Errata in *Physical Mathematics*

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Second-Edition List

Chapter 1

In Section 1.5, on page 10, in equation 1.63: The index ℓ should be k.

In Section 1.5, on page 11, equation 1.70 should be

$$\mathrm{Tr}(a') = \mathrm{Tr}(u^{-1}\, a\, u) = \mathrm{Tr}(a\, u\, u^{-1}) = \mathrm{Tr}(a).$$

(Thanks to Spencer Dimitroff.)

On page 83, a better statement of exercise 1.40 of Chapter 1 is:
The coherent state $|\{\alpha(\boldsymbol{k},\ell)\}\rangle$ is an eigenstate of the annihilation operator $a(\boldsymbol{k},\ell)$ with eigenvalue $\alpha(\boldsymbol{k},\ell)$ for each wavenumber $\boldsymbol{k}$ and polarization ℓ

$$a(\boldsymbol{k},\ell)|\{\alpha(\boldsymbol{k},\ell)\}\rangle = \alpha(\boldsymbol{k},\ell)|\{\alpha(\boldsymbol{k},\ell)\}\rangle. \tag{1}$$

The positive-frequency part $E_i^{(+)}(t,\boldsymbol{x})$ of the electric field is a sum over $\boldsymbol{k}$ and ℓ

$$E_i^{(+)}(t,\boldsymbol{x}) = \sum_{\boldsymbol{k}} \sum_{\ell=1}^{2} a(\boldsymbol{k},\ell)\, e_i(\boldsymbol{k},\ell)\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} \tag{2}$$

in which $\boldsymbol{e}(\boldsymbol{k},\ell) = \boldsymbol{k} \times \boldsymbol{\epsilon}(\boldsymbol{k},\ell)$ and $\boldsymbol{\epsilon}(\boldsymbol{k},\ell)$ is proportional to a polarization vector. Show that $|\{\alpha_k\}\rangle$ is an eigenstate of $E_i^{(+)}(t,\boldsymbol{x})$ as in (1.490) and find its eigenvalue $\mathcal{E}_i^{(+)}(t,\boldsymbol{x})$.

Chapter 2

On page 85, between equations (2.10) and (2.11)

$$\hat{\boldsymbol{e}}_j = \boldsymbol{e}_/ h_j \quad \text{should be} \quad \hat{\boldsymbol{e}}_j = \boldsymbol{e}_j / h_j.$$

On page 89, equation 2.36 should be (Thanks to Spencer Dimitroff)

$$\boldsymbol{E} = \hat{\boldsymbol{r}} \frac{\rho R}{3\epsilon_0} = \hat{\boldsymbol{r}} \frac{b}{R^2},$$

and equation 2.38 should be

$$\boldsymbol{dS} = h_i \hat{\boldsymbol{e}}_i du_i \times h_j \hat{\boldsymbol{e}}_j du_j = \sum_{k=1}^{3} \epsilon_{ijk} \, \hat{\boldsymbol{e}}_k \, h_i h_j \, du_i du_j.$$

Chapter 3

In the top line on page 114, $c_2 = 0$ should be $f_2 = 0$.

In the line immediately after equation (3.113), $c_{2n} = 0$ should be $f_{2n} = 0$.

Chapter 4

On page 135 of Section 4.3, equation 4.49 of Example 4.4 should be

$$\delta\big(g(x)\big) = -\frac{\delta'(x - x_0)}{|g''(x_0)|},$$

and as noted there, this relation holds only if x_0 is the only root of $g(x)$ and $g(x_0) = g'(x_0) = 0$, and if $f \in C^1$ and $f(x_0) = 0$.

Section 4.10, example 4.13, page 147: In equations 4.135–4.139, the prefactor τ should be τ^{-1}. The last line of equation 4.138 is then

$$L(t) = \frac{\sin(\omega t) - \omega\tau \cos(\omega t)}{1 + (\omega\tau)^2}.$$

Chapter 5

On page 171 in equation 5.95 of example 5.12 of Section 5.9, each of the two occurrences of $\hbar E$ should be $2\hbar E$.

