ordinary differential equation

$$(1 - v^2)\,\phi_{uu} = \frac{dV}{d\phi} = V_{,\phi}. \tag{7.515}$$

We multiply both sides of this equation by ϕ_u to get $(1-v^2)\,\phi_u\,\phi_{uu} = \phi_u\,V_{,\phi}$ which we can integrate to $(1-v^2)\,\frac{1}{2}\,\phi_u^2 = V + E$ or (exercise 7.41)

$$u - u_0 = \int \frac{\sqrt{1-v^2}}{\sqrt{2(E + V(\phi))}}\, d\phi \tag{7.516}$$

in which E is a constant of integration. This integral gives us $u(\phi)$ which we then invert to get $\phi(u - u_0) = \phi(x - x_0 - v(t - t_0))$, which is a lump of energy traveling with speed v.

Example 7.75 (Soliton of the ϕ^4 theory) To simplify the integration (7.516), we take as the action density

$$\mathcal{L} = \frac{1}{2}\left(\dot{\phi}^2 - \phi'^2\right) - \left[\frac{\lambda^2}{2}\left(\phi^2 - \phi_0^2\right)^2 - E\right]. \tag{7.517}$$

Our formal solution (7.516) gives

$$u - u_0 = \pm\int \frac{\sqrt{1-v^2}}{\lambda\left(\phi^2 - \phi_0^2\right)}\, d\phi = \mp\frac{\sqrt{1-v^2}}{\lambda\phi_0}\,\tanh^{-1}(\phi/\phi_0) \tag{7.518}$$

or

$$\phi(x - vt) = \mp\phi_0 \tanh\left[\lambda\phi_0 \frac{x - x_0 - v(t - t_0)}{\sqrt{1-v^2}}\right] \tag{7.519}$$

which is a soliton (or an antisoliton) at $x_0 + v(t - t_0)$. A unit soliton at rest is plotted in Fig. 7.6. Its energy is concentrated at $x = 0$ where $|\phi^2 - \phi_0^2|$ is maximal. □

7.48 Matlab Solves Differential Equations

Example 7.76 (First-order nonlinear ordinary differential equation)

```
>> syms y(t) a Y
ode = diff(y,t) == a*y*(1-y/Y);
cond = y(0) == y0;
ySol(t) = dsolve(ode,cond);
>> ySol = simplify(ySol)
ySol(t) = (Y*y0*exp(a*t))/(Y - y0 + y0*exp(a*t))
```

which is the solution (7.66) of the logistic equation (7.63). □

Example 7.77 (Second-order linear ordinary differential equation)

```
>> syms u(x) x
>> cond = u(1) == 1;
>> ode = -(1-x^2)*diff(u,x,2) + 2*x*diff(u,x,1) == 6*u;
>> uSol(x) = dsolve(ode,cond);
>> uSol = simplify(uSol)
uSol(x) = (3*x^2)/2 - 1/2
```

which is the Legendre polynomial (example 9.1) $P_2(x)$. A second boundary condition is not needed because the solution $Q_2(x)$ is singular at $x = 1$. □

Example 7.78 (Second-order linear ODE with two conditions)

```
>> syms u(x) x
>> cond1 = u(0) == 1;
>> cond2 = u(pi) == 0;
>> ode = x^2*diff(u,x,2) + 2*x*diff(u,x,1) + x^2*u ==0;
>> uSol(x) = dsolve(ode,cond1,cond2);
>> uSol = simplify(uSol)
uSol(x) = sin(x)/x
```

which is the spherical Bessel function $j_0(x)$. □

Further reading

One can learn more about differential equations in *Advanced Mathematical Methods for Scientists and Engineers* (Bender and Orszag, 1978).

Exercises

7.1 In rectangular coordinates, the curl of a curl is by definition (2.45)

$$(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}))_i = \sum_{j,k=1}^{3} \epsilon_{ijk} \partial_j (\boldsymbol{\nabla} \times \boldsymbol{E})_k = \sum_{j,k,\ell,m=1}^{3} \epsilon_{ijk} \partial_j \epsilon_{k\ell m} \partial_\ell E_m. \tag{7.520}$$

Use Levi-Civita's identity (1.535) to show that

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) - \triangle \boldsymbol{E}. \tag{7.521}$$

This formula defines $\triangle \boldsymbol{E}$ in any system of orthogonal coordinates.

7.2 Show that since the Bessel function $J_n(x)$ satisfies Bessel's equation (7.26), the function $P_{kn}(\rho) = J_n(k\rho)$ satisfies (7.25).

7.3 Show that (7.38) implies that $R_{k,\ell}(r) = j_\ell(kr)$ satisfies (7.37).

7.4 Use (7.36, 7.37), and $\Phi_m'' = -m^2 \Phi_m$ to show in detail that the product $f(r,\theta,\phi) = R_{k\ell}(r)\, \Theta_{\ell m}(\theta)\, \Phi_m(\phi)$ satisfies $-\triangle f = k^2 f$.