8

Integral equations

8.1 Differential equations as integral equations

Differential equations when integrated become integral equations with built-in boundary conditions. Thus if we integrate the first-order ODE

$$\frac{du(x)}{dx} \equiv u_x(x) = p(x)\,u(x) + q(x) \tag{8.1}$$

then we get the integral equation

$$u(x) = \int_a^x p(y)\,u(y)\,dy + \int_a^x q(y)\,dy + u(a). \tag{8.2}$$

To transform a second-order differential equation into an integral equation, we use Cauchy's identity (exercise 8.1)

$$\int_a^x dz \int_a^z dy\, f(y) = \int_a^x (x-y)\, f(y)\, dy, \tag{8.3}$$

which is a special case of his formula for repeated integration

$$\int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} f(x_n)\, dx_n \ldots dx_2\, dx_1 = \frac{1}{(n-1)!} \int_a^x (x-y)^{n-1}\, f(y)\, dy. \tag{8.4}$$

We first write the second-order ODE in self-adjoint form $-(pu')' + qu = \lambda \rho u$ as outlined in section 7.32 and then as $(pu)'' = (p'u)' + (q - \lambda\rho)u$ which we integrate twice to

$$\begin{aligned} p(x)u(x) = p(a)u(a) + (x-a)p(a)u'(a) + \int_a^x p'(y)u(y)\,dy \\ + \int_a^x dy \int_a^y dz (q(z) - \lambda\rho(z))u(z). \end{aligned} \tag{8.5}$$

We then use Cauchy's identity (8.3) to integrate this equation to

$$p(x)u(x) = f(x) + \int_a^x k(x,y)\,u(y)\,dy \tag{8.6}$$

in which $f(x) = p(a)[u(a) + (x-a)u'(a)]$ and

$$k(x,y) = p'(y) + (x-y)[q(y) - \lambda\rho(y)]. \tag{8.7}$$

Example 8.1 (Legendre's equation) The function $p(x) = 1 - x^2$ in Legendre's equation $-[(1-x^2)P_n']' = n(n+1)P_n$ vanishes at the end point $x = a = -1$ of the interval $[-1,1]$, so $f(x)$ also vanishes, and therefore formulas (8.6 and 8.7) give Legendre's integral equation as

$$(1-x^2)P_n(x) = -\int_{-1}^x [2y + n(n+1)(x-y)]\,P_n(y)\,dy. \tag{8.8}$$

□

Example 8.2 (Bessel's equation) The function $p(x) = x$ in Bessel's equation (10.11) $-[xJ_n'(x)]' + (n^2/x)J_n(x) = xJ_n(x)$ for $k = 1$ vanishes at the end point $x = a = 0$ of the interval $[0,1]$, so $f(x)$ also vanishes, and therefore since $q(x) = n^2/x$, formulas (8.6 and 8.7) give Bessel's integral equation as

$$xJ_n(x) = \int_0^x \left\{1 + (x-y)\left[n^2/y - y\right]\right\} J_n(y)\,dy. \tag{8.9}$$

□

In some physical problems, integral equations arise independently of differential equations. Whatever their origin, integral equations tend to have properties more suitable to mathematical analysis because derivatives are unbounded operators.

8.2 Fredholm integral equations

An equation of the form

$$\int_a^b k(x,y)\,u(y)\,dy = \lambda\,u(x) + f(x) \tag{8.10}$$

for $a \le x \le b$ with a given **kernel** $k(x,y)$ and a specified function $f(x)$ is an **inhomogeneous Fredholm equation of the second kind** for the function $u(x)$ and the parameter λ. (Erik Ivar Fredholm, 1866–1927).

If $f(x) = 0$, then it is a **homogeneous Fredholm equation of the**