Chapter 6

On page 199 of Section 6.7, equation 6.72 should read

$$\sum_{n=0}^{\infty} \frac{|z-z_0|^n}{n!} |f^{(n)}(z_0)| \le M \sum_{n=0}^{\infty} \frac{|z-z_0|^n}{R^n}.$$

Chapter 7

In the fourth line on page 254 in example 7.7 of Section 7.3, the inline equation for $f(\rho,\phi,z)$ should be

$$f(\rho,\phi,z) = J_n(\alpha\rho)e^{in\phi}e^{\alpha z}.$$

Also on page 254, the first sentence after equation 7.35 should be: The first and last terms are functions of r, and the two middle terms are functions of θ.

On page 255, equation 7.39 should be

$$\partial_a f = \frac{\partial f}{\partial x^a} \quad \text{and} \quad \partial^a f = \frac{\partial f}{\partial x_a}, \quad \text{so} \quad \partial^0 f = -\partial_0 f. \tag{3}$$

On page 272 of Section 7.13, equation 7.167 should be

$$\frac{dS}{dt} = L = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q_i}\dot{q}_i = \frac{\partial S}{\partial t} + \sum_i p_i \dot{q}_i.$$

On page 279, the second line should be, "in which $x_1 = \dot{x}_2$, $x_2 = \dot{x}_3$, and $x_3 = x$."

On page 281 of Section 7.20, equation 7.217 should be

$$0 = \sum_{k=0}^{n} c_k \, t^{n-k} \, \frac{d^k y}{dt^k}.$$

On page 286, just below equation 7.241 of Section 7.24, $h_0 = 1$ should be $h_2 = -1$.

On page 295, in the first line of Section 7.32, the word 'homogeneous' should be omitted.

On page 297 in equation 7.310 of Section 7.33, $1 - x^2$ should be $1 - x'^2$.

On page 318 in equation 7.446 of Section 7.41, $\delta_{n\ell}$ should be δ_{nk}.

On page 331, in the statement of exercise 7.2, $P_n(\rho)$ should be $P_{kn}(\rho)$.

Chapter 9

On page 355 in the second line of Section 9.10, $\alpha = 0$ should be $k = 0$.

On page 360 in the last sentence of the first paragraph of Section 9.14, the words "is usually (and inexplicably) called **recombination**" should be "is called **decoupling**."

On page 362 equation 9.128 of Section 9.14 should be

$$\frac{3.8 \times 10^5 \times 1100 \times 180^\circ}{\sqrt{3} \times 13.8 \times 10^9 \times \pi} = 1.0^\circ,$$

and in the following three sentences "one degree" and "1°" should be "0.6°."

On page 362, please replace the last paragraph with

We can learn a lot from the data in the CMB Figure 9.4. The radius of the maximum gravitationally compressed region at decoupling is related to the speed of "sound" in the plasma, which is nearly $c/\sqrt{3}$. The expansion of the universe since transparency has stretched the wavelength of light and reduced its temperature from 3000 K to 2.7255 K, an expansion factor $z \approx 3000/2.7255 \approx 1100$. The radius of the maximum compressed region is now longer by 1100. So is our distance from the surface of last scattering. A careful analysis based on general relativity and a model of a flat ($k = 0$) universe with cold dark matter and dark energy gives the solid curve in Fig. 9.4 which fits the CMB data for multipole moments greater than 30. The model and the data put the maximum of the power spectrum D_ℓ at $\ell \sim 200$ and at an angle of $0.6°$. This agreement tells us that space is flat. For if the universe were closed ($k = 1$), then the angle would appear bigger; and if it were open ($k = -1$), it would appear smaller. The heights and locations of the peaks in Fig. 9.4 tell us about the density of baryons, the density of dark matter, and much more (Aghanim et al., 2018).

