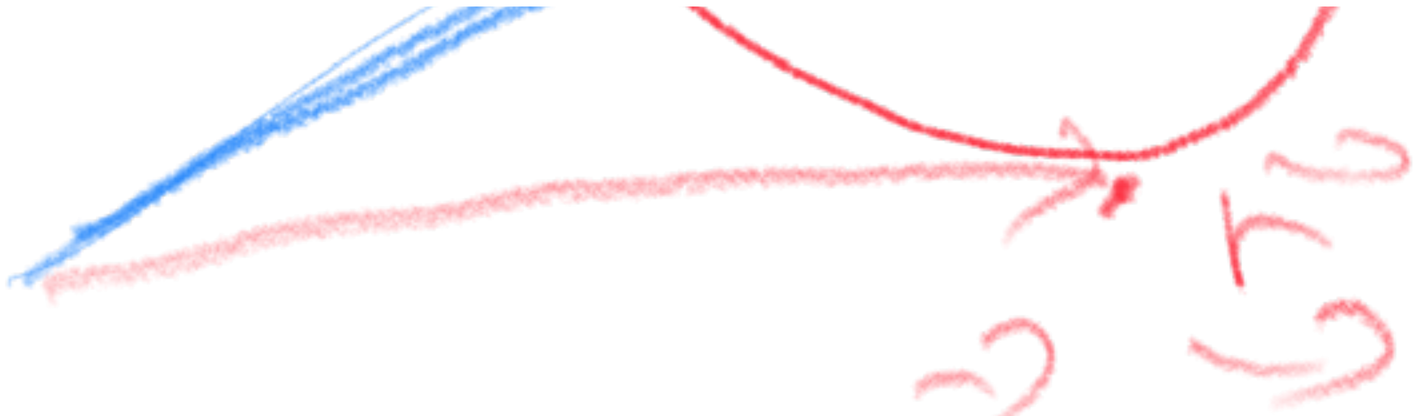
$x=0$

$$\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

$$\partial_0 J^0 + \vec{\nabla} \cdot \vec{J} = 0$$

$$\partial_0 J^0 = -\vec{\nabla} \cdot \vec{J}$$

$$\partial_0 J^0 = \frac{\partial J^0}{\partial t}$$



$$\delta r = \vec{r} \times \vec{\theta}$$

$$-\left[(1-x^2)P_n'\right]' = n(n+1)P_n$$

$$-(1-x^2)P_n'' + 2xP_n' = n(n+1)P_n$$

$$P_l'' = +\frac{2x}{(1-x^2)}P_l' - \frac{l(l+1)P_n}{(1-x^2)}$$

$$Q = \frac{l(l+1)}{(1-x^2)}$$

$$Q(x-1)^2 = \frac{(x-1)^2 l(l+1)}{(1-x)(1+x)}$$

$$= \frac{l(l+1)(x-1)}{1+x} \to 0$$

$$x \to 1$$

$$z = \frac{1}{x}$$

$$\lim_{z \to 0} \frac{2z - \frac{\frac{2}{z}}{(1 - 1/z^2)}}{z^2} = \frac{2}{z} - \frac{2}{z(z^2-1)}$$

$$= 0$$

$$Q = \frac{1}{1-x^2} = \frac{1}{1-\frac{1}{z^2}}$$

$$a_{j+1} = \frac{j(j+1)-\lambda}{(j+2)(j+1)} a_{j-1}$$

$$a_{j+2} =$$

$$b_j = c_k \cdot Y_{kj}$$

$$b = c\,Y$$

$$b\gamma^{-1} = c$$

$$\langle \chi | H | \psi \rangle = E_\psi \langle \chi | \psi \rangle$$

$$= \langle H\chi | \psi \rangle = E_\chi \langle \chi | \psi \rangle$$

$$E_\psi = E_\chi$$

unless $\langle \chi | \psi \rangle = 0$.

$$Lu = -(xu')' + \frac{n^2}{x} u = \lambda x u$$

$$L = L(\phi_i, \partial_a \phi_i)$$

$$\frac{\partial L}{\partial x^a} = 0$$

$$0 = \partial_a \left(\frac{\partial \mathcal{L}}{\partial \partial_a \phi} \delta\phi \right)$$

$$\nabla_i \phi \, \epsilon_{ijk} \theta_j \, x_k$$

$$= \epsilon_{jki} \nabla_i \phi \, \theta_j \, x_k$$

$$= \cancel{\epsilon_{kij}} \quad ''$$

$$ds^2 = \sum_{i,j=0}^{3} g_{ij} \, dx^i dx^j$$

$$x^{i'} = x^{i'}(x^0, x^1, x^2, x^3)$$

$i = 0, 1, 2, 3$