An elementary introduction to the CMB, cold dark matter, and dark energy is provided by the paper "Flat space, dark energy, and the cosmic microwave background" (*Eur. J. Phys.* 41 (2020) 3, 035603 & e-print 2002.11464).

Chapter 10

On page 370 in example 10.1: on the line after equation 10.27, $\sin(n(\theta - \theta_0))$ should be $\sin(n\theta - \theta_n)$; in equation 10.29, x should be r; and in equation 10.30, $\sin[n(\theta - \theta_0)]$ should be $\sin(n\theta - \theta_n)$.

On page 375, in example 10.3, in the first line of the first paragraph after equation 10.61, B_z should be B^z; and in the first line of the first paragraph after equation 10.63, E_z should be E^z.

On page 377, the first words of the sentence enclosing equation (10.73) should be, "The series expansion (10.10) for $J_{\ell+1/2}$."

On page 381, in equation 10.96, the last term should include a sum over m

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{\frac{2}{\pi}}\, i^\ell \, j_\ell(kr) \, Y_{\ell,m}(\theta, \phi) \, Y^*_{\ell,m}(\theta', \phi'). \tag{4}$$

The same sum over m is needed in the last term of equation 10.125 of exercise 10.26 on page 388.

Chapter 11

On page 406, Section 11.16, in equation 11.68, ${f^c}_{ab}$ should be $i\,{f^c}_{ab}$, so that equation should read

$$[t_a, t_b] = \sum_{c=1}^{n} i\,{f^c}_{ab}\,t_c \quad \text{and} \quad [t'_a, t'_b] = \sum_{c=1}^{n} i\,{f^c}_{ab}\,t'_c.$$

Similarly, on page 407, equation 11.70 should be

$$[t_a, t_b] = i\,f^c_{ab}\,t_c \quad \text{and} \quad [t'_a, t'_b] = i\,f^c_{ab}\,t'_c.$$

On page 416, in Section 11.21, in the last line of Example 11.21

$$\frac{2\ell+1}{4\pi} \quad \text{should be} \quad \frac{4\pi}{2\ell+1}$$

as in the statement (9.123) of the addition theorem.

Chapter 12

On page 452 just before equation 12.6, "explcity" should be "explicitly."

Incidentally, in matrix notation left indexes label rows, and right indexes label columns. Transposition interchanges rows and column indexes as in

$$L^{\mathsf{T}a}{}_b = L_b{}^a \quad \text{and} \quad L^{\mathsf{T}}{}_b{}^a = L^a{}_b. \tag{5}$$

On page 475, the sentence surrounding equation 13.42 should read:
The semi-euclidian d-dimensional dot products of the tangent vectors of an n-dimensional semi-riemannian manifold define its metric for $0 \le i, k \le n-1$ as

$$\begin{aligned} g_{ik}(x) = e_i(x) \cdot e_k(x) = & -\sum_{\alpha=1}^{u} e_i^\alpha(x)\, e_k^\alpha(x) + \sum_{\alpha=u+1}^{d} e_i^\alpha(x)\, e_k^\alpha(x) \\ = & \; e_{i\alpha}(x)\, e_k^\alpha(x) = e_i^\alpha(x)\, e_{\alpha k}(x). \end{aligned} \tag{6}$$

Chapter 13

On page 473, some of the i's should have been k's in equation 13.36 which should be

$$\begin{aligned} ds^2 &= dp(x) \cdot dp(x) = (e_i(x)\,dx^i) \cdot (e_k(x)\,dx^k) \\ &= e_i(x) \cdot e_k(x)\,dx^i dx^k = g_{ik}(x)\,dx^i dx^k. \end{aligned} \tag{7}$$

Similarly, on page 475, some of the i's should have been k's in equation 13.43 which should be

$$\begin{aligned} ds^2 &= dp(x) \cdot dp(x) = (e_i(x)\,dx^i) \cdot (e_k(x)\,dx^k) \\ &= e_i(x) \cdot e_k(x)\,dx^i dx^k = g_{ik}(x)\,dx^i dx^k. \end{aligned} \tag{8}$$

On page 479 in example 13.9, there should be a semicolon in the second term of equation 13.69 which should be

$$D_\ell e_i = e_{i;\ell} = e_{i,\ell} - e_{i,\ell} \cdot e^k e_k = e_{i,\ell} - e_{i,\ell} = 0. \tag{9}$$

Similarly, on page 480 in example 13.12, there should be a semicolon in the last term of the first of the three equations 13.77 which should read

$$D_\ell(A_i\,B_k) = (A_i\,B_k)_{;\ell} = A_{i;\ell}\,B_k + A_i\,B_{k;\ell}. \tag{10}$$

In standard general relativity, the Christoffel symbols are those of the Levi-Civita connection (13.87)

$$\Gamma^k{}_{i\ell} = \tfrac{1}{2}\,g^{kj}\left(g_{ji,\ell} + g_{j\ell,i} - g_{i\ell,j}\right) \tag{11}$$

which is symmetric in its lower indices

$$\Gamma^k{}_{i\ell} = \Gamma^k{}_{\ell i}. \tag{12}$$

I assumed this symmetry when writing several of the formulas of the section on curvature (13.23). In order to make these formulas consistent with non-standard general relativity, I am restating them here:

On page 485, equations 13.97 and 13.98 should be

$$\Delta V_i = a^2\left[\Gamma^k{}_{in,\ell} - \Gamma^k{}_{i\ell,n} + \Gamma^k{}_{m\ell}\,\Gamma^m{}_{in} - \Gamma^k{}_{mn}\,\Gamma^m{}_{i\ell}\right]\,V_k \tag{13}$$

and

$$R^k{}_{i\ell n} = \Gamma^k{}_{in,\ell} - \Gamma^k{}_{i\ell,n} + \Gamma^k{}_{m\ell}\,\Gamma^m{}_{in} - \Gamma^k{}_{mn}\,\Gamma^m{}_{i\ell}. \tag{14}$$

Equations 13.100 and 13.101 should be

$$\begin{aligned} D_n D_\ell V_i &= D_n \left(V_{i,\ell} - \Gamma^k{}_{i\ell}\,V_k\right) \\ &= V_{i,\ell n} - \Gamma^k{}_{i\ell,n}\,V_k - \Gamma^k{}_{i\ell}\,V_{k,n} \\ &\quad - \Gamma^j{}_{in}\left(V_{j,\ell} - \Gamma^m{}_{j\ell} V_m\right) - \Gamma^m{}_{\ell n}\left(V_{i,m} - \Gamma^q{}_{im} V_q\right) \end{aligned} \tag{15}$$

and

$$[D_n, D_\ell] V_i = \left(\Gamma^k{}_{in,\ell} - \Gamma^k{}_{i\ell,n} + \Gamma^k{}_{m\ell}\,\Gamma^m{}_{in} - \Gamma^k{}_{mn}\,\Gamma^m{}_{i\ell}\right) V_k = R^k{}_{i\ell n}\,V_k. \tag{16}$$

The matrix 13.102 should be

$$\Gamma_\ell = \begin{pmatrix} \Gamma^0{}_{0\ell} & \Gamma^0{}_{1\ell} & \Gamma^0{}_{2\ell} & \Gamma^0{}_{3\ell} \\ \Gamma^1{}_{0\ell} & \Gamma^1{}_{1\ell} & \Gamma^1{}_{2\ell} & \Gamma^1{}_{3\ell} \\ \Gamma^2{}_{0\ell} & \Gamma^2{}_{1\ell} & \Gamma^2{}_{2\ell} & \Gamma^2{}_{3\ell} \\ \Gamma^3{}_{0\ell} & \Gamma^3{}_{1\ell} & \Gamma^3{}_{2\ell} & \Gamma^3{}_{3\ell} \end{pmatrix}. \tag{17}$$

On page 490 in example 13.17 in the first line after equation 13.136, $r = L \sinh \chi / a$ should be $r = L \sinh \chi$.

On page 507, the last sentence of Section 13.36 should be replaced by: "In flat space, this equation says that the energy and momentum of the matter fields are conserved, $0 = T^{ik}_{\ ,k}$. The analog of T^{ab} for the gravitational field is not a tensor; it's a pseudotensor t^{ab} (Dirac, 1975). The divergence of $(T^{ab} + t^{ab})\sqrt{g}$ vanishes because the action does not depend upon an external spacetime point. But the spatial integrals for the energy and momentum of the gravitational field may lie at infinity or diverge." The reference is to pages 58–63 of Dirac's book, *General Theory of Relativity*.

On page 510, in the first sentence, $J = GMa/c$ should be $J = Mc\,a$. On the same page, a better way of writing equation 13.254 is

$$\begin{aligned} ds^2 = &- \frac{\Delta - a^2 \sin^2\theta}{\rho^2}\, c^2 dt^2 + \frac{2a\, \sin^2\theta\, G(q^2 - 2Mc^2\, r)}{c^3 \rho^2}\, dt\, d\phi \\ &+ \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta}{\rho^2}\, \sin^2\theta\, d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \end{aligned} \tag{18}$$

in which $\rho^2 = r^2 + a^2 \cos^2\theta$ and $\Delta = r^2 + a^2 - 2MGr/c^2 + q^2 G/c^4$. This form of the Kerr-Newman metric makes it easier to understand this addition to the text:

By writing ds^2 as $ds^2 = -\alpha dt^2 + \beta dr^2 + \gamma d\theta^2 + \delta(d\phi - \Omega dt)^2$, one sees that a rotating mass drags nearby masses along with it at the angular velocity

$$\Omega = \frac{c\,G\,a(2Mc^2 r - q^2)}{c^4 \rho^2 (r^2 + a^2) + G a^2 \sin^2\theta(2Mc^2 r - q^2)}. \tag{19}$$

On page 513 in the last line of equation 13.267, the exponent 3 should be 2 so that the line reads

$$R_{tt} = R_{00} = [D_k, D_0]^k{}_0 = \,-\,3(\dot a/a)_{,0} - 3(\dot a/a)^2 = \,-\,3\ddot a/a. \tag{20}$$

On page 515, the first line of Section 13.44 should be, "For $i = 0$, the vanishing of the covariant divergence of the energy-momentum tensor (13.236) is."

On page 520 in the last sentence of the first paragraph of Section 13.47, the words "**recombination**, a term that makes sense only if the universe is cyclic" should be "**decoupling**."

On page 521 of Section 13.47, the term $+a(t_m)$ should be added to the right-hand side of the second of equations 13.306 so that it reads

$$a(t) = \left(\frac{3H_0\sqrt{\Omega_m}\,(t - t_m)}{2}\right)^{2/3} + a(t_m). \tag{21}$$

Also, the clause after it should read, "in which t is a time well inside the era of matter and $t_m \sim$ 50,000 years."

On page 526, in Section 13.51, equation 13.319 should be

$$D_\ell\, c^a{}_k = c^a{}_{k,\ell} - \Gamma^j{}_{k\,\ell}\, c^a{}_j + \omega^a{}_{b\,\ell}\, c^b{}_k. \tag{22}$$

And in the sentence that contains equation 13.322, "So too the **spin connection** $\omega^a{}_{b\,\ell}$ is defined so as" should be "Similarly Cartan defined the **spin connection** $\omega^a{}_{b\,\ell}$ so as".

Chapter 17

On page 646, in the fourth line, “pseudomatrix” should be “pseudoinverse.”

Chapter 18

On page 651, equation 18.12 should read

$$\|x'(t) - x(t)\| = e^{\lambda t}\|x'(0) - x(0)\|. \tag{23}$$